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GROUP ACTIONS ON ALGEBRAS AND MODULE CATEGORIES

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Introduction and notations

Let k be a field and A a finite dimensional (associative with 1) k-algebra. By mod_A we denote the category of finite dimensional left A-modules. In many important situations we may suppose that A is *presented* as a quiver with relations (Q, I) (e.g. if k is algebraically closed, then A is Morita equivalent to kQ/I). We recall that if A is presented by (Q, I), then Q is a finite quiver and I is an *admissible* ideal of the path algebra kQ, that is, $J^m \subset I \subset J^2$ for some $m \geq 2$, where J is the ideal of kQ generated by the arrows of Q, see [6].

It is convenient to consider $A = kQ/I$ as a k-linear category with objects Q_0 (= vertices of Q) and morphisms given by linear maps $Q(x, y) = e_y A e_x$, where e_x is the trivial path at x (for $x, y \in Q_0$). In this categorical approach we do not need to assume that Q is finite (therefore the k-algebra kQ/I may not have unity). Ocasionally we write $A_0 = Q_0$ if we do not need to explicit the quiver Q.

The *purpose of these notes* is to present an introduction to the study of actions of groups on algebras $A = kQ/I$ and their module categories MOD_A and to consider associated constructions that have proved useful in the Representation Theory of Algebras.

A *symmetry* of the quiver Q is a permutation of the set of vertices Q_0 inducing an automorphism of Q. We denote by $\text{Aut}\,(Q)$ the group of all symmetries of Q. Those symmetries $g \in \text{Aut}\,(Q)$ inducing a morphism $g\colon kQ \to kQ$ such that $g(I) \subset I$ form the group $\text{Aut}\,(Q, I)$. In natural way, any $g \in \text{Aut}\,(Q, I)$ induces an automorphism of the module category mod_A and on the Auslander-Reiten quiver Γ_A of A (since the action commutes with the Auslander-Reiten translation τ_A of Γ_A).

In section 1, we present some basic facts about the actions of groups $G \subset \text{Aut}\,(Q, I)$ on $A = kQ/I$ (orbits, stabilizers, Burnside's lemma) and show that for a representation-finite standard algebra A, we have $\text{Aut}\,(Q, I) = \text{Aut}\,\Gamma_A$, where $\text{Aut}\,\Gamma_A$ is formed by the symmetris of Γ_A commuting with the translation τ_A. We recall that A is *standard* if A is representation-finite and for a choice of representatives of the isoclasses of indecomposables, the induced full subcategory of mod_A (denoted by $\text{ind}_A/\cong$) is equivalent to $k(\Gamma_A)$ which is the quotient of the path algebra $k\Gamma_A$ by the ideal generated by the *meshes* $\sum_{i=1}^{s} (\sigma\alpha_i) \cdot \alpha_i$ of the almost split sequences $0 \longrightarrow \tau_A X \xrightarrow{(\sigma\alpha_i)} \bigoplus_{i=1}^{s} Y_i \xrightarrow{(\alpha_i)} X \longrightarrow 0$. We recall that for $\operatorname{char} k \neq 2$, any representation-finite algebra $A = kQ/I$ is standard.

In section 2 we consider relations between the structure of $\operatorname{Aut}(Q, I)$ and the *Coxeter polynomial* of A (recall that, $p_A(t) = \det(t\,\mathrm{id} - \varphi_A)$ is the Coxeter polynomial associated to the Coxeter matrix φ_A which is $\mathbb{Z}$-invertible in case $\mathrm{g}\ell \dim A < \infty$).

In section 3 we present the main constructions associated to groups acting on algebras:

• if G acts freely on A (that is, $G \subset \operatorname{Aut}(Q, I)$ and $g(x) = x$ for a vertex x, implies $g = 1$), then the *Galois covering* $F\colon A \to A/G$ is a G-invariant functor of k-categories.

• if B is a G-graded k-category, the *smash product* $B \# G$ is a k-category accepting the free action of G. Moreover, $(B \# G)/G \cong B$.

Galois coverings were introduced by Bongartz and Gabriel [6, 7, 2] for the study of representation type of algebras. Smash products is a well-known construction in ring theory (see [1]), but only recently was observed by Cibils and Marcos [4] that it yields the inverse operation to Galois coverings. Indeed, if G acts freely on A, then A/G is a G-graded category such that $(A/G) \# G \cong A$.

In section 4, following [2], we introduce functors relating the module categories of A and A/G when G acts freely on A. The main results in these notes (section 5) relate the representation types of A and A/G (which was the original purpose of the introduction of Galois coverings). Indeed, given a Galois covering $F\colon A \to A/G = B$ with B a finite dimensional k-algebra, then B is representation-finite if and only if A is *locally representation-finite* (that is, for each $i \in A_0$, there are only finitely many indecomposable A-modules X, up to isomorphism, with $X(i) \neq 0$). The proof of this result was partially given in [7] and completed in [10], and provides an efficient tool to deal with representation-finite algebras.

The representation-infinite situation is more involved. We recall that A is said to be *tame* if for every $d \in \mathbb{N}$ there are finitely many $A - k[t]$-bimodules $M_1, \ldots, M_{s(d)}$ wich are free finitely generated as right $k[t]$-modules and such that any indecomposable A-module X with dimension d is of the form $M_i \otimes_{k[t]} (k[t]/(t-\lambda))$ for some $1 \leq i \leq s(d)$ and some $\lambda \in k$. We say that a tame algebra A is *domestic* (resp. of *polynomial growth*) if $s(d)$ can be chosen $\leq c$ (resp. $\leq d^m$ for some m) for all d. It is not hard to show that A is tame if A/G is tame for a group acting freely on A. The converse was shown to be false in [8]; nevertheless there are many interesting, general situations where it holds true.

In section 5 we give examples of results [5, 12] showing that for a Galois covering $F\colon A \to A/G$, the category A is tame if and only if A/G is tame, provided certain restrictions on the group G or on the category A/G are satisfied.

We denote by $K_0(A)$ the *Grothendieck group* of A, freely generated by representatives $S_1, \ldots, S_n$ of the simple A-modules, where $n = n(Q)$ is the number of vertices of Q. We denote P_i (resp. I_i) the projective cover (resp. injective envelope) of S_i. With the categorical approach an A-module is a functor $X\colon A \to \mathrm{Mod}_k$, and a morphism $f: X \to Y$ is a natural transformation. In case $\mathrm{g}\ell \dim A < \infty$, the Coxeter matrix $\varphi_A: K_0(A) \to K_0(A)$ is defined by $\varphi_A([P_i]) = -[I_i]$.

These notes follow closely the lectures given at the Workshop on Representation Theory in Mar del Plata, Argentina in March 2006. The *intention of the lectures* was to present an elementary introduction to the topic which would serve as a source of motivation and information on the techniques used. While we cannot provide complete proofs of every result, we tried to sketch some representative arguments. We thank the organizers of the Workshop for his hospitality.

1. The group of automorphisms of an algebra

1.1. Let $A = kQ/I$ be a finite dimensional k-algebra:

$\mathrm{Aut}\, A$ denotes the group of automorphisms of A. By $\mathrm{Aut}\,(Q)$ we denote the group of symmetries of Q and by $\mathrm{Aut}\,(Q, I)$ the group of symmetries of Q fixing I (that is, $g \in \mathrm{Aut}\,(Q)$ induces $g: kQ \to kQ$ such that $g(I) = I$).

Lemma. $\mathrm{Aut}\,(Q, I)$ *is a subgroup of* $\mathrm{Aut}\, A$. □

Each symmetry $g \in \mathrm{Aut}\,(Q)$ gives rise to a matrix $g \in \mathrm{G}\ell_{\mathbb{Z}}(n(Q))$ sending S_i to $S_{g(i)}$. This representation $\gamma: \mathrm{Aut}\,(Q) \to \mathrm{G}\ell_{\mathbb{Z}}(n(Q))$ is called the *canonical representation*.

1.2. Let $G \subset \mathrm{Aut}\,(Q, I)$ be a subgroup. For a given $i \in Q_0$, Gi denotes the *orbit* of i and subgroup of G, $G_i = \{g \in G: gi = i\}$ denotes the stabilizer of i. Clearly G_i is a subgroup of G.

Lemma. *The mapping* $Gi \to G/G_i$, $gi \mapsto gG_i$ *is a bijection from the orbit to the set of left cosets* G/G_i.

$$n(Q) = |G| \sum \frac{1}{|G_i|}$$

where the sum runs over representatives of the orbits. □

1.3. Let $K_0(A) = \mathbb{Z}^{n(Q)}$ be the Grothendieck group of A. Consider

$$\mathrm{Inv}_G(A) = \{v \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}: v^g = v \text{ for } g \in G\}$$

the $\mathbb{Q}$-space of G-invariant vectors. Then $t_0(G)$ the number of orbits of G in Q_0 equals $\dim_{\mathbb{Q}} \mathrm{Inv}_G(Q)$.

Let $S_1, \ldots, S_m$ be a set of representatives of the *irreducible* $\mathbb{C}$*-representations* of G (here m is the number of conjugacy classes of G). Let S_1 be the trivial representation.

Consider χ_β the character corresponding to S_β (that is, $\chi_\beta: G \to \mathbb{C}^*$, $g \mapsto \mathrm{tr}\, S_\beta(g)$). The characters $1 = \chi_1, \ldots, \chi_m$ form an orthonormal basis of the *class group* $X(G)$, with the scalar product $(\chi, \chi') = \frac{1}{|G|} \sum\limits_{g \in G} \chi(g)\overline{\chi'(g)}$.

Lemma. [Burnside's lemma]. $t_0(G) = \frac{1}{|G|} \sum\limits_{g \in G} \chi_\gamma(g)$ *where* $\chi_\gamma: G \to \mathbb{Q}^*$, *is the character of the canonical representation* γ.

Proof. Observe that $\chi_\gamma(g)$ is the number of fixed points of g.

$$\sum_{g\in G}\chi_\gamma(g)=\sum_{g\in G}\sum_{i\in Q_g}1=\sum_{i}\sum_{g\in G_i}1=\sum_{i\in Q_0}|G_i|=|G|\sum_{i\in Q_0}|Gi|^{-1}=|G|t_0(G)$$

where $Q_g=\{j\in Q_0\colon gj=j\}$. □

1.4. Let G be a subgroup of $\mathrm{Aut}\,(Q,I)$ then G acts on Mod_A as follows:

$$X\in\mathrm{Mod}_A \text{ and } g\in G, \text{ then } X^g\in\mathrm{Mod}_A$$

such that for $i\xrightarrow{\alpha}j$, we have $X^g(i)=X(gi)\xrightarrow{X(g\alpha)}X(gj)$. Similarly, for $f\in\mathrm{Hom}_A(X,Y)$, we define $f^g\in\mathrm{Hom}_A(X^g,Y^g)$.

Clearly, this action preserves indecomposable modules and induces an action of G of the Auslander-Reiten quiver Γ_A, satisfying:

(a) the action preserves projective, injective and simple modules;
(b) the action preserves Auslander-Reiten sequences (in particular, $(\tau_AX)^g=\tau_AX^g$);
(c) G is a subgroup of $\mathrm{Aut}\,\Gamma_A$, the group of automorphisms of the quiver Γ_A (commuting with the τ_A-structure).

For (c), let $g\in G$ be an element inducing a trivial action on $\mathrm{Aut}\,\Gamma_A$; we shall prove $g=1$. Indeed, $P_i=P_i^g=P_{gi}$ implies $gi=i$ for every vertex $i\in Q_0$.

Let $i\xrightarrow{\alpha_s}j$, $s=1,\dots,m$ be all arrows between i and j, then g establishes a permutation of the α_s. Let X_s be the 2 dimensional A-module with $X_s(\alpha_i)=\delta_{is}\colon k\to k$. Since X_s is indecomposable, $X_s^g=X_s$ or $g\alpha_s=\alpha_s$. Therefore $g=1$.

Proposition. *Let $A=kQ/I$ be an algebra satisfying:*
(a) *A is representation finite,* (b) *A is standard*
then $\mathrm{Aut}\,(Q,I)=\mathrm{Aut}\,\Gamma_A$.

Proof. We already know that $\mathrm{Aut}\,(Q,I)$ is a subgroup of $\mathrm{Aut}\,\Gamma_A$. Let $g\in\mathrm{Aut}\,\Gamma_A$ and consider the induced automorphism $\bar g$ of the mesh category $k(\Gamma_A)$. Clearly, $\bar g$ restricts to an automorphism of A (considering the full embedding $i\mapsto P_i$, $i\in Q_0$). By definition there is an automorphism $h\in\mathrm{Aut}\,(Q,I)$ inducing $\bar g$, and therefore inducing g. □

2. The canonical representation and the Coxeter matrix

2.1. Let $A=kQ/I$ and assume that $g\ell\dim A<\infty$. For example, this happens if A is triangular, that is, Q has no oriented cycles.

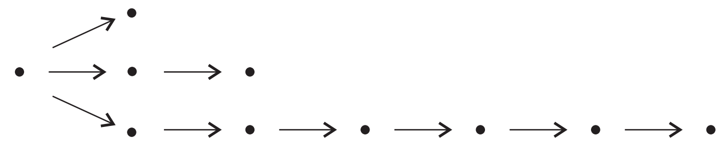
Recall that $\varphi_A\colon K_0(A)\xrightarrow{\sim}K_0(A)$, $\mathbf{dim}\,P_i\longmapsto-\mathbf{dim}\,I_i$ is the *Coxeter matrix* if A, which is $\mathbb{Z}$-invertible. In case $A=kQ$ (i.e. $I=0$), then $(\mathbf{dim}\,X)\varphi_A=\mathbf{dim}\,\tau_AX$ for any non-projective indecomposable X.

The characteristic polynomial $p_A(t)=\det\,(t\,\mathrm{Id}-\varphi_A)$ is called the *Coxeter polynomial.*

Examples:

(a) Q Dynkin type, then $\mathrm{Spec}\,\varphi_A\subset\mathbb{S}^1\setminus\{1,-1\}$.

(b) Q extended Dynkin type, then $\operatorname{Spec}\varphi_A \subset \mathbb{S}^1$ and 1 is a root of multiplicity 2.

(c) 

$p_A(t) = 1 + t - t^3 - t^4 - t^5 - t^6 - t^7 + t^9 + t^{10}$ is irreducible (over $\mathbb{Z}[t]$).

Proposition [11].

(a) *φ_A is an automorphism of the representation γ.*
(b) *If $\operatorname{Aut}(Q, I)$ is not trivial, then $p_A(t)$ is not irreducible.*

Proof. (a): It suffices to observe that $g\varphi_A = \varphi_A g$ for any $g \in G$.

(b): Let $R_1, \dots, R_m$ be a set of representatives of the irreducible $\mathbb{Q}$-representations of $G = \operatorname{Aut}(Q, I)$. Let R_1 be the trivial representation. Up to conjugation (with L)

$$\gamma^L = \bigoplus_{\alpha=1}^{m} R_\alpha^{r(\alpha)}$$

$R_\alpha \colon G \to GL_{\mathbb{Q}}(\dim R_\alpha)$, then $n = \sum\limits_{\alpha=1}^{m} r(\alpha) \dim R_\alpha$.

By Schur's lemma $\varphi_A^L = \begin{bmatrix} \varphi_1 & & 0 \\ & \ddots & \\ 0 & & \varphi_m \end{bmatrix}$, where $\varphi_\alpha \colon R_\alpha^{r(\alpha)} \to R_\alpha^{r(\alpha)}$ is an automorphism. Hence $p_A(t) = p_{\varphi_1}(t) \dots p_{\varphi_m}(t)$. Therefore if $p_A(t)$ is irreducible then $r(\alpha) = 0$ for $\alpha \geq 2$.

If $G \neq (1)$, then $t_0(G) < n$. Moreover, the characters

$$\chi_\alpha \colon G \to \mathbb{Q}^*, \quad g \mapsto \operatorname{tr} R_\alpha(g), \quad \alpha = 1, \dots, m$$

form an orthonormal basis of $X(G)$ with scalar product

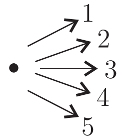
$$(\chi, \chi') = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi'(g)}.$$

Then

$$r(\alpha) = (\chi_\gamma, \chi_\alpha) = \frac{1}{|G|} \sum_{g \in G} \chi_\gamma(g) \overline{\chi_\alpha(g)}$$

for $\alpha = 1$, $r(1) = \frac{1}{|G|} \sum\limits_{g \in G} \chi_\gamma(g) = t_0(G)$.

Finally, $n = \sum\limits_{\alpha=1}^{m} r(\alpha) \dim R_\alpha = t_0(G) + \sum\limits_{\alpha=2}^{m} r(\alpha) \dim R_\alpha$, with implies the existence of $\alpha \geq 2$ with $r(\alpha) > 0$. Therefore $p_A(t)$ is not irreducible. $\square$

2.2. *Example:* $Q\colon$ 

Consider $G = A_5 \subset \operatorname{Aut} Q$ and $\gamma\colon A_5 \to G\ell\,(6)$ the canonical representation. It is not hard to calculate the character table of A_5:

conjugacy classes	$\{1\}$	(123)	$(12)(34)$	(12345)	(13524)
$\lvert Gx_i\rvert$	1	20	15	12	12
$1 = \chi_1$	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	α_1	α_2
χ_5	3	0	-1	α_2	α_1

$\alpha_1 = (1+\sqrt{5})/2$
$\alpha_2 = (1-\sqrt{5})/2$

where χ_i corresponds to S_i irreducible $\mathbb{R}$-representation of G (with S_1 the trivial representation).

Then

$$\gamma^T = \bigoplus_{\beta=1}^{s} S_\beta^{n(\beta)},\ n(1) = t_0(G) = 2$$

$$n(2) = (\chi_\gamma, \chi_2) = \frac{1}{60}\sum_{g\in G} \chi_\gamma(g)\overline{\chi_2(g)} = \frac{1}{60}[24+60-24] = 1$$

Since $\dim \gamma = 6$, $\gamma^T = S_1 \oplus S_1 \oplus S_2$ which is also a $\mathbb{Q}$-decomposition. Moreover, $p_A(t) = (1-3t+t^2)(1+t)^4$ is an irreducible factorization.

3. Constructions of algebras asociated to groups of automorphisms (coverings and smash products)

3.1. Let A be a k-category given as $A = kQ/I$. Let G be a subgroup of $\operatorname{Aut}(Q, I) \subset \operatorname{Aut} A$. We say that G *acts freely* on A if $gi = i$ for some $i \in Q_0$ implies $g = 1$.

Lemma. *Let G be a group acting freely on $A = kQ/I$. Then there exists a k-category $B = k\bar{Q}/\bar{I}$ and a functor $F\colon A \to B$ satisfying:*

(a) *(G-invariant): $Fg = F$ for every $g \in G$*
(b) *(universal G-invariant): for any functor $F'\colon A \to B'$ which is G-invariant, there exists a unique functor $\bar{F}\colon B \to B'$ such that $F' = \bar{F}F$*
(c) *$\bar{Q}_0$ is formed by the G-orbits of vertices in Q_0 and for any $a = Gi$ and $b = Gj$*

$$B(a,b) = \bigoplus_{g\in G} A(i, gj)$$

Proof. Let B be defined as in (c) with composition maps

$$B(a,b)\otimes B(b,c)=\left(\bigoplus_{g\in G}A(i,gj)\right)\otimes\left(\bigoplus_{h\in G}A(j,h\ell)\right)\to\bigoplus_{g'}A(i,g'j)=B(a,c)$$

$$(f_g)\otimes\ (f'_h)\longmapsto\left(\sum_{gh=g'}(f'_h)^g f_g\right)$$

Define $F\colon A\to B$, $f\in A(i,j)\subset\bigoplus_g A(i,gj)=B(a,b)$ in the unique possible way (since $gj=j$ implies $g=1$). □

3.2. If G acts freely on $A=kQ/I$, the functor $F\colon A\to B$ as in (2.1) is called a *Galois covering* defined by G and $B=A/G$ is a *Galois quotient* of A.

We say that a k-category B is *G-graded* if for each pair of objects a,b there is a vector space decomposition $B(a,b)=\bigoplus_{g\in G}B^g(a,b)$ such that the composition induces linear maps

$$B^g(a,b)\otimes B^h(b,c)\to B^{gh}(a,c)$$

Lemma. *$B=A/G$ is a G-graded k-category.* □

3.3. For a G-graded k-category B we define the *smash product* as the k-category $B\#G$ with objects $B_0\times G$ and for any pair $(a,g),(b,h)\in B_0\times G$, the morphisms are

$$(B\#G)((a,g),(b,h))=B^{g^{-1}h}(a,b)$$

with composition given by:

$$\begin{array}{ccccc}(B\#G)((a,g),(b,h)) & \otimes & (B\#G)((b,h),(c,t))\to & & (B\#G)((a,g),(c,t))\\ & \| & & & \|\\ B^{g^{-1}h}(a,b)\otimes B^{h^{-1}t}(b,c) & & \longmapsto & & B^{g^{-1}t}(a,c)\end{array}$$

Theorem [4]. *The category $B\#G$ accepts a free action of G such that $(B\#G)/G\cong B$.*

Moreover, if G acts freely on A, then $(A/G)\#G\cong A$.

Proof. Clearly there is a G-invariant functor $F\colon B\#G\to B$ inducing a functor $\bar{F}\colon(B\#G)/G\to B$ which is the identity on objects. Check that $\bar{F}$ is an isomorphism.

Assume G acts freely on A and let $F\colon A\to A/G$ be the induced Galois covering. Since G acts freely, there is a bijection $(A/G)_0\times G\xrightarrow[\sim]{f}A_0$ which commutes with the G-action. Moreover, if $a,b\in(A/G)_0$, $f(a,1)=i$, $f(b,1)=j\in A_0$ then

$$\begin{aligned}(A/G\#G)((a,g),(b,h)) &= A/G^{g^{-1}h}(a,b)=A(i,g^{-1}hj)=A(gi,hj)\\ &= A(f(a,g),f(b,h))\end{aligned}$$

is compatible with the composition. □

In other words, smash products and Galois quotients are inverse operations in the class of k-categories.

3.4. Galois correspondence. Let B be a k-category. There is a $1-1$ correspondence
$(A \xrightarrow{F} B$ Galois covering defined by $G) \mapsto G$ group grading B, such that for any two Galois coverings $A \xrightarrow{F} B$ defined by G and $A' \xrightarrow{F'} B$ defined by H, there exist a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ {\scriptstyle \bar{F}} \uparrow & \nearrow {\scriptstyle F'} & \\ A' & & \end{array}$$

if and only if there exists a normal subgroup $H \trianglelefteq G'$, such that $G'/H = G$.

A *Galois triple* (A, F, G) of B is a Galois covering $F \colon A \to B$ defined by the (free) action of a group G. There is a universal object in this set of Galois triples. Indeed, let $B = kQ/I$ and consider W the set of all walks in Q starting and ending at a fixed vertex b. Let $\sim$ be the equivalence relation induced by the following elementary relations:

(i) $\alpha\alpha^{-1} \sim e_y$ and $\alpha^{-1}\alpha \sim e_x$ for any arrow $x \xrightarrow{\alpha} y$;

(ii) if $\sum\limits_{i=1}^{s} \lambda_i w_i \in I(x,y)$ with $\lambda_i \in k^*$, such that for any $L \subsetneq \{1, \dots, s\}$ we have $\sum\limits_{i \in L} \lambda_i \notin I(x,y)$, then $w_i \sim w_j$ for i, j;

(iii) if $w \sim w'$, then $ww'' \sim w'w''$, whenever the products are defined.

Then $\tilde{G} = W/\!\sim$ has a group structure such that B is $\tilde{G}$-graded. The group $\tilde{G}$ is called the *fundamental group* of B.

Proposition [9]. *Let $\tilde{B} = B \# \tilde{G}$ and $\tilde{F} \colon \tilde{B} \to B$ be defined by $\tilde{G}$. The triple $(\tilde{B}, \tilde{F}, \tilde{G})$ is a* universal Galois covering, *that is, for any Galois covering $F \colon A \to B$ defined by the action of a group G, there exists a covering $\bar{F} \colon \tilde{B} \to A$ defined by $H \triangleleft \tilde{G}$ such that $\tilde{G}/H = G$.* □

4. Actions induced on module categories

4.1. Let $F \colon A \to B$ be a Galois covering of k-categories defined by the action of G. We shall denote by

MOD_A the category of left A-modules;
Mod_A those $X \in \mathrm{MOD}_A$ with $\dim_k X(i) < \infty$ for every $i \in A_0$;
mod_A those $X \in \mathrm{MOD}_A$ with $\sum\limits_{i \in A_0} \dim_k X(i) < \infty$.

There are naturally defined functors:

$$F_. \colon \mathrm{MOD}_B \to \mathrm{MOD}_A, \quad (Y : B \to \mathrm{Mod}_k) \mapsto (Y \circ F \colon A \to \mathrm{Mod}_k),$$

called the *pull-up functor*;

$$F_\lambda \colon \mathrm{MOD}_A \to \mathrm{MOD}_B, (X \colon A \to \mathrm{Mod}_k) \mapsto F_\lambda X(a) = \bigoplus_{g \in G} X(gi)$$

and such that for $f = (f_g) \in B(a,b) = \bigoplus_{g\in G} A(i,gj)$, then $F_\lambda X(f)\colon F_\lambda X(a) \to F_\lambda X(b)$, sends (a_g) to $\left(\sum_h X(f^h_{h_g^{-1}})(a_h)\right)_g$, called the *push-down functor*. We observe that F_λ is a left adjoint to $F.$. Similarly, there is a right adjoint $F_\rho\colon \mathrm{MOD}_A \to \mathrm{MOD}_B$ to $F.$. For modules $X \in \mathrm{mod}_A$, the modules $F_\lambda X$ and $F_\rho X$ coincide.

Recall that G acts on MOD_A and $X \in \mathrm{MOD}_A$ is *G-stable* if $X^g = X$ for every $g \in G$. The category of G-stable A-modules is denoted by MOD_A^G.

4.2. Proposition [7, 2]. *Let $F\colon A \to B$ be a Galois covering defined by the action of G. Then the following happens:*

(a) *The categories MOD_A^G and MOD_B are equivalent.*
(b) *For any $X \in \mathrm{MOD}_A$ and $g \in G$, we have $F_\lambda X^g \cong F_\lambda X$. Moreover, $F.F_\lambda X \xrightarrow{\sim} \bigoplus_{g\in G} X^g$ as A-modules.*
(c) *Let H be a subgroup of G and $X \in \mathrm{MOD}_A^H$. Then $F_\lambda X$ has a natural structure as kH-module. If H is a finite group and* $\mathrm{char}\, k \not| \, |H|$, *then $F_\lambda X$ decomposes as a direct sum of at least $|H|$ factors.*

Proof. (a): clear.

(b): Observe that $F_\lambda X(a) = \bigoplus_{h\in G} X(hi) \xrightarrow{\sim} F_\lambda X^g(a) = \bigoplus_{h\in G} X(hgi)$ canonically.

Hence $F.F_\lambda X(i) = F_\lambda X(Fi) = \bigoplus_{h\in G} X(hi) = \bigoplus_{h\in G} X^h(i)$ and correspondingly in morphisms.

(c): Choose a set W of representatives in G of the right cosets G/H. Then $F_\lambda X(a) = \bigoplus_{g\in G} X(i) = kH \otimes_k \left(\bigoplus_{w\in W} X(wi)\right)$ and correspondingly in morphisms.

Hence we get $\varphi\colon kH \to \mathrm{End}_B(F_\lambda X)$ a group homomorphism such that for each idempotent e of kH, $\varphi(e)$ is idempotent and there is a factorization of $F_\lambda X$.

In case H is a finite group with $\mathrm{char}\, k \not| \, |H|$, by Mashké theorem, the group algebra kH is semisimple (with $|H|$ idempotents). The result follows. □

4.3. Assume G acts freely on A and $F\colon A \to B = A/G$ is the corresponding Galois covering.

Let $X \in \mathrm{MOD}_A$, the stabilizer G_X is the subgroup of G formed by those $g \in G$ such that $X^g \cong X$. That is, $X \in \mathrm{MOD}_A^H$ if $H \subset G_X$.

Proposition [7].

a) *If $X \in \mathrm{ind}_A$ and G is torsion free, then $G_X = (1)$.*
b) *If $X \in \mathrm{ind}_A$ and $G_X = (1)$, then $F_\lambda X$ is indecomposable and for any module $Y \in \mathrm{mod}_A$ with $F_\lambda X \simeq F_\lambda Y$, then $Y \simeq X^g$ for some $g \in G$.*

Proof. (a): Let $g \in G_X$ for some $X \in \mathrm{ind}_A$, then g establishes a permutation of $\mathrm{supp}\, X$ (a finite set). Then for some $s \in \mathbb{N}$, $1 = g^s$ on $\mathrm{supp}\, X$. Since G acts freely on A, then $g^s = 1$. Since G is torsion free, then $g = 1$ and $G_X = (1)$.

(b): Assume $F_\lambda X \simeq Z \oplus Z'$, then $\bigoplus_{g\in G} X^g = F.F_\lambda X \simeq F.Z \oplus F.Z'$. Assume X is a direct summand of $F.Z \in \mathrm{MOD}_A^G$, then $\bigoplus_{g\in G} X^g \subset F.Z$ and $F.Z' = 0$. Therefore $F_\lambda X$ is indecomposable.

If $F_\lambda X \simeq F_\lambda Y$, then Y is indecomposable and $Y \simeq X^g$ for some $g \in G$. □

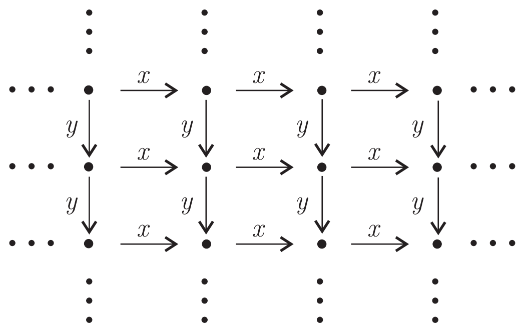
5. Coverings and the representation type of an algebra

5.1. *Examples:*

(a) Let $B = k\langle x, y\rangle/(x^2, y^3, xy, yx)$

Let $A = kQ/I$ be the infinite k-category with quiver

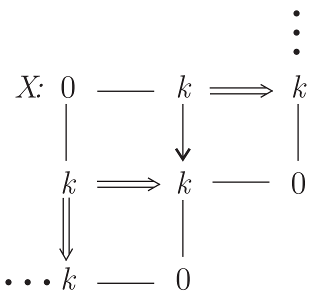
Q:



and I generated by all relations of the form x^2, y^3, xy, yx. Then we get a Galois covering $F\colon A \to B$ defined by the action of $\mathbb{Z} \times \mathbb{Z}$ on A.

The module

X:



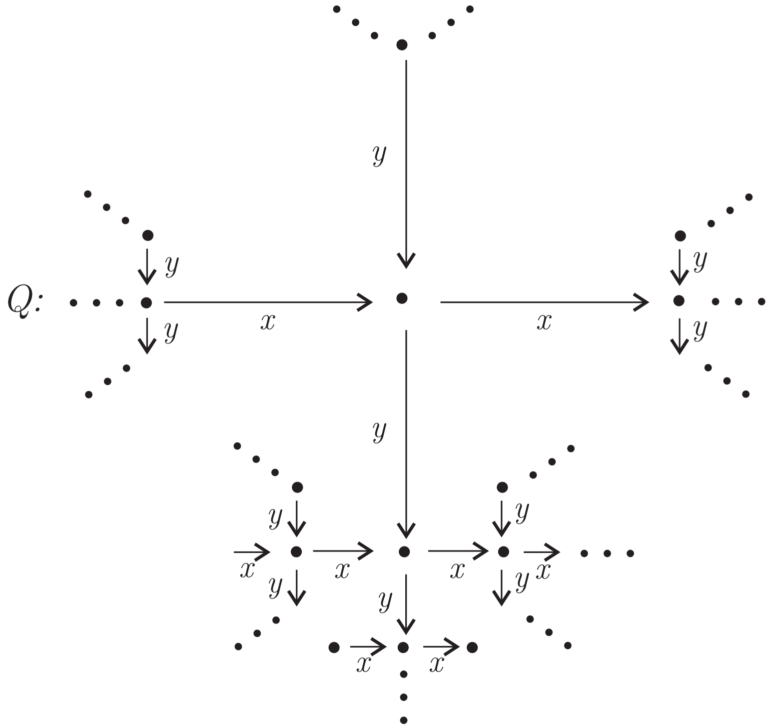
is indecomposable.

It is stable under the action of $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$, $n \mapsto (n, n)$.

The universal Galois covering of B is given by $\tilde{B} = k\tilde{Q}/\tilde{I}$ where the free group in two generators F_2 acts on $\tilde{B}$. The normal subgroup $H = \langle xy - yx\rangle$ of F_2 acts on $\tilde{B}$ inducing the covering $F\colon A \to B$.

Observe that for $X \in \mathrm{ind}_A$, we have $G_X = (1)$ and therefore $F_\lambda X \in \mathrm{ind}_B$.

(b) [13] Let B be a standard k-algebra of finite representation type. Then the Auslander-Reiten quiver Γ_B is finite and equipped with the mesh relations: $\sum_{i=1}^{n} \beta_i \circ \sigma\beta_i = 0$, for each almost split sequence $0 \longrightarrow \tau_B X \xrightarrow{(\sigma\beta_i)} \oplus Y_i \xrightarrow{(\beta_i)} X \longrightarrow 0$.



There is an universal Galois cover $\tilde{\Gamma}_B \xrightarrow{\pi} \Gamma_B$ of translation quivers defined by the action of a free group G. Moreover, $\tilde{\Gamma}_B$ is the Auslander-Reiten quiver of a k-category $\tilde{B}$ such that $k(\tilde{\Gamma}_B) = \mathrm{ind}_{\tilde{B}}$.

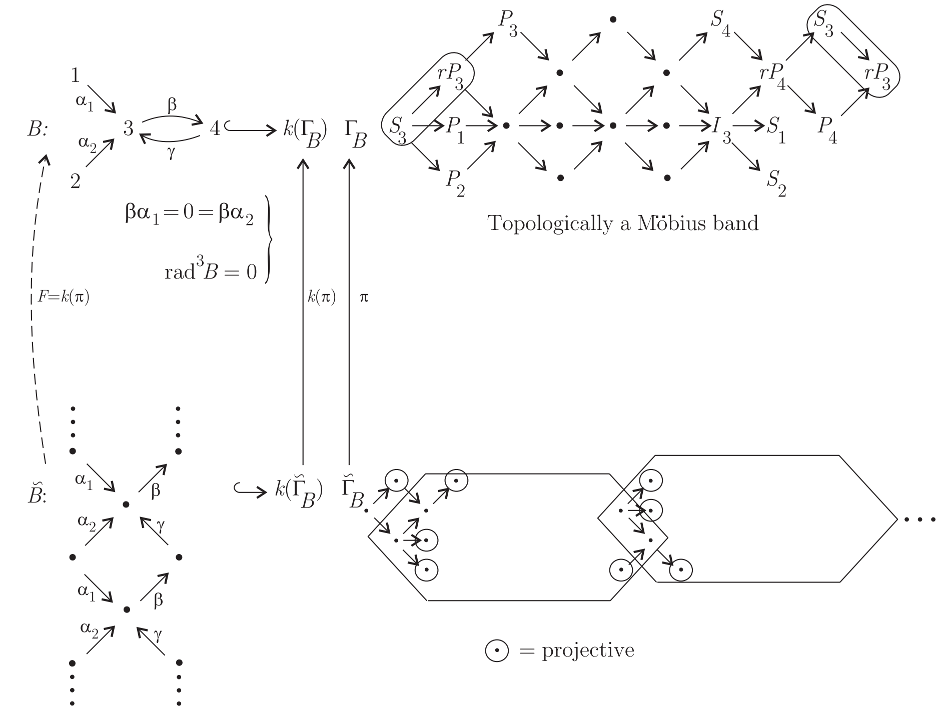
We illustrate the situation in the following *example:*

A full subcategory B' of $B = kQ/I$ is *convex* in B if $B' = kQ'/I'$ for a quiver Q' which is path closed in Q (i.e. $i_0 \to i_1 \cdots \to i_s \to i_{s+1}$ with $i_0, i_{s+1} \in Q'$ implies $i_1, \ldots, i_s \in Q'$).

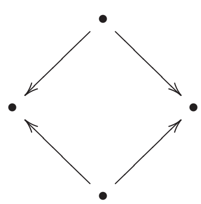
Theorem. *Let B be a representation-finite k-algebra. Then*

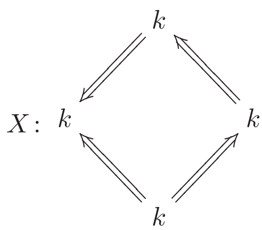
(a) [13] *There exists a universal Galois covering $\tilde{\Gamma}_B \to \Gamma_B$ defined by the action of a free group G. Moreover $k(\tilde{\Gamma}_B) = \mathrm{ind}_{\tilde{B}}$ for a k-category $\tilde{B}$. If B is standard, then $\tilde{B} \to B$ is a universal Galois covering.*

(b) [3] *For every finite convex subcategory C of $\tilde{B}$, we have $\tilde{\Gamma}_C = \Gamma_C$ and C is representation-directed (i.e. C is representation-finite and Γ_C is a preprojective component).* □



5.2. We say that the group G *acts freely on indecomposable classes* of A-modules if $X^g \simeq X$ for $X \in \mathrm{ind}_A$ implies $g = 1$. Observe that for the algebra B with quiver





the indecomposable representation X has non-trivial stabilizer.

Theorem. *Let $F : A \to A/G = B$ be a Galois covering defined by a group G*

a) [2] *If G acts freely on indecomposable classes of A-modules, then F_λ induces an injection $(\mathrm{ind}_A/\cong)/G \hookrightarrow \mathrm{ind}_B/\cong$ and preserves Auslander-Reiten sequences.*

b) [2, 10] *If B is finite (hence a k-algebra), then B is representation-finite if and only if* (i) *G acts freely on* $\mathrm{ind}_A/\cong$ *and* (ii) *A is locally representation-finite (i.e. for each $i \in A_0$, $\{X \in \mathrm{ind}_A : X(i) \neq 0\}$ is finite). In that case, $\Gamma_A/G \simeq \Gamma_B$.*

Proof. (a): Let $X \in \mathrm{ind}_A$, then we know $G_X = (1)$ and $F_\lambda X \in \mathrm{ind}_B$. Moreover $F_\lambda X \simeq F_\lambda Y$ implies $Y \simeq X^g$ for some $g \in G$. For an almost split sequence $\eta\colon 0 \longrightarrow \tau_A X \xrightarrow{\alpha} E \xrightarrow{\beta} X \longrightarrow 0$ with $X \in \mathrm{ind}_A$ we get an exact sequence

$$F_\lambda \eta\colon 0 \longrightarrow F_\lambda \tau_A X \longrightarrow F_\lambda E \longrightarrow F_\lambda X \longrightarrow 0$$

with $F_\lambda X$, $F_\lambda \tau_A X$ indecomposable. For $f\colon Y \to F_\lambda X$ a non-invertible epi in mod_B, we get a map

$$\begin{array}{ccc} & & F.Y \\ & & \big\downarrow {\scriptstyle F.f} \\ \bigoplus\limits_{g\in G} E^g = F.F_\lambda E & \xrightarrow{F.F_\lambda\beta} & F.F_\lambda X = \bigoplus\limits_{g\in G} X^g \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow {\scriptstyle \pi_1} \\ E & \xrightarrow{\quad\beta\quad} & X \end{array}$$

There is some $g \in \mathrm{Hom}_A(F.Y, E)$ such that $\beta g = \pi_1 F.(f)$. From the adjunction $(F., F_\rho)$ we get

$$\mathrm{Hom}_A(F.Y, E) \xrightarrow{\sim} \mathrm{Hom}_B(Y, F_\rho E) = \mathrm{Hom}_B(Y, F_\lambda E),\ g \mapsto g'$$

Then $\beta g' = f$ and $F_\lambda \eta$ is almost split sequence in mod_B.

(b): Assume B is representation-finite and $Y_1, \dots, Y_s$ are representatives of $\mathrm{ind}_B/\cong$. Let $X \in \mathrm{ind}_B$ and suppose $g \in G$ is such that $X^g \simeq X$. We shall prove that $g = 1$. Let $\tilde{B} \xrightarrow{F} B$ be a covering defined by a free group H as in (5.1), that is, we get commutative diagrams:

$$\begin{array}{ccc} \tilde{B} & \hookrightarrow & \mathrm{ind}_{\tilde{B}} = k(\tilde{\Gamma}_B) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow {\scriptstyle k(\pi)} \\ B & \hookrightarrow & \mathrm{ind}_B = k(\Gamma_B) \end{array} \qquad \begin{array}{ccc} \tilde{\Gamma}_B & \overset{\tilde{g}}{\dashrightarrow} & \tilde{\Gamma}_B \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow {\scriptstyle \pi} \\ \Gamma_B & \overset{g}{\dashrightarrow} & \Gamma_B \end{array}$$

Then g induces an automorphism $\tilde{g}$ on $\tilde{\Gamma}_B$ such that $g\pi = \pi\tilde{g}$ and $\tilde{g}X' = X'$ for some $X' \in \tilde{\Gamma}_B$ with $\pi X' = X$. We get an automorphism g' of $\tilde{B}$ acting freely (since $g = Fg'$ acts freely on B).

Let C be the full subcategory of $\tilde{B}$ formed by $\mathrm{supp}\, X$. It is easy to see that C is convex in $\tilde{B}$ and therefore Γ_C is a preprojective component. Since $g'X' = X'$, then we get $h \in \mathrm{Aut}\, C$ with $(X')^h \simeq X'$. Since h commutes with τ_C, there is a projective C-module P_x with $P_x^h = P_x$, which is a contradiction unless $h = 1$.

We check that A is locally representation-finite: let $i \in A_0$ and let $(X_\alpha)_{\alpha\in\Delta}$ be all indecomposable A-modules which are direct summands of $F.Y_j$ for some $1 \le j \le s$ and $X_\alpha(i) \neq 0$. This set is finite [indeed, $\bigoplus\limits_{\Delta'} X_\alpha = F.Y_j$, then $\bigoplus F_\lambda X_\alpha = F_\lambda F.Y_j = \bigoplus Y_\ell^{(I_\ell)}$ and $X_\alpha \in \mathrm{ind}_A$, for all α. Since $\dim_k F.Y_j(i) = \dim_k Y_j(Fi) < \infty$, the set is finite].

Finally, let $Y \in \mathrm{ind}_B$. Then $F_\lambda F.Y = \oplus F_\lambda X_\alpha$ and

$$\begin{array}{ccc} \mathrm{Hom}_A(F.Y, F.Y) & \simeq \mathrm{Hom}_B(Y, F_\rho F.Y) = & \mathrm{Hom}_B(Y, F_\lambda F.Y) \\ 1_{F.Y} & \longmapsto & e\colon Y \Subset F_\lambda F.Y \end{array}$$

hence $Y \simeq F_\lambda X_\alpha$ for some α. Therefore $\Gamma_B = \Gamma_A/G$. □

5.3. Let $F\colon A \to A/G$ be a Galois covering with group G. In case A/G is a representation-finite k-algebra, we have seen that F_λ covers all indecomposable modules. The situation is different for representation-infinite algebras. We shall briefly discuss the new occuring phenomena. By $\mathrm{ind}_1 A/G$ we denote the full subcategory of $\mathrm{ind}_{A/G}$ formed by all objects isomorphic to $F_\lambda M$ for some $M \in \mathrm{ind}_A$ (these modules are called *A/G-modules of the first kind*). The remaining indecomposable modules (called *of the second kind*) form the category $\mathrm{ind}_2 A/G$.

If $M \in \mathrm{mod}_A^H$ is such that $\mathrm{supp}\, M$ is not finite but $\mathrm{supp}\, M/H$ is finite, then M is called a *weakly G-periodic* module.

Lemma [5]. *Let $F\colon A \to A/G$ be a Galois covering such that G acts freely on $\mathrm{ind}_A/\cong$. For $X \in \mathrm{ind}_{A/G}$ the following conditions hold:*

(1) *$X \in \mathrm{ind}_1 A/G$ if and only if $F.X \cong \bigoplus_{i\in I} Z_i$, where all $Z_i \in \mathrm{mod}_A$.*

(2) *$X \in \mathrm{ind}_2 A/G$ if and only if $F.X \cong \bigoplus_{i\in I} Y_i$, where each Y_i is weakly G-periodic.*

Proof. (1): If $X \cong F_\lambda M$ for some $M \in \mathrm{mod}_A$, then $F.X \cong \bigoplus_{g\in G} M^g$. Assume now that $Z \in \mathrm{ind}_A$ is a direct summand of $F.X$. Let $j \in \mathrm{Hom}_A(Z, F.X)$ and $p \in \mathrm{Hom}_A(F.X, Z)$ be such that $pj = 1_Z$. We get morphisms $j^g\colon Z^g \to (F.X)^g \cong F.X$ and $p^g\colon (G.X)^g \to Z^g$ which yield a pair of maps $j'\colon \bigoplus_{g\in G} Z^g \to F.X$ and $p'\colon F.X \to \bigoplus_{g\in G} Z^g$. Since $Z^g \not\cong Z$ for $g \neq h$, then the endomorphism $p' \circ j'$ of $\bigoplus_{g\in G} Z^g$ is invertible.

Since $F.$ induces an equivalence of categories $\mathrm{MOD}_{A/G} \cong \mathrm{MOD}_A^G$, then $F_\lambda Z$ is a direct summand of X. Thus $X \cong F_\lambda Z$.

(2): Let $X \in \mathrm{mod}_2 A/G$. By the proof of (1), $F.X = \bigoplus_{i\in I} Y_i$ where $Y_i \in \mathrm{Ind}_A \setminus \mathrm{ind}_A$. Then we have to show that an indecomposable direct summand Y of $F.X$ is weakly G-periodic. Indeed, let $\mathcal{U}$ be a set of representatives of the cosets of G with respect to G_Y. Then $\bigoplus_{g\in\mathcal{U}} Y^g$ is a direct summand of $F.X$. Since $\mathrm{supp}\, Y \subset \mathrm{supp}\, X$ is contained in a finite number of G-orbits, then $(\mathrm{supp}\, Y)/G_Y$ is finite. The converse follows from (1). □

5.4. A category $A = kQ/I$ is called *locally support finite* if for each $x \in Q_0$, the full subcategory A_x of A, consisting of the vertices of all $\mathrm{supp}\, M$ with $M \in \mathrm{ind}_A$ and $M(x) \neq 0$, is finite.

Proposition. *Let $F\colon A \to A/G$ be a Galois covering. Then*

a) *If G acts freely on $\mathrm{Ind}_A/\cong$, then $F_\lambda\colon \mathrm{mod}_A \to \mathrm{mod}_{A/G}$ induces a bijection $F_\lambda\colon (\mathrm{ind}_A/\cong)/G \to (\mathrm{ind}_{A/G}/)\cong$.*

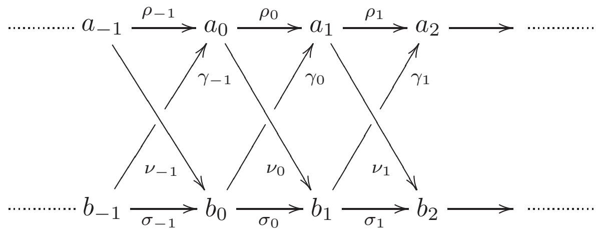
b) *If A is locally support finite, then $\mathrm{Ind}_A = \mathrm{ind}_A$. In particular if G acts freely on $\mathrm{ind}_A/\cong$, then F_λ induces a bijection between $(\mathrm{ind}_A/\cong)/G$ and $(\mathrm{ind}_A/G)/\cong$. Moreover, in this case A/G is tame (resp. domestic, poynomial growth) if and only if so is A. For the Auslander-Reiten quivers we have $\Gamma_{A/G} = \Gamma_A/G$.*

Proof. (a): Follows from (4.2) since there are no weakly G-periodic indecomposable A-modules.

(b): Let $Y \in \mathrm{Ind}_A$ and consider $x \in Q_0$ with $Y(x) \neq 0$. Let $\hat{A}_x$ be the (finite) full subcategory of A consisting in the objects $y \in Q_0$ such that $A(y,z) \neq 0$ or $A(z,y) \neq 0$ for some $z \in A_x$. Let Z be an indecomposable direct summand of the restriction $Y|\hat{A}_x$ such that $Z(x) \neq 0$. Hence $\operatorname{supp} Z \subset A_x$ and it is easy to check that Z is a direct summand of Y in mod_A.

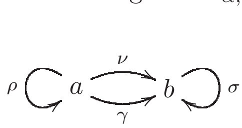
Thus $Y = Z \in \mathrm{ind}_A$. The last claim is easy to prove. □

Example: The category $A_{\alpha,\beta}$ given by the quiver with relations



$$\left.\begin{array}{rcl} \sigma_{i+1}\sigma_i &=& \alpha\nu_{i+1}\gamma_i \\ \nu_{i+1}\rho_i &=& \beta\sigma_{i+1}\nu_i \\ \gamma_{i+1}\nu_i &=& \rho_{i+1}\rho_i \\ \rho_{i+1}\gamma_i &=& \gamma_{i+1}\sigma_i \end{array}\right\}$$

with $(\alpha,\beta) \neq (1,1)$, is locally support finite (why?). Moreover the group $\mathbb{Z}$ generated by the action $(a_i \mapsto a_{i+1}, b_i \mapsto b_{i+1})$ acts freely on $A_{\alpha,\beta}$ and on $\mathrm{ind}_{A_{\alpha,\beta}}/\cong$. Hence the Galois covering $F\colon A_{\alpha,\beta} \to \bar{A}_{\alpha,\beta}/\mathbb{Z}$ yields a bijection $F_\lambda\colon (\mathrm{ind}_{A_{\alpha,\beta}}/\cong)/\mathbb{Z} \to (\mathrm{ind}_{\bar{A}_{\alpha,\beta}})/\cong$. The algebra $\bar{A}_{\alpha,\beta}$ is given by the quiver with relations



$$\left.\begin{array}{rcl} \sigma^2 &=& \alpha\gamma\nu \\ \rho\nu &=& \beta\nu\sigma \\ \nu\gamma &=& \rho^2 \\ \gamma\rho &=& \sigma\gamma \end{array}\right\}$$

Since $A_{\alpha,\beta}$ is tame (resp. polynomial growth for $\alpha\beta \neq 1$), so is $\bar{A}_{\alpha,\beta}$.

5.5. Given a natural number $p \geq 0$, a group G is said to be *p-residually finite* if for each finite subset $S \subset G \setminus \{1\}$ there is a normal subgroup of finite index $H \lhd G$ such that $H \cap S = \emptyset$ and $p \not| \ (G/H)$. For example, free groups are p-residually finite.

Theorem [12]. *Let $F\colon (\tilde{Q},\tilde{I}) \to (Q,I)$ be a Galois covering given by the action of a p-residually finite group G, where $p = \operatorname{char} k$. Assume that $\tilde{A} = k\tilde{Q}/\tilde{I}$ is locally support finite. Then $\tilde{A}$ is tame if and only if $A = kQ/I$ is tame.*

Proof. Without loss of generality, we assume that Q is finite. We denote also by $F\colon \tilde{A} \to A$ the induced covering functor. We divide the proof in several steps.

(1) Let $F_\lambda\colon \text{mod}_{\tilde{A}} \to \text{mod}_A$ be the push-down functor. Consider a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of finite full subcategories of $\tilde{A}$ such that $\bigcup\limits_n \Gamma_n = \tilde{A}$ and if $\tilde{A}(x,\Gamma_n) \neq 0$ or $\tilde{A}(\Gamma_n, x) \neq 0$, then $x \in \Gamma_{n+1}$. The restriciton functor $\varepsilon.^n\colon \text{mod}_{\tilde{A}} \to \text{mod}_{\Gamma_n}$ has a left adjoint $\varepsilon^n_\lambda\colon \text{mod}_{\Gamma_n} \to \text{mod}_{\tilde{A}}$ such that $\varepsilon.^n\varepsilon^n_\lambda = \text{id}_{\text{mod}_{\Gamma_n}}$. Therefore, the functor

$$F_n = F_\lambda \varepsilon^n_\lambda\colon \text{mod}_{\Gamma_n} \to \text{mod}_A$$

is right exact, $n \in \mathbb{N}$.

(2) Let $Y \in \text{ind}_A$, we shall prove that Y is a direct summand of $F_\lambda F.Y$. Indeed, since $\tilde{A}$ is locally support-finite, by (5.3), the pull-up $F.Y$ decomposes as $F.Y \simeq \bigoplus\limits_{i\in L} X_i$, where $X_i \in \text{ind}_{\tilde{A}}$. Thus $F_\lambda F.Y \simeq \bigoplus\limits_{i\in L} F_\lambda X_i$. We proceed in two steps.

(2.1) There exists a normal subgroup H of G with finite index not divisible by p, such that the Galois covering $\bar{F}\colon \tilde{A} \to \tilde{A}/H$ induces a bijection $\bar{F}_\lambda\colon (\text{ind}_{\tilde{A}}/\simeq)/H \to (\text{ind}_{\tilde{A}}/H)/\simeq$.

Since A is finite, the set of vertices of $\tilde{Q}$ is a disjoint union of a finite number of G-orbits Gx_i, $i = 1,\dots,s$. With the notation introduced above, we consider the full subcategory $R_i = \tilde{A}_x$, of $\tilde{A}$. Let S be the set of all elements $g \in G/\{1\}$ such that $g(\hat{\hat{R}}_i) \cap \hat{\hat{R}}_i \neq \emptyset$, for some $i \in \{1,\dots,s\}$. Since G acts freely on $\tilde{Q}$, then S is finite and there is a normal subgroup H of G such that G/H is finite of order not divisible by p and $H \cap S = \emptyset$. Hence $h(\hat{\hat{R}}_i) \cap \hat{\hat{R}}_i = \emptyset$, $i = 1,\dots,s$. By (5.1), the induced Galois covering $\bar{F}\colon \tilde{A} \to \tilde{A}/H$ yields an injection $\bar{F}_\lambda(\text{ind}_{\tilde{A}}/\simeq)/H \to (\text{ind}_{\tilde{A}}/H)/\simeq$. We show that this map is also surjective. Let $M \in \text{ind}_{\tilde{A}}/H$ and $b = \bar{F}(x_j)$ a vertex of $\tilde{Q}/H$ such that $M(b) \neq 0$. The restriction $F' = \bar{F}|: \hat{R}_j \to F(\hat{R}_j)$ is a bijection and there is an indecomposable $\hat{R}_j$-module $Y = F'N$ with $Y(x_j) \neq 0$ and such that N is an indecomposable direct summand of the restriction M' of M to $F(\hat{R}_j)$. We conclude that $Y \in \text{ind}_{\tilde{A}}$ and $N \in \text{ind}_{\tilde{A}}/H$, showing that $M = N = F_\lambda Y$.

(2.2) For the proof of (2), consider a normal subgroup H of G as in (2.1) and a factorization of Galois coverings

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{F} & A = \tilde{A}/G \\ {\scriptstyle F}\downarrow & \nearrow_{F'} & \\ \tilde{A} = \tilde{A}/H & & \end{array}$$

Then $F_\lambda = F'_\lambda \bar{F}_\lambda$ and $F. = \bar{F}.F.'$. For an indecomposable $Y \in \text{ind}_A$, we get

$$F_\lambda F.Y = F'_\lambda(\bar{F}_\lambda \bar{F}.)F.'Y$$

Consider the indecomposable decomposition $F.'Y = \bigoplus\limits_{i=1}^{t} Y_i$ with $\bar{F}_\lambda \bar{F}.Y_i \simeq Y_i$ for $i = 1,\dots,t$. Then $F_\lambda F.Y \simeq F'_\lambda F.'Y$ and the result follows from (4.2). In particular, Y is a direct summand of $F_\lambda X_j$ for some $j \in L$.

(3) There is some $n \in \mathbb{N}$ and a function $f \colon \mathbb{N} \to \mathbb{N}$ such that

$$F_n \colon \mathrm{mod}_{\Gamma_n} \to \mathrm{mod}_A$$

has the property that for every $Y \in \mathrm{ind}_A$ with $\dim_k Y = d$, there exists some $X \in \mathrm{ind}_{\Gamma_n}$ with $\dim_k X \leq f(d)$ and such that Y is a direct summand of $F.X$.

Indeed, the set of vertices $\tilde{Q}_0$ is the disjoint union $\bigcup_{i=1}^{s} Gx_i$, then there is some $n \in \mathbb{N}$ such that Γ_n contains all the categories $\hat{R}_i$ for $R_i = \tilde{A}_{x_i}$, $i = 1, \dots, s$. By (2), Y is a direct summand of $F_\lambda Z$ for some $Z \in \mathrm{ind}_{\tilde{A}}$. For some $g \in G$, $\mathrm{supp}\, Z^g \subset \Gamma_n$ and $X = \varepsilon.^n Z^g \in \mathrm{ind}_{\Gamma_n}$ satisfies that Y is a direct summand of $F_n X$. Clearly, $f(d) = |\Gamma_n| d$ (where $|\Gamma_n|$ denotes the number of objects of Γ_n) satisfies the desired property.

(4) Assume that $\tilde{A}$ is tame, we show that A is tame. Indeed, let $n \in \mathbb{N}$ and $f \colon \mathbb{N} \to \mathbb{N}$ be as in (3). Since F_n is right exact (1), there is a $A - F_n$-bimodule N such that $F_n = N \otimes_{\Gamma_n} (-)$. Let $z \in \mathbb{N}^{Q_0}$.

Since Γ_n is tame, there is a family $M_1, \dots, M_t$ of $\Gamma_n - k[T]$-bimodules which are finitely generated free as $k[T]$-right modules and such that any indecomposable Γ_n-module X with $\dim_k X \leq f(d)$ is isomorphic to $M_i \otimes_{k[T]} S$ for some $1 \leq i \leq t$ and S a simple $k[T]$-module. By (3), for every $Y \in \mathrm{ind}_A$ with $\dim_k Y = d$, there exists some $1 \leq i \leq t$ and S a simple $k[T]$-module such that Y is a direct summand of $(N \otimes_{\Gamma_n} M_i) \otimes_{k[T]} S$. It is easy to see that this (aparently) weaker property is equivalent to the tameness of A.

(5) Assume now that A is tame. It is enough to show that each Γ_n is tame. Consider the right exact functors

$$H_n = \varepsilon.^n F. \colon \mathrm{mod}_A \to \mathrm{mod}_{\Gamma_n}.$$

Let $Y \in \mathrm{ind}_{\Gamma_n}$. Thus $X = F_\lambda \varepsilon_\lambda^n Y \in \mathrm{mod}_A$ and

$$H_n X = \varepsilon^n \left(\bigoplus_{g \in G} (\varepsilon_\lambda^n Y)^g \right) \cong \bigoplus_{g \in S} \varepsilon.^n (\varepsilon_\lambda^n Y)^g = Y \oplus \left(\bigoplus_{g \in S \setminus \{1\}} \varepsilon.^n (\varepsilon_\lambda^n Y)^g \right),$$

where S is the finite set of $g \in G$ such that $\mathrm{supp}\, (\varepsilon_\lambda^n Y)^g \cap \Gamma_n \neq \emptyset$. As in (3), (4) we get that Γ_n is tame. □

5.6. We consider several *examples:*

(a) Let $F \colon \tilde{A} \to A$ be a Galois covering induced from a Galois covering of quivers with relations and defined by the action of a group G. Assume that A is locally support-finite.

- If G is a finite group such that $p = \mathrm{char}\, k$ does not divide the order of G, then (5.5) applies and $\tilde{A}$ is tame if and only if so is A.
- If G is a free group, then both (5.2) and (5.7) apply.

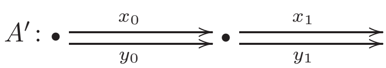
(b) Consider example (5.4). For $(\alpha, \beta) \neq (1, 1)$, the category $A_{\alpha,\beta}$ is tame and locally support finite. Hence the algebra $\bar{A}_{\alpha,\beta}$ is tame.

(c) [8] Consider the Galois covering

$$F\colon A = A_{1,1}^{(2)} \to \bar{A} = A_{1,1}^{(2)}/\mathbb{Z}_2$$

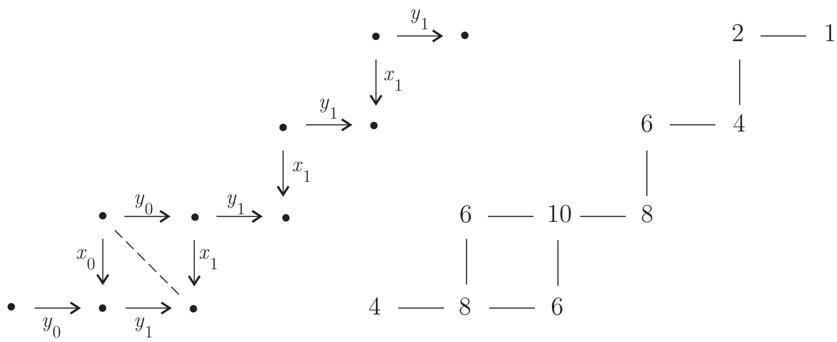
given in (2.2.c) and assume that $\operatorname{char} k = 2$. We know that A is tame. We show that $\bar{A}$ is a wild algebra.

Set $x_0 = \alpha_0 + \beta_0$, $y_0 = \beta_0$, $x_1 = \alpha_1 + \beta_1$, $y_1 = \beta_1$. Then $\bar{A}$ is isomorphic to the algebra A' given by the quiver with relations.



$$\left.\begin{array}{rcl} x_1x_0 & = & 0 \\ y_1x_0 & = & x_1y_0 \end{array}\right\}$$

The universal covering $\tilde{A}'$ constructed as in (2.3) admits a full convex subcategory Δ as follows



Since Δ is wild (observe that the Tits form q_Δ takes value -1 in the indicated vector), then $\tilde{A}'$ is wild. Then A is also wild.

(d) The algebra $\bar{A}$ of example (c) provides another example of a wild algebra whose Tits form $q_{\bar{A}}$ is weakly non-negative. Indeed,

$$q_{\bar{A}}(a,b,c) = (a-b+c)^2.$$

5.7. We consider briefly the situation of coverings $F\colon A \to A/G$ where A is not necessarily locally support-finite.

Let $F\colon A \to A/G$ be a Galois covering. A *line* in A is a full convex subcategory isomorphic to the path category of a linear quiver (of type $\mathbb{A}_n$, $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$). A *line* L is *G-periodic* if its stabilizer $G_L = \{g \in G\colon gL = L\}$ is non-trivial (then L is of type $\mathbb{A}_\infty^\infty$).

Let L be a G-periodic line in A, we construct an indecomposable weakly G-periodic A-module B_L by setting $B_L(x) = k$ for $x \in L$ and $B_L(x) = 0$ for $x \notin L$ and $B_L(\alpha) = \mathrm{id}$ for any arrow α in L. Then $G_{B_L} \simeq G_L$ is isomorhic to $\mathbb{Z}$. Let W_x be a set of representatives of the G_{B_L}-orbits in Gx, for any object x in A. Then $F_\lambda B_L$ is a $A/G - k[T,T^{-1}]$-bimodule such that for each $x \in A$, $F_\lambda B_L(x)$ is a free $k[T,T^{-1}]$-module of rank $\sum\limits_{y \in W_x} \dim_k B_L(y)$. We consider the functor

$$\phi^L = F_\lambda B_L \otimes_{k[T,T^{-1}]} (-)\colon \mathrm{mod}_{k[T,T^{-1}]} \to \mathrm{mod}_{A/G}.$$

Let $\mathcal{L}$ be the set of all lines in A and $\mathcal{L}_0$ be the set of representatives of the G-orbits in $\mathcal{L}$.

Theorem [5]. *Let $F\colon A \to A/G$ be a Galois covering such that G acts freely on $(\mathrm{ind}_A)/\simeq$. Assume that for any weakly G-periodic A-module X, $\operatorname{supp} X$ is a line. Then*

a) *Every module in $\mathrm{ind}_2 A/G$ is of the form $\phi^L(V)$ for some $L \in \mathcal{L}_0$ and some indecomposable $k[T,T^{-1}]$-module V.*

b) $\Gamma_{A/G} = \Gamma_A/G \vee \bigvee_{L\in\mathcal{L}_0} \Gamma_{k[T,T^{-1}]}$, *where $\Gamma_{k[T,T^{-1}]}$ is the translation quiver of finite dimensional indecomposable $k[T,T^{-1}]$-modules, consisting of a k^*-family of stable tubes of rank one.*

c) *A is tame if and only if so is A/G.* □

References

[1] M. Beattie. *A generalization of the smash product of a graded ring.* J. Pure Appl. Algebra 52 (1988) 219–229.

[2] K. Bongartz and P. Gabriel. *Covering spaces in Representation Theory.* Invent. Math. 65 (3) (1982) 331–378.

[3] O. Butscher and P. Gabriel. *The standard form of a representation-finite algebra.* Bull. Soc. Math. France 111 (1983) 21–40.

[4] C. Cibils and E. Marcos. *Skew categories. Galois coverings and smash product of a category over a ring.* Preprint (2004).

[5] P. Dowbor and A. Skowroński. *Galois coverings of representation infinite algebras.* Comment. Math. Helv. 62 (1987) 311–337.

[6] P. Gabriel. *Auslander-Reiten sequences and representation-finite algebras.* In 'Representation of Algebras', Springer LNM 831 (1975) 1–71.

[7] P. Gabriel. *The universal cover of a representation-finite algebra.* In 'Representation of Algebras'. Springer LNM 903 (1981) 68–105.

[8] Ch. Geiss and J. A. de la Peña. *An interesting family of algebras.* Archiv. Math. 60 (1993) 25–35.

[9] R. Martínez-Villa and J. A. de la Peña. *The universal cover of a quiver with relations.* J. Pure and Appl. Algebra 30 (3) (1983) 277–292.

[10] R. Martínez-Villa and J. A. de la Peña. *Automorphisms of representation-finite algebras.* Invent. Math. 72 (1983) 359–362.

[11] J. A. de la Peña. *Symmetries of quivers and the Coxeter transformation.* J. Algebra 169 (1994) 200–215.

[12] J. A. de la Peña. *Tame algebras: some fundamental notions.* Ergänzungsreihe 95–010 (1995) Universität Bielefeld.

[13] Ch. Riedtmann. *Darstellungsköcher, Überlagerungen und zurück.* Comment. Math. Helv. 55 (1980) 199–224.

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