REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 48, Número 2, 2007, Páginas 21–45

LECTURES ON ALGEBRAS

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Mar del Plata, Argentina, March 2006

ABSTRACT. The purpose of this note is to give a fast introduction to some problems of homological and geometrical nature related to finitely dimensional representations of finitely generated, and especially, finitely dimensional algebras over a field. Some of these results can also be extended to the situation where the field is not algebraically closed, and some of the results can even be extended to the situation where one is considering algebras over a commutative artin ring. For the results which hold true in the most general situation the proofs become most elegant since they depend on using length arguments only and thereby forgetting about the nature of a field altogether.

1. Introduction and notation

Let $K = \overline{K}$ be an algebraically closed field and let Λ a finitely generated associative K-algebra with 1.

Examples:

1. $\Lambda = K$, the field itself which is the simplest example of a K-algebra.
2. $\Lambda = K[X]$, the polynomial algebra in one variable over the field K.
3. $\Lambda = K\langle X, Y\rangle$, the free algebra in two non-commuting variables over the field K.
4. $\Lambda = K[X_1, X_2, ..., X_n]$, the polynomial algebra in n commuting variables over the field K.
5. $\Lambda = M_d(K)$, the algebra of $d \times d$-matrices over the field K.
6. For a K-vector space V, $\Lambda = \mathrm{End}_K(V)$ is the algebra of K-linear endomorphisms of V where addition is the usual addition of K-linear homomorphisms and the multiplication is given by composition of K-linear homomorphisms.
7. For a K-vector space V and $f_1, f_2, ..., f_n$ elements of $\mathrm{End}_K(V)$, one lets $\Lambda = K\langle f_1, f_2, ..., f_n\rangle$ be the subalgebra of $\mathrm{End}_K(V)$ generated by the elements $f_1, f_2, ..., f_n$.
8. For a K-vector space V and a subspace $V_1 \subseteq V$, one can look at the algebra $\Lambda = \mathrm{End}_K(V; V_1) = \{f \in \mathrm{End}_K(V) \mid f(V_1) \subseteq V_1\}$. By choosing a complement V_1' to V_1 in V and decomposing V as $V_1' \oplus V_1$ as a K-vector space, one can identify this algebra Λ with the matrix algebra

$$\{\begin{pmatrix} f & 0 \\ h & g \end{pmatrix} \mid f \in \mathrm{End}_K(V_1'),\ g \in \mathrm{End}_K(V_1) \text{ and } h \in \mathrm{Hom}_K(V_1', V_1)\}.$$

2000 *Mathematics Subject Classification* 16D10, 16E30, 16G10, 16G20.

In this matrix algebra the addition is induced from the usual addition of K-linear homomorphisms and the multiplication is induced from the composition of K-linear homomorphisms.

9. Now, this last example can be extended to more than one subspace. So let V be a K-vector space and V_i subspaces for $i = 1, 2, ..., n$; then $\Lambda = \text{End}(V; V_1, V_2, ..., V_n)$ is defined as the subalgebra $\{f : V \to V \mid f(V_i) \subseteq V_i$ for $i = 1, 2, ..., n\}$ of $\text{End}_K(V)$. However, it is not so easy in general to describe this algebra as an algebra of matrices.

10. If one in example 9 has a K-vector space V and subspaces $V_1 \subset V_2 \subset ... \subset V_n = V$ and the dimension of the K-vector spaces are given by $\dim V_i = i$, then this algebra can be identified with the algebra of lower triangular matrices over K. This identification can be obtained by choosing a basis for V by starting with a basis for V_1 and then successively extending a basis from V_i to V_{i+1} until one has a basis for the whole space V.

Finitely generated K-algebras can also be given by generators and relations over the ground field K. This way one obtains the algebra as a quotient of a finitely generated free algebra $K\langle X_1, X_2, ..., X_n\rangle$ over the ground field K by an ideal I.

11. $\Lambda = K\langle X, Y\rangle/\langle XY - YX\rangle$ which is isomorphic to $K[X, Y]$, the polynomial algebra in two variables over the field K.

12. $\Lambda = K\langle X, Y\rangle/\langle X^2, XY + YX, Y^2\rangle$ which is isomorphic to $\bigwedge(X, Y)$, the exterior algebra in two variables over the field K.

13. $\Lambda = K\langle X, Y\rangle/\langle XY - YX + 1\rangle = A_1(K)$ which is called the first Weil algebra over the field K. This K-algebra is isomorphic to the K-subalgebra of the endomorphism algebra of the K-vector space $K[T]$ generated by the two K-linear endomorphisms where one is given as multiplication by T, and the other one is derivation with respect to T.

Another way of giving an algebra is to start with a vector space over K with a basis $\{v_i\}_{i\in I}$, and then give a multiplication table for the base elements, and then extend the multiplication by bilinearity to all elements of the K-vector space V. Let $(V, \{v_i\}_{i\in I})$ be a K-vector space with a basis. A multiplication table is then given as $v_i v_j = \sum_{k\in I} a_{i,j}^k v_k$ with the elements $a_{i,j}^k \in K$ where for each pair of indices i, j the set $\{k \mid a_{i,j}^k \neq 0\}$ is finite. For this to become an associative K-algebra with 1, severe restrictions on the $a_{i,j}^k$, the structure constants, have to be imposed.

14. Let V be a K-vector space with v_i where i runs through the nonnegative integers, as a basis. Define a multiplication on the base elements by $v_i v_j = v_{i+j}$ and extend this multiplication by bilinearity to all elements of V. This makes V into an associative K-algebra with v_0 as the identity element. By identifying v_i with X^i one gets an isomorphism between the K-algebra V and $K[X]$, the polynomial algebra in one variable over the field K.

15. Another source of this form of K-algebras is the group algebras. Let G be a group written multiplicatively. Let V be the K-vector space of all maps f from G to K such that $G \setminus f^{-1}(0)$ is finite. Then for each g in G one has the map f_g given by $f_g(h) = 1$ for $h = g$ and zero for $h \neq g$. The collection of all these maps as g

runs through G becomes a basis for V. One can then form the multiplication table $f_g f_h = f_{gh}$ which makes V into a K-algebra with f_e as the unit element, where e is the identity element of the group G. Usually one writes an element of V as $\sum_{g\in G} a_g g$ where a_g are elements of K and each f_g is substituted with g, and only a finite number of the elements a_g as g runs through G are nonzero. The notation used for the group algebra is often KG or $K[G]$.

16. Another important source of examples of algebras where the multiplication is given by a multiplication table for a basis, is the path algebra of a quiver.

A quiver Q is an oriented graph. Here one lets Q_0 denote the set of vertices, and one lets Q_1 denote the set of oriented edges. The oriented edges are also often called arrows as the name indicates and one usually represents a quiver by a collection of vertices and arrows drawn in the plain.

Example:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Now one can look at the set of all oriented paths in this oriented graph including the paths of length zero at each vertex, and use them as a basis for an algebra. By concatenating paths one makes a multiplication table for these base elements and in this way one obtains the path algebra.

For the example of the quiver above, this will be a six dimensional algebra, with basis e_1, e_2, e_3, α, β and $\beta\alpha$. Here e_1, e_2 and e_3 represent the paths of length zero at the vertices 1, 2 and 3 respectively. One has to make a convention about how to represent a path and here one is using the convention that an oriented path is ordered from right to left. The multiplication table for this algebra is rather long, but for the convenience of the reader the complete table is included.

$$e_1 \cdot e_1 = e_1,\ e_1 \cdot e_2 = 0,\ e_1 \cdot e_3 = 0,\ e_1 \cdot \alpha = 0,\ e_1 \cdot \beta = 0,\ e_1 \cdot \beta\alpha = 0,$$

$$e_2 \cdot e_1 = 0,\ e_2 \cdot e_2 = e_2,\ e_2 \cdot e_3 = 0,\ e_2 \cdot \alpha = \alpha,\ e_2 \cdot \beta = 0,\ e_2 \cdot \beta\alpha = 0,$$

$$e_3 \cdot e_1 = 0,\ e_3 \cdot e_2 = 0,\ e_3 \cdot e_3 = e_3,\ e_3 \cdot \alpha = 0,\ e_3 \cdot \beta = \beta,\ e_3 \cdot \beta\alpha = \beta\alpha,$$

$$\alpha \cdot e_1 = \alpha,\ \alpha \cdot e_2 = 0,\ \alpha \cdot e_3 = 0,\ \alpha \cdot \alpha = 0,\ \alpha \cdot \beta = 0,\ \alpha \cdot \beta\alpha = 0,$$

$$\beta \cdot e_1 = 0,\ \beta \cdot e_2 = \beta,\ \beta \cdot e_3 = 0,\ \beta \cdot \alpha = \beta\alpha,\ \beta \cdot \beta = 0,\ \beta \cdot \beta\alpha = 0,$$

$$\beta\alpha \cdot e_1 = \beta\alpha,\ \beta\alpha \cdot e_2 = 0,\ \beta\alpha \cdot e_3 = 0,\ \beta\alpha \cdot \alpha = 0,\ \beta\alpha \cdot \beta = 0,\ \beta\alpha \cdot \beta\alpha = 0.$$

Here $e_1 + e_2 + e_3$ is the identity element.

For this simple example, the path algebra is isomorphic to the K-algebra of lower three by three matrices over K. To see this let e_{ij} be the matrix with 1 in place ij and zero otherwise. Then an isomorphism can be given by sending e_1 in the path algebra to the matrix e_{11}, e_2 in the path algebra to the matrix e_{22}, e_3 in the path algebra to the matrix e_{33}, α in the path algebra to the matrix e_{21}, β in the path algebra to the matrix e_{32} and $\beta\alpha$ in the path algebra to the matrix e_{31}. An easy calculation now shows that this is a K-algebra isomorphism from the path algebra of this quiver to the algebra of lower three by three matrices over K.

An important problem which often appears in representation theory is to determine whether two algebras are isomorphic or not isomorphic. This can sometimes be decided by giving an explicit algebra isomorphism between them, but sometimes one can still prove that two algebras have to be isomorphic without ever in principle being able to give such an isomorphism explicitly.

Example. The algebra $\mathbb{C}[X]/\langle 1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\frac{X^4}{4!}+\frac{X^5}{5!}\rangle$ is isomorphic to the algebra $\mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$. However, finding an explicit isomorphism requires that one can find the zeros of the polynomial $1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\frac{X^4}{4!}+\frac{X^5}{5!}$. However, this is one of the degree five complex polynomials where the zeros cannot be expressed with the help of radicals of the coefficients. By calculating the difference between this polynomial and its derivative one obtains the following: $1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\frac{X^4}{4!}+\frac{X^5}{5!}-(1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\frac{X^4}{4!}+\frac{X^5}{5!})'=\frac{X^5}{5!}$. Therefore the polynomial $1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\frac{X^4}{4!}+\frac{X^5}{5!}$ and its derivative have no common factors. This implies that the polynomial $1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\frac{X^4}{4!}+\frac{X^5}{5!}$ has no multiple roots, and therefore by the Chinese remainder theorem there is an isomorphism between the algebra $\mathbb{C}[X]/\langle 1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\frac{X^4}{4!}+\frac{X^5}{5!}\rangle$ and the algebra $\mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$, as claimed.

2. Representations

If an algebra Λ is given, one way or another, one wants to get as much information about the algebra as possible. Some of this information can be obtained by considering the algebra homomorphisms from Λ to other better understood algebras. For example knowing all algebra homomorphisms from Λ to $M_d(K)$, the ring of $d\times d$-matrices over K, gives a lot of information.

Examples: 1. $\Lambda = K$. There is only one K-algebra homomorphism $\phi : \Lambda \to M_d(K)$ for each d, and that is given by $\phi(\lambda)=\lambda I_d$ where I_d is the $d\times d$-identity matrix.

2. $\Lambda = K[X]$. A K-algebra homomorphism from $K[X]$ to $M_d(K)$ is completely determined by the image of X, and any matrix in $M_d(K)$ can be the image of X. And hence, $\{\phi : K[X]\to M_d(K) \mid \phi$ a K-algebra homomorphism $\}$ can be identified with $M_d(K)$.

3. $\Lambda = K\langle X,Y\rangle$. A K-algebra homomorphism $\phi : K\langle X,Y\rangle \to M_d(K)$ is completely determined by its value at X and Y, and any two matrices A and B in $M_d(K)$ can be such an image. Hence the set of K-algebra homomorphisms can be identified with $M_d(K)\times M_d(K)$.

4. $\Lambda = K[X,Y]$. Now a K-algebra homomorphism is completely determined by the images A and B of X and Y respectively, and in addition these two matrices A and B have to commute. Hence the set of K-algebra homomorphism from $K[X,Y]$ to $M_d(K)$ can be identified with the set $\{(A,B)\in M_d(K)\times M_d(K)\mid AB=BA\}$.

For $d=1$ this gives all ordered pairs of matrices since any pair of 1×1-matrices commutes. And for $d=2$ this gives the set of ordered pairs of 2×2-matrices

$$\{(A,B)=\left(\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\right)\mid$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \}.$$

By multiplying out the left hand side of this equation one obtains the following four polynomial equations in 8 unknowns:

$$a_{11}b_{11} + a_{12}b_{21} - b_{11}a_{11} - b_{12}a_{21} = 0,$$

$$a_{11}b_{12} + a_{12}b_{22} - b_{11}a_{12} - b_{12}a_{22} = 0,$$

$$a_{21}b_{11} + a_{22}b_{21} - b_{21}a_{11} - b_{22}a_{21} = 0,$$

$$a_{21}b_{12} + a_{22}b_{22} - b_{21}a_{12} - b_{22}a_{22} = 0.$$

These four polynomial equations in the 8 unknowns a_{ij} and b_{ij}, $i, j \in \{1, 2\}$ are expressing exactly that the two matrices A and B are commuting.

Here the first and the fourth equation becomes the same, so the system consisting of these four equations is redundant.

For a general d for this example one gets d^2 quadratic equations in $2d^2$ unknowns, and one obtains a bijection between the set of K-algebra homomorphisms from $K[X_1, X_2]$ to $M_d(K)$ and the set of zeros of these polynomial equations, i.e. the set of K-algebra homomorphisms is in a bijection with a subvariety of the set of pairs of matrices defined as the set of common zeros of all these quadratic polynomials.

5. $\Lambda = M_d(K)$. Let G be an invertible $d \times d$-matrix and define $\phi_G : M_d(K) \to M_d(K)$ by $\phi_G(H) = GHG^{-1}$. Then ϕ_G is a K-algebra homomorphism, and by Skolem-Noether's theorem all K-algebra homomorphisms from $M_d(K)$ to $M_d(K)$ are given in this way. Since $\phi_G \circ \phi_{G^{-1}} = I_{M_d(K)}$ they are all isomorphisms.

Letting $\mathrm{Gl}_d(K)$ denote the group of invertible $d \times d$-matrices, one gets a group homomorphism from $\mathrm{Gl}_d(K)$, the group of invertible $d \times d$-matrices, onto the group of K-algebra automorphisms of $M_d(K)$. The kernel of this group homomorphism is the set of invertible matrices commuting with all invertible matrices. This is the set of nonzero scalar matrices. Hence the group of K-algebra automorphisms of $M_d(K)$ is naturally isomorphic to $\mathrm{Gl}_d(K)/K^*$.

13. $\Lambda = K\langle X, Y\rangle/\langle XY - YX + 1\rangle$. A K-algebra homomorphism ϕ from $K\langle X, Y\rangle/\langle XY - YX + 1\rangle$ to $M_d(K)$ is completely determined by the values $\phi(X)$ and $\phi(Y)$ of X and Y respectively, and in addition these two matrices $\phi(X)$ and $\phi(Y)$ have to satisfy the equation

$$\phi(X)\phi(Y) - \phi(Y)\phi(X) + I_d = 0$$

where I_d is the $d \times d$-identity matrix. However, the trace of the matrix $\phi(X)\phi(Y) - \phi(Y)\phi(X)$ is zero; and therefore the trace of the $d \times d$-identity matrix has to be zero. For this to be the case one has to have that the characteristic of the field K divides d. So if the characteristic of K does not divide d there are no K-algebra homomorphisms from Λ to $M_d(K)$. On the other hand, if the characteristic of the field K divides d, then an easy calculation will show that there always exists such K-algebra homomorphisms from Λ to $M_d(K)$.

3. Geometry and group actions

In this section the action of the group $\mathrm{Gl}_d(K)$ on the varieties of representations will be introduced, and some facts about this action will be given.

For Λ a finitely generated K-algebra and a fixed natural number d, let $\mathrm{rep}_d\,\Lambda = \{f : \Lambda \to M_d(K) \mid f \text{ a } K\text{-algebra homomorphism }\}$.

Examples:

1. $\Lambda = K$. One has that $\mathrm{rep}_d(\Lambda)$ consists of one element, the K-algebra homomorphism sending λ to λI_d where I_d is the $d \times d$-identity matrix.

2. $\Lambda = K[X]$. One has that $\mathrm{rep}_d\,K[X] \simeq M_d(K)$ where the map from the right to the left is given by evaluation, i.e. a matrix A is sent to the K-algebra homomorphism which takes the value $f(A)$ on a given polynomial f.

3. $\Lambda = K\langle X,Y\rangle$. One has that $\mathrm{rep}_d\,K\langle X,Y\rangle \simeq M_d(K) \times M_d(K)$ where the map from the right to the left takes a pair of matrices (A,B) to the K-algebra homomorphism sending a given polynomial f in two non commuting variables to the matrix $f(A,B)$.

4. $\Lambda = K[X,Y]$. One has that $\mathrm{rep}_d\,K[X,Y] \simeq \{(A,B) \in M_d(K) \times M_d(K) \mid AB = BA\}$, where again the map from right to left is given by sending a polynomial f to the matrix obtained by evaluating the polynomial f in two commuting variables in the pair of matrices (A,B) obtaining $f(A,B)$.

13. $\Lambda = K\langle X,Y\rangle/\langle XY - YX + 1\rangle$. If $K = \mathbb{C}$ then $\mathrm{rep}_d\,\Lambda$ is empty. In general one has that $\mathrm{rep}_d\,\Lambda$ is nonempty if and only the characteristic of the field K divides the number d.

5. $\Lambda = M_d(K)$. One has that $\mathrm{rep}_d\,M_d(K) \simeq \mathrm{Gl}_d(K)/K^*$ where the map from right to left is given by sending a coset AK^* of an invertible matrix A to the K-algebra homomorphism given by sending a matrix B to the matric ABA^{-1}.

Since composition of two K-algebra homomorphisms is again a K-algebra homomorphism, one can let $\mathrm{Gl}_d(K)/K^*$ act on $\mathrm{rep}_d(\Lambda)$ just by composition with the corresponding automorphism on $M_d(K)$. $\Psi : \mathrm{Gl}_d(K)/K^* \times \mathrm{rep}_d(\Lambda) \to \mathrm{rep}_d(\Lambda)$ is given by $\Psi(AK^*, f) = \phi_A \circ f$ where $\phi_A(B) = ABA^{-1}$ where A is any representative from the coset AK^*.

Often one suppresses K^* and one is speaking about an action of $\mathrm{Gl}_d(K)$ on $\mathrm{rep}_d\,\Lambda$.

Now let Λ be a finitely generated K-algebra. Then one has that Λ is isomorphic to $K\langle X_1, X_2, ..., X_n\rangle/I$ for an ideal I in the free algebra $K\langle X_1, X_2, ..., X_n\rangle$. Using such an isomorphism as an identification, an element f of $\mathrm{rep}_d\,\Lambda$ is completely determined by the values on the elements $X_1, X_2, ..., X_n$ where one denotes the residues of the variables $X_1, X_2, ..., X_n$ also with the symbol $X_1, X_2, ..., X_n$ respectively. Therefore the n-tuple $(f(X_1), f(X_2), ..., f(X_n)) \in M_d(K) \times M_d(K) \times .... \times M_d(K)$ can be identified with f. In the Examples 2, 3, 4 and 13, n has been 1, 2, 2 and 2 respectively. In Examples 2 and 3, one obtained the whole space of matrices and the whole space of ordered pairs of matrices as possible values respectively, while in example 4 one obtained that the set of possible ordered pairs of matrices were given as the common zeros of d^2 quadratic polynomials in $2d^2$ variables. While in

Example 13 with $K = \mathbb{C}$ one obtained the empty set, which is also the common zeros of a set of polynomials.

In general, the possible n-tuples in $M_d(K)^n$ which represent K-algebra homomorphisms are given by polynomial equations. These can be described in the following way: For each $\alpha \in I$, $\alpha(A_1, A_2, ..., A_n) = 0$ gives a set of polynomial equations on the nd^2 entries in the matrices $A_1, A_2, ..., A_n$. Next taking all these equations as α runs through the ideal I, gives (usually) an infinite collection of polynomial equations on the entries in the matrices $A_1, A_2, ..., A_n$, and these equations determine an affine variety which is in bijection with $\operatorname{rep}_d \Lambda$. Hence

$$\operatorname{rep}_d \Lambda \simeq \{(A_1, A_2, ..., A_n) \in M_d(K)^n \mid \alpha(A_1, A_2, ..., A_n) = 0, \forall \alpha \in I\},$$

which shows that $\operatorname{rep}_d \Lambda$ can be thought of as a subvariety of the affine space $M_d(K)^n$. Furthermore, if one uses the identification between $\operatorname{rep}_d \Lambda$ and the set of n-tuples of matrices satisfying all the polynomial equations above, where an $f \in \operatorname{rep}_d \Lambda$ is identified with the n-tuple of matrices $(f(X_1), f(X_2), ..., f(X_n))$, then for each $A \in \operatorname{Gl}_d(K)$ the n-tuple corresponding to the K-algebra homomorphism $\phi_A \circ f$ is the n-tuple $(Af(X_1)A^{-1}, Af(X_2)A^{-1}, ..., Af(X_n)A^{-1})$. This shows that the subvariety of n-tuples of matrices representing the set of K-algebra homomorphisms from Λ to $M_d(K)$ is closed with respect to conjugation with matrices from $\operatorname{Gl}_d(K)$. In fact it shows that $\operatorname{rep}_d \Lambda$ with the given action of $Gl_d(K)$ and the set of matrices representing the K-algebra homomorphisms with the action from $Gl_d(K)$ by conjugation are isomorphic as $\operatorname{Gl}_d(G)$-sets.

For each $m \in \operatorname{rep}_d \Lambda$ there is associated a d-dimensional Λ-module M_m. This module is K^d as a K-vector space, and for each $\lambda \in \Lambda$ and $v \in M_m = K^d$ the multiplication of λ with v is given as $\lambda v = m(\lambda)v$ where the multiplication on the right side is regular matrix multiplication when each element of K^d is considered as a $d \times 1$-matrix.

Two elements m and m' in $\operatorname{rep}_d \Lambda$ represent isomorphic Λ-modules M_m and $M_{m'}$ if and only if m and m' belong to the same orbit under the action of $\operatorname{Gl}_d(K)$ on $\operatorname{rep}_d \Lambda$. Hence, one obtains a bijection between the set of isomorphism classes of Λ-modules of dimension d as K-vector spaces and the set of $\operatorname{Gl}_d(K)$-orbits in $\operatorname{rep}_d \Lambda$.

Here are some facts about the set of $\operatorname{Gl}_d(K)$ orbits in $\operatorname{rep}_d \Lambda$ for a K-algebra Λ generated by n elements.

1. For each m in $\operatorname{rep}_d \Lambda$ the orbit $\operatorname{Gl}_d(K)m$ is open in its Zariski closure $\overline{\operatorname{Gl}_d(K)m} = \{m' \in \operatorname{rep}_d \Lambda \mid p(m') = 0$ whenever p is a polynomial in nd^2 variables such that $p(\operatorname{Gl}_d(K)m) = 0\}$.

2. For each m in $\operatorname{rep}_d \Lambda$ the closure of the orbit of m, $\overline{\operatorname{Gl}_d(K)m}$, is a union of orbits.

3. The dimension of the complement of an orbit in its closure is less than the dimension of the orbit, i.e. $\dim(\overline{\operatorname{Gl}_d(K)m} \setminus \operatorname{Gl}_d(K)m) < \dim \overline{\operatorname{Gl}_d(K)m}$. The following formula for the dimension hold: $\dim \operatorname{Gl}_d(K)m = d^2 - \dim \operatorname{End}_\Lambda(M_m)$. Here dimension is referring to the Krull-dimension of the varieties.

Now one looks at isomorphism classes of Λ-modules having dimension d as K-vector spaces, and denotes each of them also by a simple letter M. For two isomorphism classes M and N, one says that M degenerates to N if the orbit in $\operatorname{rep}_d \Lambda$ corresponding to N is contained in the closure of the orbit of the isomorphism class corresponding to M. One writes this by $M \leq_{\deg} N$. Because of statement 3 above, this becomes a partial order on the set of isomorphism classes of Λ-modules having dimension d as K-vector spaces.

Example:

2. $\Lambda = K[X]$ and $d = 2$. Then $\operatorname{rep}_2 K[X]$ consists of all 2 by 2 matrices over K. The orbits under the $\mathrm{Gl}_2(K)$-action are basically determined by the eigenvalues of the matrices. Most of the orbits can be parameterized by $\{\{\alpha, \beta\} \mid \alpha \in K,\ \beta \in K,\ \alpha \neq \beta\}$ where the correspondence is given in a one-to-two fashion by taking the orbit of the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ which has dimension $4 - 2 = 2$ according to the above formula. These orbits are closed.

Then to each α in K there are two orbits, the orbit of the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, which consists of a single element and is hence of dimension $0 = 4 - 4$, and the orbit of the matrix $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$. This orbit is of dimension $4 - 2 = 2$, and its closure contains in addition to the orbit itself, the orbit (one single element) of $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$.

If one wants to look at higher dimensions than two for this example, one can list the eigenvalues of the matrix, and since conjugate matrices have the same eigenvalues, all matrices in an orbit have the same eigenvalues counted with multiplicities. The eigenvalues together with their multiplicity will therefore parameterize the orbits in a certain combinatorial way.

To do this, one first has to look at the situation where there is only one eigenvalue. Letting $\alpha \in K$, one then has the set $\operatorname{rep}_d K[X]^\alpha$ which is the subset of $\operatorname{rep}_d \Lambda$ consisting of matrices having α as the sole eigenvalue. Then the orbits in $\operatorname{rep}_d \Lambda^\alpha$ will correspond to partitions of d, $d_1 \geq d_2 \geq .... \geq d_d \geq 0$ where $\sum_{i=1}^d d_i = d$. Each d_i corresponds to a Jordan normal form of a matrix of size d_i with eigenvalue α, and the partition $d_1 \geq d_2 \geq ... \geq d_d$ then corresponds to the representation of dimension d with Jordan blocks of size d_i with eigenvalue α along the diagonal. Or equivalently, this corresponds to the direct sum of representations of dimension d_i given by a Jordan normal form of size d_i with eigenvalue α. Then a $K[X]$-module corresponding to the partition $d_1 \geq d_2 \geq ... \geq d_d$ degenerates to the $K[X]$-module corresponding to the partition $d'_1 \geq d'_2 \geq ... \geq d'_d$ if and only if $\sum_{i=1}^t d_i \geq \sum_{i=1}^t d'_i$ for all t.

Now this information can be put together by first sorting by eigenvalues and their multiplicities, and then using the order given by partitions for each eigenvalue individually, to get a complete picture of the degeneration order in $\operatorname{rep}_d K[X]$ for any d as the example with $d = 2$ illustrates.

4. Degeneration and exact sequences

As one shall see in this section, there is a close relationship between exact sequences and degenerations. The starting point for this relationship is the following old result which is included in order to illustrate some of the techniques used to obtain degenerations.

Proposition 4.1. *Given a finitely generated K-algebra Λ and an exact sequence $0 \to N' \to M \to N \to 0$ of Λ-modules which are finitely dimensional as K-vector spaces, then $M \leq_{deg} N \oplus N'$.*

Proof. Let $\Lambda = K\langle X_1, X_2, ..., X_n\rangle/I$ be a presentation of the finitely generated K-algebra. Assume the dimension of M as a K-vector space is d which is then the same as the dimension of $N' \oplus N$ as a K-vector space. Produce a basis for M by first choosing a basis for N', and then extend this basis to a basis for the whole of M. This produces a complement V of N' in M such that $M = N' \oplus V$ as a K-vector space. Then identify the part of the basis for M coming from V with a basis for N through the map from M to N given in the exact sequence. Since N' is closed with respect to multiplication with each elements from Λ one gets that $\phi_M(\lambda) = \begin{pmatrix} \phi_{N'}(\lambda) & 0 \\ \alpha_\lambda & \beta_\lambda \end{pmatrix}$ for each λ in Λ. Further, β_λ induces a Λ-module structure on V which by the identification of the basis in V with a basis of N makes V equal to N as a Λ-module. Hence $\beta_\lambda = \phi_N(\lambda)$ so the matrix representing multiplication with each λ in Λ is with respect to this basis given as $\phi_M(\lambda) = \begin{pmatrix} \phi_{N'}(\lambda) & 0 \\ \alpha_\lambda & \phi_N(\lambda) \end{pmatrix}$ for some K-linear homomorphism $\alpha_\lambda : V \to N'$. Therefore the n-tuple of matrices representing the module M is

$$(\phi_M(X_1), \phi_M(X_2), ..., \phi_M(X_n)) =$$

$$\left(\begin{pmatrix} \phi_{N'}(X_1) & 0 \\ \alpha_{X_1} & \phi_N(X_1) \end{pmatrix}, \begin{pmatrix} \phi_{N'}(X_2) & 0 \\ \alpha_{X_2} & \phi_N(X_2) \end{pmatrix}, ..., \begin{pmatrix} \phi_{N'}(X_n) & 0 \\ \alpha_{X_n} & \phi_N(X_n) \end{pmatrix} \right).$$

Now let $t \neq 0$ be an element of K and compose ϕ_M with the inner automorphism determined by the matrix $\begin{pmatrix} t^{-1}I_{d'} & 0 \\ 0 & I_{d''} \end{pmatrix}$ where d' is the dimension of N' as a K-vector space and d'' is the dimension of V as a K-vector space. From this one can see that the part of the line (a point if all α_{X_i} are zero)

$$\left(\begin{pmatrix} \phi_{N'}(X_1) & 0 \\ t\alpha_{X_1} & \phi_N(X_1) \end{pmatrix}, \begin{pmatrix} \phi_{N'}(X_2) & 0 \\ t\alpha_{X_2} & \phi_N(X_2) \end{pmatrix}, ..., \begin{pmatrix} \phi_{N'}(X_n) & 0 \\ t\alpha_{X_n} & \phi_N(X_n) \end{pmatrix} \right)$$

when $t \neq 0$ is in the orbit corresponding to M. Any polynomial that vanishes on this part of the line will also have to vanish on the n-tuple of matrices corresponding to $t = 0$. Therefore the tuple of matrices corresponding to $t = 0$ in $\operatorname{rep}_d \Lambda$ is in the closure of the orbit corresponding to M. However, the tuple of matrices corresponding to the value $t = 0$ is

$$\left(\begin{pmatrix} \phi_{N'}(X_1) & 0 \\ 0 & \phi_N(X_1) \end{pmatrix}, \begin{pmatrix} \phi_{N'}(X_2) & 0 \\ 0 & \phi_N(X_2) \end{pmatrix}, ..., \begin{pmatrix} \phi_{N'}(X_n) & 0 \\ 0 & \phi_N(X_n) \end{pmatrix} \right),$$

which represent the K-algebra homomorphism corresponding to the direct sum of the two Λ-modules N' and N. $\square$

This result was the starting point for the degeneration theory of modules and goes back to M. Artin in the 60'ties.

As a corollary of this result one obtains the following.

Corollary 4.2. *Let Λ be a finitely generated K-algebra and M a Λ-module in $\text{rep}_d \Lambda$. Then the module M is semisimple if the orbit of M in $\text{rep}_d \Lambda$ is closed.*

Proof. Assume that the orbit of M in $\text{rep}_d \Lambda$ is closed. Then for each Λ-submodule N of M, the module $N \oplus M/N$ is in the closure of the orbit of M by Artin's result. By assumption the orbit of M is its own closure. Therefore M has to be isomorphic to $N \oplus M/N$. However, this forces N to be a direct summand of M and hence M is semisimple since each submodule of M is then a direct summand. $\square$

Later on, one will see that the result of this corollary can be extended to give a complete characterization of the closed orbits in $\text{rep}_d \Lambda$ for a finitely generated K-algebra Λ since the closed orbits in $\text{rep}_d \Lambda$ are exactly the orbits of semisimple Λ-modules which has dimension d as K-vector spaces.

However, not all degenerations can be obtained in this way since there are examples where an indecomposable Λ-module degenerates properly into another indecomposable Λ-module, while each degeneration $M \leq_{deg} N$ coming from a short exact sequence with all terms being non-zero as above, have the property that N have to be a decomposable Λ-module.

Proposition 4.3. *Let Λ be a finitely generated K-algebra. If there exists an exact sequence $0 \to A \to A \oplus M \to N \to 0$ of Λ-modules with A, M and N finitely dimensional as K-modules, then $\dim M = \dim N$ and $M \leq_{deg} N$.*

This result was obtained by Chr. Riedtmann and could be used directly to give examples where an indecomposable Λ-module was degenerating properly into some other indecomposable Λ-module.

An outline of the proof of this proposition will be postponed. Instead a couple of examples are given in order to show how this result can be applied to obtain degenerations.

The first application is an example originally coming from representations of algebra given by a quiver which is illustrating that an indecomposable Λ-module can degenerate properly into another indecomposable Λ-module.

Example: Consider the K-subalgebra Λ of the K-algebra of lower 3×3-matrices over the field K given as the set of matrices

$$\Lambda = \{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & c \end{pmatrix} \mid a, b, c, d, e \in K \}.$$

Then the left ideal I generated by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ in Λ is as a left Λ-module projective and is isomorphic to $P_2 = \Lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Furthermore, one has that

$$\Lambda = P_1 \oplus P_2 = \Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \Lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now one has the non split exact sequence

$$0 \to I \to \Lambda \to \Lambda/I \to 0$$

with $\Lambda = P_1 \oplus P_2$ which by identifying I and P_2 through an Λ-isomorphism between them gives an exact sequence

$$0 \to P_2 \to P_1 \oplus P_2 \to \Lambda/I \to 0.$$

This is showing that $P_1 \leq_{deg} \Lambda/I$. It is not hard to show that both P_1 and Λ/I are indecomposable Λ-modules. Furthermore, since I is in the radical of Λ, Λ/I is not a projective Λ-module, and therefore P_1 and Λ/I can not be isomorphic as left Λ-modules.

Here is another example where one can use the result of Riedtmann to prove that any finitely dimensional Λ-module degenerates to a semisimple Λ-module.

Let Λ be a finitely generated K-algebra, and let M be a Λ-module which is finitely dimensional as a K-vector space. Let $A = \operatorname{rad} M \oplus \operatorname{rad}^2 M \oplus .... \oplus \operatorname{rad}^{n-1} M$ where $\operatorname{rad}^i M$ denotes the i-th power of the radical of the Λ-module M and $\operatorname{rad}^n M = 0$. Consider now the exact sequence

$$0 \to A \xrightarrow{f} A \oplus M \to$$
$$M/\operatorname{rad} M \oplus \operatorname{rad} M/\operatorname{rad}^2 M \oplus ... \oplus \operatorname{rad}^{n-1} M/\operatorname{rad}^n M \to 0$$

where the left hand Λ-homomorphism f is induced from the inclusions $\operatorname{rad}^i M \subset \operatorname{rad}^{i-1} M$ for $i = 1,\ 2,\ ...\ , n-1$. This shows that M degenerates into a semisimple Λ-module having the same composition factors as M counted with multiplicities.

One could also have seen this last result by an inductive argument using just exact sequences as in the result of M. Artin.

Proof. Here is an indication of a proof of Riedtmann's result in Proposition 4.3

Consider an exact sequence

$$0 \to A \xrightarrow{\binom{f}{g}} A \oplus M \to N \to 0,$$

where $f : A \to A$ and $g : A \to M$ are Λ-homomorphisms. Next consider for each λ in K the exact sequence

$$0 \to A \xrightarrow{\binom{f-\lambda I_A}{g}} A \oplus M \to N_\lambda \to 0$$

of Λ-modules where I_A is the identity on A and N_λ is the cokernel of the Λ-homomorphism $\binom{f-\lambda I_A}{g}$. Now $\binom{f-\lambda I_A}{g}$ is a split monomorphism for almost all λ in K which is a consequence of the fact that $f - \lambda I_A$ is an isomorphism for all λ not an eigenvalue of f. Next choose a subspace of $A \oplus M$ of the same dimension as M and which only meets a finite number of the subspaces $\binom{f-\lambda I_A}{g} A$ nontrivially, including the case with $\lambda = 0$. Now for each λ in K where the intersection of $\binom{f-\lambda I_A}{g} A$ and V is trivial, one obtains an induces Λ-module structure on V by projecting $A \oplus M$ onto V along $\binom{f-\lambda I_A}{g} A$. For each of these admissible values of λ which is not an eigenvalue of f one obtains an induced Λ-module structure on V isomorphic to M, while for the value $\lambda = 0$ one obtains an induced Λ-module structure isomorphic to N. This shows that $M \leq_{deg} N$. □

The converse of Riedtmann's result also holds as has been proved by G. Zwara. For a proof of this statement given in the next proposition, the reader is referred to the paper [Z2].

Proposition 4.4. *Let Λ be a finitely generated K-algebra and M and N in* $\operatorname{rep}_d \Lambda$ *with $M \leq_{deg} N$. Then there exists an exact sequence* $0 \to A \to A \oplus M \to N \to 0$ *of Λ-modules where A is finitely dimensional as a K-module.*

These two results give a complete algebraical description of degeneration, and hence one can take these as the starting point generalizing the notion of degeneration also to the situations where one is not working over an algebraically closed field, or even to the situations considering just a finitely generated K-algebra over a commutative artin ring K, and where one is using lengths instead of dimensions. So from now on, unless otherwise stated, K will be a commutative artin ring, Λ will be a finitely generated algebra over K, and the dimension as a K-vector space will be substituted by the length as a K-module and denoted by ℓ_K or just ℓ if there can be no confusion. One can also loosely introduce the set $\operatorname{rep}_d \Lambda$ as the set of Λ-modules of length d considered as K-modules.

In this note the version of the exact sequence

$$0 \to A \to A \oplus M \to N \to 0$$

has been given, but there exists such a sequence if and only if there exists a sequence

$$0 \to N \to B \oplus M \to B \to 0.$$

So one can choose to work with either side as one desires.

5. Virtual degenerations

In this section the notion of degeneration is generalized. Since the Krull-Remark-Schmidt-theorem holds for finite length modules, one has uniqueness in decompositions of such a module into indecomposable summands up to isomorphism. However, as one will soon see, this decomposition does not behave nicely with respect to degenerations. This is because one cannot cancel common direct summands from modules M and N when $M \leq_{deg} N$, and obtain a degeneration of the remaining complements. Here is an interesting example that demonstrates this.

Consider $\Lambda_q = K\langle X, Y\rangle/\langle X^2,\ Y^2, XY + qYX\rangle$ where K is an algebraically closed field and q a non-zero element of K. Λ_q is then a self injective local K-algebra of dimension four. Then look at $\operatorname{rep}_4 \Lambda = \{(A, B) \in M_4(K)^2 \mid A^2 = B^2 = AB + qBA = 0\}$. The regular representation ${}_\Lambda\Lambda$ itself belongs to $\operatorname{rep}_4 \Lambda_q$ and the dimension of its orbit under the action of $\mathrm{Gl}_4(K)$ is given as $4^2 - \dim_K \operatorname{End}_{\Lambda_q}(\Lambda_q) = 4^2 - 4 = 12$. Next consider, for each $0 \neq (a, b) \in K^2$, the 2-dimensional Λ_q-module $M_{a,b} = \Lambda_q/\langle aX + bY\rangle$ where also the residue of X and Y in Λ_q are denoted by X and Y respectively. Now $M_{a,b}$ is isomorphic to $M_{a',b'}$ if and only if the determinant of the matrix $\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)$ is zero. Further, if the determinant of the matrix $\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)$ is non-zero, the dimension of the endomorphism ring of $M_{a,b} \oplus M_{a',b'}$ is 6. Hence, the orbit of each of these Λ_q-modules $M_{a,b} \oplus M_{a'b'}$ when the determinant of the matrix $\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)$ is nonzero, has dimension $16 - 6 = 10$. This together with the two parameters (a, b) and (a', b') shows that all these orbits make up a geometric object of dimension 12. Therefore not all orbits of these Λ_q-modules $M_{a,b} \oplus M_{a',b'}$ can be in the closure of the orbit corresponding to ${}_{\Lambda_q}\Lambda_q$ since this orbit is also, as calculated above, of dimension 12. However, for each $0 \neq (a, b) \in K^2$ there is an exact sequence

$$0 \to M_{a,b} \to \Lambda_q \oplus S \to \Lambda_q/\operatorname{soc}\Lambda_q \to 0$$

where S is the simple Λ_q-module and $\operatorname{soc}\Lambda_q$ denotes the socle of Λ_q which is also isomorphic to S. Therefore one has that $\Lambda_q \oplus S \leq_{deg} M_{a,b} \oplus (\Lambda_q/\operatorname{soc}\Lambda_q)$. Now since each $M_{a',b'}$ is annihilated by $\operatorname{soc}\Lambda_q$ and in addition is cyclic, there is an exact sequence

$$0 \to S \to \Lambda_q/\operatorname{soc}\Lambda_q \to M_{a',b'} \to 0.$$

By transitivity of degenerations one gets that $\Lambda_q \oplus S \leq_{deg} M_{a,b} \oplus M_{a',b'} \oplus S$ for all choices for $0 \neq (a, b)$ and $0 \neq (a', b')$. So even if one does not have $\Lambda_q \leq_{deg} M_{a,b} \oplus M_{a',b'}$ one obtains a degeneration by adding the same direct summand to both sides. This shows that in the degeneration order, one cannot in general cancel common direct summands.

This example was first considered for this purpose by J. Carlson where he used the case $q = -1$.

Let again Λ be a finitely generated K-algebra with K an algebraically closed field. One says that a Λ-module M in $\operatorname{rep}_d \Lambda$ virtually degenerates to a Λ-module N in $\operatorname{rep}_d \Lambda$ if there exists a finitely dimensional Λ-module B such that $M \oplus B$ degenerates to $N \oplus B$.

By Zwara's result, this is the same as saying that there exact an sequence $0 \to A \to A \oplus B \oplus M \to B \oplus N \to 0$ of Λ-modules where A and B are finitely dimensional as K-vector spaces.

Again, using the characterization of virtual degenerations given by short exact sequences, one can extend the notion of virtual degeneration to situations where one is working over a finitely generated algebra Λ over a commutative artin ring K instead of working with finitely generated algebras over an algebraically closed field.

Proposition 5.1. *Let Λ be a finitely generated K-algebra. If the Λ-module M virtually degenerates to the Λ-module N, i.e. there is an exact sequence of Λ-modules*

$$0 \to A \to A \oplus B \oplus M \to B \oplus N \to 0$$

which are of finite length as K-modules, then $\ell(\mathrm{Hom}_\Lambda(M, X)) \leq \ell(\mathrm{Hom}_\Lambda(N, X))$ for each Λ-module X which has finite length as a K-module.

Proof. Consider an exact sequence

$$0 \to A \to A \oplus B \oplus M \to B \oplus N \to 0$$

as in the proposition, and let X be a Λ-module which has finite length as a K-module. Then apply the functor $\mathrm{Hom}_\Lambda(\ ,X)$ to this sequence and obtaining the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(B \oplus N, X) \to \mathrm{Hom}_\Lambda(A \oplus B \oplus M, X) \to \mathrm{Hom}_\Lambda(A, X)$$

of K-modules. Then counting lengths as K-modules, one obtains the inequality:

$$\ell(\mathrm{Hom}_\Lambda(B \oplus N, X)) + \ell(\mathrm{Hom}_\Lambda(A, X)) \geq \ell(\mathrm{Hom}_\Lambda(A \oplus B \oplus M, X)).$$

Now by subtracting $\ell(\mathrm{Hom}_\Lambda(A \oplus B, X))$ from both sides of this inequality one obtains the desired result $\ell(\mathrm{Hom}_\Lambda(M, X)) \leq \ell(\mathrm{Hom}_\Lambda(N, X))$ for each Λ-modules X which has finite length as a K-module. □

One can also obtain this result from geometric considerations in the case one is working over an algebraically closed field.

In light of this result one introduces another relation on $\mathrm{rep}_d \Lambda$, called the *hom* order, which also turns out to be a partial order on the set $\mathrm{rep}_d \Lambda$ by a result of M. Auslander.

For two Λ-modules M and N in $\mathrm{rep}_d \Lambda$ one says that $M \leq_{hom} N$ if $\ell(\mathrm{Hom}_\Lambda(X, M)) \leq \ell(\mathrm{Hom}_\Lambda(X, N))$ for all Λ-modules X of finite length as K-modules.

As already mentioned, this order is called the hom order. This order is also symmetric in the sense that $\ell(\mathrm{Hom}_\Lambda(X, M)) \leq \ell(\mathrm{Hom}_\Lambda(X, N))$ for all Λ-modules X of finite length as K-modules if and only if $\ell(\mathrm{Hom}_\Lambda(M, Y)) \leq \ell(\mathrm{Hom}_\Lambda(N, Y))$ for all Λ-module Y of finite length as K-modules.

¿From the propositions above one then has the following implications: $M \leq_{deg} N \Longrightarrow M \leq_{vdeg} N \Longrightarrow M \leq_{hom} N$. By the example of J. Carlson one knows that the first implication is not an equivalence in general. So far no one has come up with an example showing that the last implication is not an equivalence, and hence it is an open problem if the second implication is in fact an equivalence.

Recall that a finitely generated K-algebra Λ with K a commutative artin ring is called an artin algebra if Λ as a K-module has finite length. Such an artin algebra is said to be of finite representation type if there are only a finite number of isomorphism classes of indecomposable Λ-modules. The algebra in the example of J. Carlson is artinian, but it is not of finite representation type. One can see that directly. However, that the Λ_qs above are not of finite representation type can also be deduced from the next result stating that for artin algebras of finite representation type the degeneration order, the virtual degeneration order and the hom order all agree. This was first proved by G. Zwara. For general background on representation theory of artin algebras see [ARS].

Theorem 5.2. *If Λ is an artin K-algebra of finite representation type, and M and N are two Λ-modules of the same length as K-modules, then the three following statements are equivalent:*

1. $M \leq_{deg} N$
2. $M \leq_{vdeg} N$
3. $M \leq_{hom} N$

Proof. A complete proof of this is based on the study of the finitely presented functors on the category of finitely generated Λ-modules and hence belongs to the Auslander-Reiten theory for artin algebras. Here only a sketch will be given. That statement *1* implies statement *2* is obvious, and that statement *2* implies statement *3* has already been proven, so the only part left which has to be proved is that statement *3* implies statement *1* when Λ is an artin K-algebra of finite representation type.

So, assume there are two Λ-modules M and N of the same length as K-modules such that $M \leq_{hom} N$. If M is isomorphic to N there is nothing to prove. Hence one can assume that M and N are not isomorphic modules. Then one can consider the contravariant functors $\mathrm{Hom}_\Lambda(\ ,N)$ and $\mathrm{Hom}_\Lambda(\ ,M)$ on the category of finitely generated Λ-modules. Since Λ is of finite representation type, these functors are of finite length and hence artinian. Next consider the category $\mathrm{mod}\,\mathrm{mod}\,\Lambda$ of finitely presented contravariant functors on the category of finitely generated Λ-modules. For each F in $\mathrm{mod}\,\mathrm{mod}\,\Lambda$ one lets $[F]$ denote the element associated with the functor F in the Grothendieck group of $\mathrm{mod}\,\mathrm{mod}\,\Lambda$. Since Λ is of finite representation type, there is a finitely presented functor G such that $[G] = [\mathrm{Hom}_\Lambda(\ ,N)] - [\mathrm{Hom}_\Lambda(\ ,M)]$. Now a fact which will be proven in the next lemma is that for a pair of nonzero Λ-modules N and M of the same length as K-modules, with no common nonzero direct summand and with $M \leq_{hom} N$, there is always a direct summand N' of N such that $\ell(\mathrm{Hom}_\Lambda(N',M)) < \ell(\mathrm{Hom}_\Lambda(N',N))$. Using this fact and that the functor $\mathrm{Hom}_\Lambda(\ ,N)$ is artinian, one can now prove that the functor $\mathrm{Hom}_\Lambda(\ ,N)$ has a subfunctor H with the same composition factors as $\mathrm{Hom}_\Lambda(\ ,M)$ counted with multiplicity, i.e. $[H] = [\mathrm{Hom}_\Lambda(\ ,M)]$. To see this, consider all subfunctors F of $\mathrm{Hom}_\Lambda(\ ,N)$ such that $\ell F(X) \geq \ell\,\mathrm{Hom}_\Lambda(X,M)$ for all Λ-modules X of finite length as K-module. This set is nonempty since $\mathrm{Hom}_\Lambda(\ ,N)$ is a member, and hence this set has a minimal element. Let H be one of these minimal elements. Choose a minimal projective presentation of the functor H which has projective dimension one since it is a submodule of a projective functor, $0 \to \mathrm{Hom}_\Lambda(\ ,A) \to \mathrm{Hom}_\Lambda(\ ,B) \to H \to 0$. If $[H] \neq [\mathrm{Hom}_\Lambda(\ ,M)]$, one obtains that $M \oplus B \leq_{hom} A$ with $M \oplus B \not\simeq A$. But then, by first cancelling common direct summands one gets from the fact which is proved in the next lemma, that there exists an indecomposable summand A' of A where one has that $\ell\,\mathrm{Hom}_\Lambda(A',A) > \ell\,\mathrm{Hom}_\Lambda(A',M \oplus B)$. Hence, since $\mathrm{Hom}_\Lambda(\ ,B)$ is contained in the radical of the functor $\mathrm{Hom}_\Lambda(\ ,A)$, the image H' of the subfunctor $\mathrm{rad}\,\mathrm{Hom}_\Lambda(\ ,A') \oplus \mathrm{Hom}_\Lambda(\ ,A'')$ of $\mathrm{Hom}_\Lambda(\ ,A)$ in H when

$A = A' \oplus A''$, also have the property that $\ell H'(X) \geq \ell \operatorname{Hom}_\Lambda(X, M)$ for all Λ-modules X of finite length as an K-module. This contradicts the minimality of H and completes the proof of the claim that $\operatorname{Hom}_\Lambda(\ , N)$ has a subfunctor H such that $[H] = [\operatorname{Hom}_\Lambda(\ , M)]$.

Now since the global dimension of the category of finitely presented functors is at most two, one takes a minimal projective resolution of $\operatorname{Hom}_\Lambda(\ , N)/H$ and obtains an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(\ , X) \to \operatorname{Hom}_\Lambda(\ , Y) \to \operatorname{Hom}_\Lambda(\ , N'') \to \operatorname{Hom}_\Lambda(\ , N)/H \to 0$$

with N'' a direct summand of N and

$$[\operatorname{Hom} \Lambda(\ , N)/F] = [\operatorname{Hom}_\Lambda(\ , N)] - [\operatorname{Hom}_\Lambda(\ , M)].$$

Hence one gets that

$$[\operatorname{Hom}_\Lambda(\ , X)] + [\operatorname{Hom}_\Lambda(\ , N'')] - [\operatorname{Hom}_\Lambda(\ , Y)]$$
$$= [\operatorname{Hom}_\Lambda(\ , N)] - [\operatorname{Hom}_\Lambda(\ , M)],$$

or equivalently,

$$[\operatorname{Hom}_\Lambda(\ , X)] + [\operatorname{Hom}_\Lambda(\ , N'')] + [\operatorname{Hom}_\Lambda(\ , M)]$$
$$= [\operatorname{Hom}_\Lambda(\ , Y)] + [\operatorname{Hom}_\Lambda(\ , N)].$$

The representatives of the projective functors $[\operatorname{Hom}_\Lambda(\ , A)]$ with A running through a fixed set of representatives of the isomorphism classes of the finitely generated indecomposable Λ-modules is a basis for the Grothendieck group of the finitely presented functors. This group is a free abelian group. From this one obtains that

$$X \oplus N'' \oplus M \simeq Y \oplus N.$$

Now since one can assume that N and M have no common nonzero direct summand, M is isomorphic to a direct summand of Y, and $X = X' \oplus X''$ with $N \simeq X'' \oplus N''$, and hence also $Y \simeq M \oplus X'$. Therefore adding a copy of X'' to Y and to N'' with the identity homomorphism between these copies of X'' one obtains the exact sequence $0 \to X \to X \oplus M \to N \to 0$ showing that $M \leq_{deg} N$. □

Here is the statement, including a proof, of the proper inequality in the length of homomorphism spaces which was used in the proof of the theorem above. This result is due to K. Bongatz.

Lemma 5.3. *Let Λ be an finitely generated K-algebra and M and N two nonzero Λ-modules in $\operatorname{rep}_d \Lambda$ such that $M \leq_h omN$ and such that M and N have no nonzero common direct summand. Then there exist an indecomposable summand N' of N such that $\ell(\operatorname{Hom}_\Lambda(N', M)) < \ell(\operatorname{Hom}_\Lambda(N', N))$.*

Proof. Let $f_1, f_2, ..., f_n$ be a generating set for the finite length K-module $\operatorname{Hom}_\Lambda(M, N)$ and consider the exact sequence

$$M \overset{(f_1, f_2, \ldots, f_n)^{tr}}{\longrightarrow} N^n \longrightarrow C \longrightarrow 0$$

where C is the cokernel of the given map induced by the f_i, $i = 1, 2, ..., n$. Now apply $\mathrm{Hom}_\Lambda(\ , M)$ and $\mathrm{Hom}_\Lambda(\ , N)$ to this sequence and obtain the sequences

$$0 \to \mathrm{Hom}_\Lambda(C, N) \to \mathrm{Hom}_\Lambda(N^n, N) \overset{\mathrm{Hom}_\Lambda((f_1, f_2, ..., f_n)^{tr}, N)}{\longrightarrow} \mathrm{Hom}_\Lambda(M, N)$$

and

$$0 \to \mathrm{Hom}_\Lambda(C, M) \to \mathrm{Hom}_\Lambda(N^n, M) \overset{\mathrm{Hom}_\Lambda((f_1, f_2, ..., f_n)^{tr}, M)}{\longrightarrow} \mathrm{Hom}_\Lambda(M, M)$$

The K-homomorphism $\mathrm{Hom}_\Lambda((f_1, f_2, ..., f_n)^{tr}, N)$ is by construction an epimorphism. Now assume that the equality $\ell(\mathrm{Hom}_\Lambda(N, N)) = \ell(\mathrm{Hom}_\Lambda(M, N))$ is satisfied. Letting $\mathrm{Im}(\mathrm{Hom}_\Lambda((f_1, f_2, ..., f_n)^{tr}, M))$ denote the image of the morphism $\mathrm{Hom}_\Lambda((f_1, f_2, ..., f_n)^{tr}, M)$, one gets from the exact sequences above that $\ell\,\mathrm{Im}(\mathrm{Hom}_\Lambda((f_1, f_2, ..., f_n)^{tr}, M)) = \ell(\mathrm{Hom}_\Lambda(N^n, M)) - \ell(\mathrm{Hom}_\Lambda(C, M)) \geq \ell(\mathrm{Hom}_\Lambda(N^n, N)) - \ell(\mathrm{Hom}_\Lambda(C, N)) = \ell(\mathrm{Hom}_\Lambda(M, N)) \geq \ell(\mathrm{Hom}_\Lambda(M, M))$, which shows that $\mathrm{Hom}_\Lambda((f_1, f_2, ..., f_n)^{tr}, M)$ is an epimorphism. Therefore one also obtains that the identity from M to M factors through the map $(f_1, f_2, ..., f_n)^{tr}$ and hence $(f_1, f_2, ..., f_n)^{tr}$ is itself a split monomorphism. This shows that the two modules M and N have to have some common nonzero direct summands. Hence if M and N have no common nonzero direct summands and $M \leq_{hom} N$, then $\ell(\mathrm{Hom}_\Lambda(N, M)) < \ell(\mathrm{Hom}_\Lambda(N, N))$, and hence there is at least one nonzero direct summand N' of N where one has a proper inequality $\ell(\mathrm{Hom}_\Lambda(N', M)) < \ell(\mathrm{Hom}_\Lambda(N', N))$ completing the proof of the lemma. □

If one analysis the proof of the theorem above, one also get that if $M \leq_{hom} N$ and $\ell\,\mathrm{Hom}_\Lambda(X, M) = \ell\,\mathrm{Hom}_\Lambda(X, N)$ is satisfied for all but a finite number of indecomposable Λ-modules X which are of finite length as K-modules, then $M \leq_{deg} N$.

Also for the algebra $\Lambda = K[X]$, the degeneration order and the hom order coincides on $\mathrm{rep}_d\, K[X]$. This can be seen from the description on the degeneration order given earlier in this note and a simple calculation of the hom order.

One also has that the hom order descends to all extension groups for an artin K-algebra Λ as the next result shows.

Proposition 5.4. *Let Λ be an artin K-algebra and M and N two Λ-modules of the same length as K-modules. Then $M \leq_{hom} N$ implies that $\ell(\mathrm{Ext}^i(X, M)) \leq \ell(\mathrm{Ext}^i(X, N))$ and $\ell(\mathrm{Ext}^i(M, Y)) \leq \ell(\mathrm{Ext}^i(N, Y))$ for all Λ-modules X and Y of finite length and for all $i \geq 0$.*

Proof. Since Λ is assumed to be an artin K-algebra, one can consider an exact sequence

$$0 \to \Omega X \to P \to X \to 0$$

with P projective and both P and ΩX of finite length. Applying $\mathrm{Hom}_\Lambda(\ , M)$ and $\mathrm{Hom}_\Lambda(\ , N)$ to this sequence one obtains the following two exact sequences

$$0 \to \mathrm{Hom}_\Lambda(X, M) \to \mathrm{Hom}_\Lambda(P, M) \to \mathrm{Hom}_\Lambda(\Omega X, M) \to \mathrm{Ext}^1_\Lambda(X, M) \to 0$$

and

$$0 \to \mathrm{Hom}_\Lambda(X, N) \to \mathrm{Hom}_\Lambda(P, N) \to \mathrm{Hom}_\Lambda(\Omega X, N) \to \mathrm{Ext}^1_\Lambda(X, N) \to 0.$$

From these sequences one obtains that

$$\ell(\mathrm{Hom}_\Lambda(P,M)) + \ell(\mathrm{Ext}^1_\Lambda(X,M)) = \ell(\mathrm{Hom}_\Lambda(X,M)) + \ell(\mathrm{Hom}_\Lambda(\Omega X,M))$$
$$\leq \ell(\mathrm{Hom}_\Lambda(X,N)) + \ell(\mathrm{Hom}_\Lambda(\Omega X,N)) = \ell(\mathrm{Hom}_\Lambda(P,N)) + \ell(\mathrm{Ext}^1_\Lambda(X,N)).$$

However, M and N being from $\mathrm{rep}_d\,\Lambda$ with $M \leq_{hom} N$ implies that $\ell(\mathrm{Hom}_\Lambda(P,M)) = \ell(\mathrm{Hom}_\Lambda(P,N))$ for all projective Λ-modules of fine length. Subtracting this number from both sides of the inequality above gives then $\ell(\mathrm{Ext}^1_\Lambda(X,M)) \leq \ell(\mathrm{Ext}^1_\Lambda(X,N))$ for all Λ-modules X of finite length. That $\ell(\mathrm{Ext}^i_\Lambda(X,M)) \leq \ell(\mathrm{Ext}^i_\Lambda(X,N))$ for all $i \geq 0$ now follows by induction. A dual argument gives the other half of the statement. □

This result is also due to K. Bongartz.

The following consequences on the projective and injective dimensions of Λ-modules are immediate:

Corollary 5.5. *Let Λ be an artin K-algebra and M and N two finitely generated Λ-modules of the same length as K-modules. If $M \leq_{hom} N$, then the projective dimension of M is less than or equal to the projective dimension of N, and the injective dimension of M is less than or equal to the injective dimension of N.*

For those who want to work with tensor products and torsion groups instead of groups of homomorphisms and extensions, the following translation may be of interest.

Proposition 5.6. *Let Λ be a finitely generated K-algebra and M and N two nonzero Λ-modules of the same length as K-modules. Then $M \leq_{hom} N$ if and only if $\ell(X \otimes_\Lambda M) \leq \ell(X \otimes_\Lambda N)$ for all right Λ-modules X of finite length as K-modules.*

Proof. Let X be a right Λ-module which has finite length as a K-module. Then one has the following equality for all left Λ-modules Y which have finite length as K-modules. $\ell(X \otimes_\Lambda Y) = \ell(D(X \otimes_\Lambda Y))$ where D is a duality on the category of finitely generated K-modules over the commutative artin ring K. Using the isomorphisms $D(X \otimes_\Lambda N) \simeq \mathrm{Hom}_\Lambda(N, DX)$ and $D(X \otimes_\Lambda M) \simeq \mathrm{Hom}_\Lambda(M, DX)$ one then gets $\ell(X \otimes_\Lambda M) = \ell(\mathrm{Hom}_\Lambda(M, DX))$ and $\ell(\mathrm{Hom}_\Lambda(N, DX)) = \ell(X \otimes_\Lambda N)$. The statement in the proposition is then an immediate consequence of this. □

Restricting to artin K-algebras Λ this inequality also descends to all torsion groups which is now stated without a proof.

Proposition 5.7. *Let Λ be an artin K-algebra and M and N two Λ-modules of the same length as K-modules such that $M \leq_{hom} N$, then $\ell(\mathrm{Tor}^\Lambda_i(X,M)) \leq \ell(\mathrm{Tor}^\Lambda_i(X,N))$ for all finitely generated right Λ-modules X and all $i \geq 0$.*

6. Degenerations, hom order and discrete invariants

The degeneration order and the hom order has also some impact on discrete invariants associated with modules. One has already observed that projective dimension and injective dimension behaves nicely with respect to these orders as a corollary of Bongatz' result. These numerical invariants are rather coarse, and some interesting cases of degenerations occur between modules which cannot be separated by these invariants. Therefore it is interesting to look at the induced degeneration order, the induced virtual degeneration order and the induced hom orders on sets of modules where these discrete invariants stay the same.

Some consequences of the hom order, and which are therefore also consequences of the degeneration and the virtual degeneration orders, are the following.

Proposition 6.1. *Let Λ be a finitely generated K-algebra and M and N two Λ-modules of finite length as K-modules such that $M \leq_{hom} N$ Then the following holds true:*

1. The annihilator of M is contained in the annihilator of N.

2. $\ell(M/\operatorname{rad}^i M) \leq \ell(N/\operatorname{rad}^i N)$ for all i.

3. $\ell(\operatorname{soc}^i M) \leq \ell(\operatorname{soc}^i N)$ for all i. Here soc^i denotes the i'th socle of the Λ-module.

Proof. From both general considerations and the assumption of the proposition one has the following equalities and inequalities:

$$\ell(\operatorname{Hom}_\Lambda(\Lambda/\operatorname{ann} M, M)) = \ell M = \ell N \leq \ell(\operatorname{Hom}_\Lambda(\Lambda/\operatorname{ann} M, N))$$

. Now the following isomorphism $\operatorname{Hom}_\Lambda(\Lambda/\operatorname{ann} M, N) \simeq \{n \in N \mid (\operatorname{ann} M)n = 0\}$ is satisfied, and hence this last set has to be the whole of N because of the length inequality. Therefore $(\operatorname{ann} M)N = 0$ which finishes the proof of part *1*.

For the Λ-module $\Gamma_i = (\Lambda/\operatorname{ann} M)/\operatorname{rad}^i(\Lambda/\operatorname{ann} M)$, one can by evaluating a Λ-homomorphism f in $\operatorname{Hom}_\Lambda(\Gamma_i, X)$ in the residue of the identity, identify $\operatorname{soc}^i X$ with $\operatorname{Hom}_\Lambda(\Gamma_i, X)$ for all Λ-modules X annihilated by $\operatorname{ann} M$. Applying this to M and N using *1*, one gets that part *3* is satisfied.

Part *2* is proven in a similar way. Let $\Gamma = \Lambda/\operatorname{ann} M$ and consider the Λ-module $\Gamma/\operatorname{rad}^i \Gamma$. Then from Proposition 5.6 one obtains that

$$\ell(M/\operatorname{rad}^i M) = \ell((\Gamma/\operatorname{rad}^i \Gamma) \otimes_\Lambda M)$$

$$\leq \ell((\Gamma/\operatorname{rad}^i \Gamma) \otimes_\Lambda N) = \ell(N/\operatorname{rad}^i N)$$

which completes the proof of the whole proposition. □

This shows that when one wants to consider the hom order, one can always cut down to an artin algebra since $\Lambda/\operatorname{ann} M$ will always be an artin algebra when M is a Λ-module of finite length as a K-module.

Let

$$0 \to A \to A \oplus M \oplus B \to N \oplus B \to 0$$

be an exact sequence of Λ-modules which are all of finite length as K-modules. By letting $\Gamma = \Lambda/\operatorname{ann} M$ one has that $\Gamma \otimes_\Lambda M$ is isomorphic to M and from the first part of Proposition 6.1 one also has that $\Gamma \otimes_\Lambda N$ is isomorphic to N. By applying the functor $\Gamma \otimes_\Lambda -$ to the above sequence one obtains the exact sequence

$$\Gamma \otimes_\Lambda A \to \Gamma \otimes_\Lambda (A \oplus M \oplus B) \to \Gamma \otimes_\Lambda (N \oplus B) \to 0.$$

By using the identification $\Gamma \otimes_\Lambda X \simeq X/\operatorname{ann} MX$ this is translated into an exact sequence

$$A/\operatorname{ann} M \to A/\operatorname{ann} MA \oplus M \oplus B/\operatorname{ann} MB \to N \oplus B/\operatorname{ann} MB \to 0.$$

Now by counting lengths as K-modules one sees that the left hand homomorphism in this last sequence has to be a monomorphism, and hence one has an exact sequence

$$0 \to \Gamma \otimes_\Lambda A \to \Gamma \otimes_\Lambda (A \oplus M \oplus B) \to \Gamma \otimes_\Lambda (N \oplus B) \to 0$$

of Γ-modules which are all of finite lengths as K-modules. This shows that the orders $M \leq_{deg} N$ and $M \leq_{vgen} N$ introduced so far are also preserved by reducing to only $\Lambda/\operatorname{ann} M$-modules.

One can now give the characterization of closed orbits in $\operatorname{rep}_d \Lambda$ for a finitely generated algebra over an algebraically closed field K.

Corollary 6.2. *Let Λ be a finitely generated K-algebra where K is an algebraically closed field and d a natural number. Then the orbit of a Λ-module M in $\operatorname{rep}_d \Lambda$ is closed if and only if M is semisimple.*

Proof. That the orbit of M is closed in $rep_d\Lambda$ forces M to be semisimple is already established as a corollary of Artin's result. One has that if M is a semisimple module, then $\Lambda/\operatorname{ann} M$ is a semisimple ring. Therefore if M is semisimple and M degenerates into N, then N is annihilated by $\operatorname{ann} M$ and is hence itself semisimple. From the comments above one then has that M degenerates into N as a $\Lambda/\operatorname{ann} M$-module. Hence, there is an exact sequence $0 \to A \to A \oplus M \to N \to 0$ as $\Lambda/\operatorname{ann} M$-modules, which have to split since $\Lambda/\operatorname{ann} M$ is semisimple. This forces M to be isomorphic to N, and therefore the orbit of M is closed in $\operatorname{rep}_d \Lambda$. $\square$

For modules M and N in $\operatorname{rep}_d \Lambda$ one has the following list of partial orders on subsets of $\operatorname{rep}_d \Lambda$:

1. $M \leq_{tdeg} N$ if $M \leq_{deg} N$ and $M/\operatorname{rad} M$ is isomorphic to $N/\operatorname{rad} N$, i.e the Λ-modules M and N have isomorphic tops and M degenerates to N.
2. $M \leq_{ldeg} N$ if $M \leq_{deg} N$ and $\operatorname{rad}^i M/\operatorname{rad}^{i+1} M$ is isomorphic to $\operatorname{rad}^i N/\operatorname{rad}^{i+1} N$ for all i, i.e. the Λ-modules M and N have isomorphic radical layers and M degenerates to N.

The notation $M \leq_{tvdeg} N$, $M \leq_{lvdeg} N$, $M \leq_{thom} N$ and $M \leq_{lhom} N$ is defined in a similar way using the virtual degeneration order and hom order instead of degeneration order respectively.

This list is not exhaustive since one could also add socle layers or even use the subsets of $\operatorname{rep}_d \Lambda$ having the same lengths for the extension groups with a fixed set of Λ-modules on one or the other side, or both sides. Here the semisimple Λ-module $\Lambda/\operatorname{rad}\Lambda$ is the most obvious one to be used. This will be giving subsets of Λ-modules of a fixed length as K-modules where the terms in the projective and the injective resolutions are the same respectively, but where the Λ-homomorphisms in these resolutions may be different. These are generalizations of the requirement that the tops and the socles being isomorphic for the two Λ-modules being compared, respectively.

There is the following result obtained by B. Huisgen-Zimmermann regarding degenerations between two Λ-modules M and N which have simple tops and where the radical layers of the two modules are isomorphic. [H-Z]

Proposition 6.3. *If M is a Λ-module of finite length as a K-module with $M/\operatorname{rad} M$ simple and $M \leq_{ldeg} N$, then M is isomorphic to N.*

This result can be deduced as a corollary of the following more general result using the hom order instead of the degeneration order.

Proposition 6.4. *If M is a Λ-module with $M/\operatorname{rad} M$ simple and $M \leq_{lhom} N$, then M is isomorphic to N.*

Proof. The proof of this proposition goes by induction on the Loewy length of M.

If the Loewy length of M is one, then M is simple and hence also N has to be simple. Further since $\ell(\operatorname{Hom}_\Lambda(M,M)) \leq \ell(\operatorname{Hom}_\Lambda(M,N))$, there is at least one nonzero Λ-homomorphism from M to N. This is then showing that M is isomorphic to N.

So, let the Loewy length of M be $n+1$ and assume by induction that the statement hold for each X' with Loewy length less than or equal to n. Now assume $M \leq_{lhom} N$ with M having simple top. As has already been observed, $\operatorname{ann} MN = 0$ so one can reduce to the artin K-algebra $\Lambda/\operatorname{ann} M$.

Now consider the two $\Lambda/\operatorname{ann} M$-modules $M' = M/\operatorname{rad}^n M$ and $N' = N/\operatorname{rad}^n N$ of Loewy length n. Then from the assumption in the proposition one has that $\operatorname{rad}^i M'/\operatorname{rad}^{i+1} M'$ is isomorphic to $\operatorname{rad}^i N'/\operatorname{rad}^{i+1} N'$ for all i. One also has that for each $\Lambda/\operatorname{ann} M$-module X of finite length as a K-module that the following equalities and inequalities hold:

$$\begin{aligned}\ell(\operatorname{Hom}_\Lambda(N',X)) = \ell(\operatorname{Hom}_\Lambda(N',\operatorname{soc}^n X)) = \ell(\operatorname{Hom}_\Lambda(N,\operatorname{soc}^n X))\\ \geq \ell(\operatorname{Hom}_\Lambda(M,\operatorname{soc}^n X)) = \ell(\operatorname{Hom}_\Lambda(M',X)).\end{aligned}$$

Therefore one obtains that $M' \leq_{lhom} N'$. Hence M' is isomorphic to N' by the induction assumption.

Next, since the top of M is simple the following equalities and inequalities hold:

$$\begin{aligned}\ell(\operatorname{Hom}_\Lambda(M,M)) &= \ell(\operatorname{Hom}_\Lambda(M,\operatorname{rad} M)) + \ell(\operatorname{End}_\Lambda(M)/\operatorname{rad}\operatorname{End}_\Lambda(M))\\ &= \ell(\operatorname{Hom}_\Lambda(M',\operatorname{rad} M)) + \ell(\operatorname{End}_\Lambda(M)/\operatorname{rad}\operatorname{End}_\Lambda(M))\\ &= \ell(\operatorname{Hom}_\Lambda(N',\operatorname{rad} M)) + \ell(\operatorname{End}_\Lambda(M)/\operatorname{rad}\operatorname{End}_\Lambda(M))\\ &\geq \ell(\operatorname{Hom}_\Lambda(N',\operatorname{rad} M)).\end{aligned}$$

However, if N is not isomorphic to M, then again since the top of M is simple, and N' is isomorphic to M' the following equalities hold:

$$\begin{aligned}\ell(\operatorname{Hom}_\Lambda(N,M)) = \ell(\operatorname{Hom}_\Lambda(N,\operatorname{rad} M))\\ = \ell(\operatorname{Hom}_\Lambda(N',\operatorname{rad} M)) = \ell(\operatorname{Hom}_\Lambda(M',\operatorname{rad} M))).\end{aligned}$$

In general one has that when the Loewy length of M is $n+1$ then $\ell(\mathrm{Hom}_\Lambda(M', \mathrm{rad}\, M))$ is properly less than $\ell(\mathrm{Hom}_\Lambda(M,M))$, and therefore by the last identities one gets that $\ell(\mathrm{Hom}_\Lambda(N,M)$ is properly less than $\ell(\mathrm{Hom}_\Lambda(M,M)$. This contradicts the assumption that $M \leq_{hom} N$. Hence M has to be isomorphic to N which completes the proof of the proposition. □

This result about the hom order was obtained in a joint paper with Anita Valenta [SV].

7. Modules determined by their tops and first syzygies

One has seen in the previous section that sometimes some invariants are completely determining a module up to isomorphism. In this section one will look at a result of this type for artin algebras obtained by R. Bautista and E. Perez and also present a generalization of their result discovered by C. M. Ringel at this conference.

Let Λ be an artin K-algebra, and let rad denote the radical of Λ. For a Λ-module M, let ΩM denote its first syzygy, i.e. there is an exact sequence

$$0 \to \Omega M \to P \to M \to 0$$

with $P \to M$ a projective cover. The purpose of the rest of this note is to give an elementary proof of the following theorem obtained by R. Bautista and E. Perez in [BP], [S]. This is a generalization of earlier results of D. Voigt [V] for finitely dimensional algebras over an algebraically closed field.

Theorem 7.1. *Let Λ be an artin K-algebra, M and N two finitely generated Λ-modules such that $\mathrm{Ext}^1_\Lambda(M,M)=0$ and $\mathrm{Ext}^1_\Lambda(N,N)=0$. Then M is isomorphic to N if and only if $M/\mathrm{rad}\, M$ is isomorphic to $N/\mathrm{rad}\, N$ and ΩM is isomorphic to ΩN.*

The proof of this fact presented here is based on elementary homological algebra and length considerations over the commutative artin ring K.

The next lemma is a consequence of Nakayama's lemma and is well known.

Lemma 7.2. *Let X be a non-zero finitely generated Λ-module and $f : X^n \to X$ a Λ-epimorphism. Then there is an indecomposable Λ-module X' and Λ-homomorphisms $g : X' \to X^n$ and $h : X \to X'$ such that $hfg = 1_{X'}$, the identity on X'.*

This lemma can be applied to give the result after the following proposition has been proven.

Proposition 7.3. *Let M and N be finitely generated Λ-modules such that $\mathrm{Ext}^1_\Lambda(M,M)=0$, $\mathrm{Ext}^1_\Lambda(N,N)=0$, $M/\mathrm{rad}\, M$ is isomorphic to $N/\mathrm{rad}\, N$ and ΩM is isomorphic to ΩN. Then there is a Λ-epimorphism $M^n \to N$ for some natural number n.*

Proof. Let M and N satisfy the assumption in the proposition. Without loss of generality one can assume that M (and hence N) is non-zero. One has exact sequences $0 \to \Omega M \to P \to M \to 0$ and $0 \to \Omega M \to P \to N \to 0$ of Λ-modules with P a projective Λ-module. Then for each finitely generated Λ-module X one obtains exact sequences

$$0 \to \mathrm{Hom}_\Lambda(M,X) \to \mathrm{Hom}_\Lambda(P,X) \to \mathrm{Hom}_\Lambda(\Omega M,X) \\ \to \mathrm{Ext}^1_\Lambda(M,X) \to 0$$

and

$$0 \to \mathrm{Hom}_\Lambda(N,X) \to \mathrm{Hom}_\Lambda(P,X) \to \mathrm{Hom}_\Lambda(\Omega M,X) \\ \to \mathrm{Ext}^1_\Lambda(N,X) \to 0$$

This shows that the identities

$$(1) : \ell(\mathrm{Hom}_\Lambda(N,X)) - \ell(\mathrm{Ext}^1_\Lambda(N,X)) \\ = \ell(\mathrm{Hom}_\Lambda(P,X)) - \ell(\mathrm{Hom}_\Lambda(\Omega M,X)) \\ = \ell(\mathrm{Hom}_\Lambda(M,X)) - \ell(\mathrm{Ext}^1_\Lambda(M,X))$$

hold for all finitely generated Λ-modules X.

Next, let $\tau_M(N)$ be the trace of M in N which is the sum of all images of all Λ-homomorphisms from M to N. Then there is an n such that there is an exact sequence

$$(2) : 0 \to K \to M^n \to \tau_M(N) \to 0$$

which stays exact after applying the functor $\mathrm{Hom}_\Lambda(M,-)$. Hence one obtains the exact sequence

$$(3) : 0 \to \mathrm{Hom}_\Lambda(M,K) \to \mathrm{Hom}_\Lambda(M,M^n) \to \mathrm{Hom}_\Lambda(M,\tau_M(N)) \to \\ \mathrm{Ext}^1_\Lambda(M,K) \to \mathrm{Ext}^1_\Lambda(M,M^n)$$

with the third homomorphism being surjective and the last K-module being zero. Hence one deduces that $\mathrm{Ext}^1_\Lambda(M,K) = 0$ and therefore the following equality holds:

$$(4) : \ell(\mathrm{Hom}_\Lambda(M,\tau_M(N))) = \ell(\mathrm{Hom}_\Lambda(M,M^n)) - \ell(\mathrm{Hom}_\Lambda(M,K)).$$

Next, apply the functor $\mathrm{Hom}_\Lambda(N,-)$ to the sequence (2) and one obtains the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(N,K) \to \mathrm{Hom}_\Lambda(N,M^n) \to \mathrm{Hom}_\Lambda(N,\tau_M(N)) \to \\ \mathrm{Ext}^1_\Lambda(N,K) \to \mathrm{Ext}^1_\Lambda(N,M^n).$$

From this sequence one gets that the following inequality holds: $\ell(\mathrm{Hom}_\Lambda(N,\tau_M(N))) \geq \ell(\mathrm{Hom}_\Lambda(N,M^n)) - \ell(\mathrm{Ext}^1_\Lambda(N,M^n)) - (\ell(\mathrm{Hom}_\Lambda(N,K)) - \ell(\mathrm{Ext}^1_\Lambda(N,K)))$. Using the identity (1) twice with $X = M^n$ and $X = K$ respectively, one obtaines that the right hand side of this inequality is equal to $\ell(\mathrm{Hom}_\Lambda(M,M^n)) - \ell(\mathrm{Ext}^1_\Lambda(M,M^n)) - (\ell(\mathrm{Hom}_\Lambda(M,K)) - \ell(\mathrm{Ext}^1_\Lambda(M,K)))$. This last number is equal to $\ell(\mathrm{Hom}_\Lambda(M,M^n)) - \ell(\mathrm{Hom}_\Lambda(M,K))$ since $\ell(\mathrm{Ext}^1_\Lambda(M,M^n)) = 0$ and $\ell(\mathrm{Ext}^1_\Lambda(M,K)) = 0$. Now $\ell(\mathrm{Hom}_\Lambda(M,M^n)) -$

$\ell(\mathrm{Hom}_\Lambda(M,K))$ is equal to $\ell(\mathrm{Hom}_\Lambda(M,\tau_M(N)))$ by equality (4) above. Furthermore, $\mathrm{Hom}_\Lambda(M,\tau_M(N))$ can be identified with $\mathrm{Hom}_\Lambda(M,N)$ through the inclusion $\tau_M(N) \subseteq N$. Using the equality (1) with $X = N$, one gets that $\ell(\mathrm{Hom}_\Lambda(M,N)) \geq \ell(\mathrm{Hom}_\Lambda(N,N))$. From this one obtains all in all that $\ell(\mathrm{Hom}_\Lambda(N,\tau_M(N))) \geq \ell(\mathrm{Hom}_\Lambda(M,\tau_M(N))) = \ell(\mathrm{Hom}_\Lambda(M,N)) \geq \ell(\mathrm{Hom}_\Lambda(N,N))$. Since $\tau_M N$ is a Λ-submodule of N one has that $\ell(\mathrm{Hom}_\Lambda(N,\tau_M(N))) \leq \ell(\mathrm{Hom}_\Lambda(N,N))$ and therefore these two last inequalities give that the following equality holds: $\ell(\mathrm{Hom}_\Lambda(N,\tau_M(N))) = \ell(\mathrm{Hom}_\Lambda(N,N))$. From this equality one deduces that all Λ-homomorphisms from N to N factors through $\tau_M N$. However, the identity from N to N can factors through $\tau_M N$ only when $\tau_M(N) = N$. This finishes the proof of the proposition. □

One can now give the promised proof of the theorem of R. Bautista and E. Perez.

Proof. So let M and N be two Λ-modules such that $\mathrm{Ext}^1_\Lambda(M,M) = 0$ and $\mathrm{Ext}^1_\Lambda(N,N) = 0$. If M is isomorphic to N then clearly $M/\operatorname{rad} M$ is isomorphic to $N/\operatorname{rad} N$ and ΩM is isomorphic to ΩN.

For the converse, assume that $M/\operatorname{rad} M$ is isomorphic to $N/\operatorname{rad} N$ and that ΩM is isomorphic to ΩN. One can without loss of generality assume that M (and hence also N) is non-zero. From Proposition 7.3 one then has an Λ-epimorphism $f : M^n \to N$ for some natural number n and also an Λ-epimorphism $g : N^m \to M$ for some natural number m. Hence, there is a composition of Λ-epimorphisms $N^{mn} \to M^n \to N$. Then Lemma 7.2 ensures that there is an indecomposable Λ-module X and Λ-homomorphisms $X \to N^{mn}$ and $N \to X$ such that the composition

$$X \to N^{mn} \to M^n \to N \to X$$

is the identity on X. This shoves that X is isomorphic to an indecomposable direct summand of both M and N. Hence M and N have some common indecomposable direct summand and the result now follows by induction on the number of direct summands in a direct sum decomposition of M. □

During the meeting in Mar del Plata, C. M. Ringel observed that the following more general statement is valid.

Theorem 7.4. *Let Λ be an artin K-algebra and M a finitely generated Λ-module such that $\mathrm{Ext}^1_\Lambda(M,M) = 0$. If N is a finitely generated Λ-module such that there exists finitely generated Λ-modules U and V, and exact sequences $0 \to U \to V \to M \to 0$ and $0 \to U \to V \to N \to 0$, then there exists a finitely generated Λ-module X and an exact sequence $0 \to X \to X \oplus M \to N \to 0$, i.e. $M \leq_{deg} N$.*

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Recibido: 7 de noviembre de 2006
Aceptado: 3 de junio de 2007