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INTRODUCTION TO KOSZUL ALGEBRAS

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Dedicated to Héctor A. Merklen and María Inés Platzeck on their birthday.

Abstract. Las álgebras de Koszul fueron inventadas por Priddy [P] y han tenido un enorme desarrollo durante los últimos diez años, el artículo de Beilinson, Ginsburg y Soergel [BGS] ha sido muy influyente. En estas notas veremos los teoremas básicos de Álgebras de Koszul usando métodos de teoría de anillos y módulos, como se hizo en los artículos [GM1],[GM2], después nos concentraremos en el estudio de las álgebras Koszul autoinyectivas, primero las de radical cubo cero y posteriormente el caso general y por último aplicaremos los resultados obtenidos al estudio de las gavillas coherentes sobre el espacio proyectivo.

1. GRADED ALGEBRAS

In this lecture we recall the basic notions and definitions that will be used throughout this mini-course. We will always denote the base field by K. We will say that an algebra R is a *positively graded K-algebra*, if

- $R = R_0 \oplus R_1 \oplus \ldots$,
- $R_i R_j \subseteq R_{i+j}$, for all i, j, and
- $R_0 = K \times \ldots \times K$
- R_1 is a finite dimensional K-vector space

We will say R *is locally finite* if in addition, for each i, R_i is a finite dimensional K-vector space.

The elements of R_i are called homogeneous of degree i, and R_i is the degree i component of R. For example, the path algebra of any quiver is a graded algebra, whose degree i component is the K-vector space spanned by all the paths of length i. We can associate to each positively graded algebra a quiver $\mathcal{Q}$ with set of vertices $\{v_i\}_i$, where v_i corresponds to the primitive idempotent $e_i = (0, \ldots, 1, \ldots, 0)$ of R having 1 in the i-th entry and zero everywhere else. The arrows $\{\alpha\}$ of $\mathcal{Q}$ with $\alpha\colon v_i \to v_j$ correspond to the elements of some basis $\{a\}$ of $e_j R_1 e_i$. Note that there is a homomorphism of graded K-algebras

$$\Phi\colon\ K\mathcal{Q} \longrightarrow R$$

given by $\Phi(v_i) = e_i$ and $\Phi(\alpha) = a$.

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The morphism Φ is surjective if and only if $R_iR_j = R_{i+j}$ for all i and j. In such a case we say *R is a graded quiver algebra.*

If R is a graded K-algebra, we will denote by J (and also by R^+) the radical (also called the graded Jacobson radical) of R. It is the homogeneous ideal $J = R^+ = R_1 \oplus R_2 \oplus \dots$. It is also easy to see that J equals the intersection of all the maximal homogeneous ideals of R. Let $\mathcal{Q}$ be a finite quiver. A homogeneous ideal I in $K\mathcal{Q}$ is called *admissible* if $I \subseteq (K\mathcal{Q}^+)^2$. An element ρ of such an ideal is called a *uniform relation* if $\rho = \sum_{j=1}^{n} c_j\gamma_j$ where each coefficient c_j is a non zero element of K, and where γ_j are paths having the same lengths, all starting at a common vertex, and also ending at a common vertex. A uniform relation $\rho = \sum_{j=1}^{n} c_j\gamma_j$ is *minimal* if no proper subset $S \subset \{1, \dots, n\}$ yields a uniform relation $\rho = \sum_{j\in S} b_j\gamma_j$ of the ideal. It is easy to show that every admissible ideal can be generated by a set of uniform relations. In view of this discussion it is easy to prove the following characterization of positively graded algebras:

Proposition 1.1. *Let R be a K-algebra. Then R is a graded quiver algebra if and only if there exists a finite quiver $\mathcal{Q}$ and a homogeneous admissible ideal I of $K\mathcal{Q}$ generated by a set of minimal uniform relations such that $R \cong K\mathcal{Q}/I$.* $\square$

The algebra $R \cong K\mathcal{Q}/I$ is called *quadratic* if all the minimal uniform relations are homogeneous of degree two. This is equivalent to saying that if we let $I = I_2 \oplus I_3 \oplus \dots$ where $I_n = (K\mathcal{Q})_n \cap I$, then I is generated by I_2. Whether $R = K\mathcal{Q}/I$ is graded or not, we will always set $\Bbbk = R/J$.

Examples 1.2. *(1) The polynomial algebra $K[x_1, \dots, x_n]$ in n variables can be described as the algebra whose quiver $\mathcal{Q}$ consists of a single vertex v and n loops $\{x_1, \dots, x_n\}$ at that vertex. The path algebra of $\mathcal{Q}$ is the free algebra in n variables, and the ideal of relations is generated by all the differences $x_ix_j - x_jx_i$, and we have*

$$K[x_1, \dots, x_n] \cong \frac{K\langle x_1, \dots, x_n\rangle}{\langle x_ix_j - x_jx_i\rangle}$$

(2) Every monomial relations algebra is positively graded.

(3)Let R be the algebra of the quiver $\mathcal{Q}$:

$$\begin{tikzcd}
& & 2 \arrow[drr, "\beta"] & & \\
1 \arrow[urr, "\alpha"] \arrow[dr, "\delta"'] & & & & 5 \\
& 3 \arrow[rr, "\varepsilon"'] & & 4 \arrow[ur, "\tau"'] &
\end{tikzcd}$$

and let $R = K\mathcal{Q}/\langle \beta\alpha - \tau\varepsilon\delta\rangle$. Then R is not graded by path lengths.

Given a quadratic algebra R, its *quadratic dual* $R^!$ is obtained in the following way: let $V = (K\mathcal{Q})_2$ and $V^{op} = (K\mathcal{Q}^{op})_2$ be the subspaces of $K\mathcal{Q}$ and $K\mathcal{Q}^{op}$ respectively, spanned by the paths of length two. More precisely, let the set $\{\gamma\}$ be a basis of V consisting of the length two paths, and let the set $\{\gamma^{op}\}$ denote a dual basis of V^{op}. We have a bilinear form

$$<,>\colon V \times V^{op} \to K$$

defined on bases elements as follows:

$$<\gamma_i, \gamma_j^{op}>= \begin{cases} 0 & \text{if } \gamma_i \neq \gamma_j, \\ 1 & \text{otherwise.} \end{cases}$$

Let $I_2^{\perp}$ denote the orthogonal subspace

$$I_2^{\perp} = \{v \in V^{op} | <u,v>= 0 \ \ \forall u \in V\}$$

and let $\langle I_2^{\perp} \rangle$ be the two-sided ideal of $K\mathcal{Q}^{op}$ generated by $I_2^{\perp}$. Then the quadratic dual of R (or its *shriek algebra*) is $R^! = K\mathcal{Q}^{op}/\langle I_2^{\perp}\rangle$.

Example 1.3. *Let R denote the polynomial algebra in n variables, that is*

$$R = \frac{K\langle x_1, \dots, x_n\rangle}{\langle x_i x_j - x_j x_i\rangle}$$

In this case we can choose as a basis for I_2 the set of relations $\{\rho_{i,j}\}$ where $\rho_{i,j} = x_i x_j - x_j x_i$ for each $i < j$. Let us compute the orthogonal of I_2. First, it is obvious that for each k, x_k^2 is in $I_2^{\perp}$. Choose now an element $\mu = \sum_{k \neq l} c_{kl} x_k x_l \in I_2^{\perp}$. If $k < l$ we have $<\rho_{k,l}, \mu>= c_{kl} - c_{lk}$, and if $k > l$ we have $<\rho_{l,k}, \mu>= c_{lk} - c_{kl}$. Therefore $c_{kl} \neq 0$ for all $k \neq l$ and we must also have $c_{kl} = c_{lk}$. It follows easily now that

$$I_2^{\perp} = \langle x_k^2, x_i x_j + x_j x_i\rangle_{k, i<j}$$

and that the shriek algebra

$$R^! = \frac{K\langle x_1, \dots, x_n\rangle}{\langle x_k^2, x_i x_j + x_j x_i\rangle_{k,i<j}}$$

is isomorphic to the exterior algebra in n variables over K.

To each positively graded K-algebra R we associate two quadratic algebras. Firstly, if we write $\overline{R} = K\mathcal{Q}/I$ for a homogeneous admissible ideal I of the path algebra, we construct the quadratic (quotient) algebra $\widetilde{R} = K\mathcal{Q}/\langle I_2\rangle$, and the other quadratic algebra is simply $R^!$. We will be particularly interested in the case where R itself is a quadratic algebra, so that $R = \widetilde{R}$.

We need a few more definitions and basic facts. We denote by Mod_R, and mod_R the categories of all the R-modules, and of the finitely generated R-modules respectively. By $Mod_R^{\mathbf{Z}}$ and by $\mathrm{mod}_R^{\mathbf{Z}}$ we denote the category of graded (finitely generated graded respectively) R-modules. The category $\mathrm{mod}_R^{\mathbf{Z}}$ is a full subcategory of the category l.f.R of *locally finite* graded R-modules, that is the graded modules $\bigoplus_{i \in \mathbf{Z}} M_i$. where each M_i is a finite dimensional K-vector space. The morphisms in the categories $Mod_R^{\mathbf{Z}}$ and $\mathrm{mod}_R^{\mathbf{Z}}$ are the degree zero homomorphisms, that is homomorphisms $f\colon M \to N$ such that $f(M_i) \subseteq N_i$ for each i. Their space is denoted $Hom_R(M,N)_0$ for M and N graded modules, If M and N are graded R-modules and N is finitely generated then we have

$$Hom_R(M,N) = \bigoplus_{n\in\mathbf{Z}} Hom_R(M,N)_n$$

where $Hom_R(M,N)_n$ denotes the degree n homomorphisms, that is those homomorphisms $f\colon M \to N$ such that $f(M_i) \subseteq N_{i+n}$ for each i.

If M is a graded module we define its *graded shift* $M[n]$ as the graded module whose i-th component is $M[n]_i = M_{i+n}$ for all integers i. Note that if we forget the grading then all the graded shifts are isomorphic as R-modules, but they are all non isomorphic if $n \neq 0$ in the category of graded modules. We have the obvious identifications

$$Hom_R(M,N)_n = Hom_R(M,N[n])_0 = Hom_R(M[-n],N)_0$$

for each integer n. We will also consider the truncation $M_{\geq n}$ of M. It is graded submodule given by:

$$(M_{\geq n})_t = \begin{cases} 0 & \text{if } t < n. \\ M_t & \text{if } t \geq n. \end{cases}$$

Throughout this lecture $Mod_R^{\mathbf{Z},+}$ will denote the full subcategory of $Mod_R^{\mathbf{Z}}$ consisting of the graded modules bounded from below, that is of all the graded modules of the form $M = \oplus_{i\in\mathbf{Z}} M_i$ where $M_j = 0$ for all j less than some integer k_0. Similarly $Mod_R^{\mathbf{Z},-}$ denotes the full subcategory of $Mod_R^{\mathbf{Z}}$ consisting of those graded modules M bounded from above, that is such that $M_j = 0$ for all j greater than some integer t_0, and $Mod_R^{\mathbf{Z},b}$ will denote the graded modules bounded from above and from below. The corresponding bounded subcategories l.f.^-R, l.f.^+R and l.f.^bR are defined in the obvious way too. It is well-known that the category $Mod_R^{\mathbf{Z}}$ has enough projective and injective objects, and that the global dimensions of Mod_R and of $Mod_R^{\mathbf{Z}}$ are equal. There exists a duality

$$D\colon \mathit{l.f.R} \longrightarrow \mathit{l.f.R}$$

given by $D(M)_j = Hom_K(M_{-j},K)$. The category $\mathrm{mod}_R^{\mathbf{Z}}$ of finitely generated graded R-modules is contained in l.f.^+R hence its dual $D(\mathrm{mod}_R^{\mathbf{Z}})$ is the category of finitely cogenerated graded R^{op}-modules. Therefore, via this duality every object of $D(\mathrm{mod}_R^{\mathbf{Z}})$ is a submodule of a finitely cogenerated injective R^{op}-module. In particular we see that the finitely cogenerated graded injective R-modules are in l.f.^-R.

Let us note that the graded version of Nakayama's lemma holds, that is if we let $M \in Mod_R^{\mathbf{Z},+}$, then $JM = M$ if and only if $M = 0$. Consequently, the graded Jacobson radical (that is the intersection of the graded maximal submodules) of a module M in $Mod_R^{\mathbf{Z},+}$ is just JM, and adapting the standard methods one can also prove that projective covers exist in $Mod_R^{\mathbf{Z},+}$.

Since we come from the representation theory of finite dimensional algebras, we are particularly interested in finite dimensional algebras $R = K\mathcal{Q}/I$ and in the category $\mathrm{mod}\, R$ of finitely generated R-modules. By the Jordan-Hölder theorem, every finitely generated module M can be realized by a finite sequence of extensions of simple R-modules. Therefore the Yoneda algebra $E(R)$ also called the cohomology ring of R,

$$E(R) = \oplus_{t\geq 0} Ext^t_R(\mathbb{k}, \mathbb{k})$$

should contain plenty of relevant information about the representation theory of R. Note that $E(R)$ is a positively graded K-algebra where the multiplication is induced by the Yoneda product and the finite dimensionality of R also ensures that each graded component $E(R)_i$ is finite dimensional.

Example 1.4. *An interesting situation in which we deal with the cohomology ring of an algebra is in this context the trivial extension $\Lambda = T(K\mathcal{Q}) = K\mathcal{Q} \ltimes D(K\mathcal{Q})$, in the case the quiver $\mathcal{Q}$ is a connected bipartite graph that is not a Dynkin diagram.*

The algebra $T(K\mathcal{Q})$ has quiver $\hat{\mathcal{Q}} = (\hat{\mathcal{Q}}_0, \hat{\mathcal{Q}}_1)$ where $\hat{\mathcal{Q}}_0 = \mathcal{Q}_0$, and $\hat{\mathcal{Q}}_1 = \mathcal{Q}_1 \cup \mathcal{Q}_1^{op}$ and $T(K\mathcal{Q}) \cong K\hat{\mathcal{Q}}/I$ where I is the ideal generated by relations $\alpha\alpha^{op} - \beta\beta^{op}$ and $\alpha\beta$ where $\beta \neq \alpha^{op}$.

We will prove later that the Yoneda algebra $E(\Lambda)$ of Λ is the preprojective algebra $\Gamma = K\hat{\mathcal{Q}}/L$ with the same quiver as Λ and ideal L generated by all the quadratic relations of the form $\sum_{\alpha, s(\alpha)=v} \alpha\alpha^{op}$ where v runs over the vertices of the completed quiver $\hat{\mathcal{Q}}$.

The preprojective algebras were introduced by Gelfand and Ponomarev [GP], who proved:

- *The preprojective algebra Γ is finite dimensional if and only if $\mathcal{Q}$ is Dynkin.*

Baer, Geigle and Lenzing [BGL] proved the following result:

- *If $\mathcal{Q}$ is not a Dynkin diagram, then Γ is noetherian if and only if $\mathcal{Q}$ is a Euclidean diagram.*

The theorem proves that, in our case, actually the Yoneda algebra of Λ controls the representation type of Λ.

Definition 1.5. *Let R be a graded quiver K-algebra with graded Jacobson J and let $E(R)$ be its cohomology ring. We say that R is a Koszul algebra, if as an algebra, $E(R)$ is generated in degrees 0 and 1.*

The notion of Koszul algebra was introduced by S. Priddy in his study of the Steenrod algebras. Koszul algebras appear in many areas of algebra, in algebraic geometry, and their study has intensified significantly in the last ten years or so.

Examples 1.6. *(1) Hereditary algebras and radical square zero algebras are Koszul.*

(2) The tensor product of two Koszul algebras is Koszul.

(3) If R is a Koszul algebra and G a finite group of automorphsims of R is a Koszul K-algebra, such that characteristic of K does not divide the order of G, then the skew group algebra $R * G$ is Koszul [M1].

(4) Both the polynomial algebra $K[x_1, \ldots, x_n]$ and the exterior algebra $\dfrac{K\langle x_1, \ldots, x_n\rangle}{\langle x_k^2, x_ix_j + x_jx_i\rangle_{k,i<j}}$ in n variables are Koszul algebras.

(5) If $\mathcal{Q}$ is not a Dynkin diagram, then the preprojective algebra of $K\mathcal{Q}$ and the trivial extension $T(K\mathcal{Q}) = K\mathcal{Q} \ltimes D(K\mathcal{Q})$ are Koszul.

If R is a positively graded K-algebra, then it is easy to show that the vector space dimension of $Ext^1_R(\Bbbk, \Bbbk)$ is finite since $D(J/J^2)$ and $E(R)_1 = Ext^1_R(\Bbbk, \Bbbk)$ are isomorphic as K-vector spaces . Therefore, if R is a Koszul algebra, then $E(R)$ is again a (locally finite) positively graded algebra. There is a functor $F\colon Mod^{\mathbf{Z}}_R \to Mod^{\mathbf{Z}}_{E(R)}$ given by

$$F(M) = \oplus_{t \geq 0} Ext^t_R(M, \Bbbk)$$

and if M is finitely generated and has a finitely generated projective resolution, then $F(M)$ is locally finite, but not necessarily finitely generated over $E(M)$.

We end the first lecture with a more standard definition of Koszul algebras which, as will see later is equivalent to the previous one. Let $R = K\mathcal{Q}/I$ be a graded K-algebra. A graded R-module M is *linear or Koszul* if it has a graded projective resolution:

$$\ldots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

such that for each j, the projective module P_j is finitely generated by a set of homogeneous elements of degree j. Then we will say that R is a Koszul algebra if each graded simple R-module is linear, that is if R/J is a linear R-module. The following theorem is a summary of some of the main properties of Koszul algebras and linear modules. We will sketch the proofs of some parts in the following lectures.

Theorem 1.7. *$R = K\mathcal{Q}/I$ be a graded K-algebra.*

(1) If R is a Koszul algebra, then R is quadratic. Moreover, its Yoneda algebra $E(R)$ is isomorphic as a graded algebra to the quadratic dual $R^!$.

(2) R is a Koszul algebra if and only if R^{op} is Koszul.

(3) If M is a linear R-module, then $JM[1]$ and $\Omega M[1]$ are also linear, where ΩM denotes the first szyzygy.

(4) If R is Koszul, then so is its Yoneda algebra $E(R)$, and then R and $E(E(R))$ are isomorphic as graded K-algebras.

(5) Let R be now a Koszul algebra, and let K_R and $K_{E(R)}$ denote the full subcategories of $\mathrm{mod}^{\mathbf{Z}}_R$ and of $\mathrm{mod}^{\mathbf{Z}}_{E(R)}$ consisting of the linear R-modules (linear $E(R)$-modules respectively.) Then the functor $F\colon Mod^{\mathbf{Z}}_R \to Mod^{\mathbf{Z}}_{E(R)}$ induces a duality between the categories of linear modules $F\colon \mathcal{K}_R \to \mathcal{K}_{E(R)}$. Moreover, for each linear R-module M, $F(JM) = \Omega F(M)[1]$. □

We should talk now about a very nice consequence of this theorem. Assume that R is a Koszul algebra and M is a linear R-module. Recall that the Loewy length of M is finite if $J^t M = 0$ for some $t > 0$. Then parts (3) and (6) of the above theorem tell us that a linear module M has finite Loewy length if and only if the module $F(M)$ has finite projective dimension over the Yoneda algebra $E(R)$. Moreover, we get the following:

Theorem 1.8. *Let R be a Koszul algebra with Koszul dual $S = E(R)$. Then R is finite dimensional algebra over K if and only if S has finite graded global dimension.* □

2. KOSZUL DUALITY

We will talk now about Koszul duality and we will sketch proofs of some parts of theorem 1.7. stated at the end of the first lecture. We start with the following:

Lemma 2.1. *Let $R = K\mathcal{Q}/I$ be a graded quiver algebra and let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of graded R-modules. Then

(1) *If B is generated in degree zero, then so is C.*
(2) *If A and C are generated in degree zero, then so is B.*
(3) *If B is generated in degree zero, then A is generated in degree zero if and only if $JB \cap A = JA$.*

Proof. (1) is obvious.

(2) Let P_A and P_C be projective covers of A and of C respectively. By assumption, they are both generated in degree zero. It is then easy to show that $P_A \oplus P_C$ is a projective module mapping onto B, and since it is generated in degree zero, so is B.

(3) We always have $JA \subseteq A \cap JB$. Assume now that A is generated in degree zero, so $A_i = R_i A_0$. Let x_i be a homogeneous element of degree i in the intersection $A \cap JB$, so $i \geq 1$ since B is generated in degree zero. Thus $x_i \in A_i \subseteq J^i A$. Conversely, assume that $JA = A \cap JB$. Then we have the following commutative diagram with exact rows: since $JA = A \cap JB$, we have an exact commutative diagram: since $JA = A \cap JB$, we have an exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & JA \arrow[r] \arrow[d] & JB \arrow[r] \arrow[d] & JC \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/JA \arrow[r] \arrow[d] & B/JB \arrow[r] \arrow[d] & C/JC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom sequence is a sequence of semisimple modules generated in degree zero, so the top of A lies entirely in degree zero. The graded version of Nakayama's lemma tells us now that the projective cover of A is generated in degree zero, hence A is also generated by its degree zero part. □

We have the following immediate consequence:

Corollary 2.2. *Let R be a graded quiver K-algebra and let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of graded R-modules, all generated in degree zero, then for each nonnegative integer k we have $J^k B \cap A = J^k A$. □

We will now introduce two generalizations of the notion of linear module that make sense even in the non graded case. First, recall that a ring R is *semiperfect* if every finitely generated module has a projective cover.

Definition 2.3. *Let M be a finitely generated module over a quiver algebra $R = KQ/I$ and assume that M has a minimal finitely generated projective resolution*

$$\ldots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

1. *The module M is quasi-Koszul if $\Omega^k M \cap J^2 P_{k-1} = J\Omega^k M$ for each $k > 0$.*
2. *The module M is weakly Koszul if $\Omega^k M \cap J^{t+1} P_{k-1} = J^t \Omega^k M$ for each $t \geq 1$ and $k > 0$.*

So, every weakly Koszul module is quasi-Koszul, and in the graded case we also have that every shift of a linear module is weakly and thus also quasi-Koszul. To see this, let M be a linear module with projective cover P_0. Therefore JP_0 and ΩM are both generated in degree one and by applying corollary 2.2. we get $J^{t+1} P_0 \cap \Omega M = J^t \Omega M$ for each positive integer t. Since every syzygy of a linear module is linear, the rest follows by induction.

We have the following characterization:

Theorem 2.4. *Let $R = KQ/I$ be a graded quiver algebra and let M be a finitely generated R-module generated in degree zero. The following statements are equivalent:*

1. *M is quasi-Koszul.*
2. *M is weakly Koszul.*
3. *M is linear.*

Proof. We showed earlier that every linear module is weakly Koszul so we only need prove that under our assumptions, if M is quasi-Kosul then it must be linear. But M being quasi-Koszul implies that we have $J^2 P_0 \cap \Omega M = J\Omega M$ where P_0 is a projective cover of M and we also have an exact sequence

$$0 \to \Omega M \to JP_0 \to JM \to 0$$

From the first lemma of this lecture we deduce that ΩM is generated in degree one. Induction takes care of the rest. □

We have the following properties of weakly Koszul modules:

Lemma 2.5. *Let $R = KQ/I$ be a semiperfect or a a graded quiver algebra and let $0 \to A \to B \to C \to 0$ be a short exact sequence such that $J^tB \cap A = J^tA$ for all $t \geq 0$. Then:*

(1) *If A and C are weakly Koszul then so is B.*
(2) *If A and B are weakly Koszul, then so is C.*

Proof. The proof of the lemma is an exercise in diagram chasing. First, we use the fact that $J^tB \cap A = J^tA$ to infer that we have the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J^tA \arrow[r] \arrow[d] & J^tB \arrow[r] \arrow[d] & J^tC \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/J^tA \arrow[r] \arrow[d] & B/J^tB \arrow[r] \arrow[d] & C/J^tC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In particular, the sequence $0 \to A/JA \to B/JB \to C/JC \to 0$ is exact and the projective cover P_0^B of B is the direct sum of the projective covers P_0^A of A and P_0^C of C, hence we have an induced commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega A \arrow[r] \arrow[d] & \Omega B \arrow[r] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JP_0^A \arrow[r] \arrow[d] & JP_0^B \arrow[r] \arrow[d] & JP_0^C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JA \arrow[r] \arrow[d] & JB \arrow[r] \arrow[d] & JC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It is now easy to see that by taking radicals we have for each $t \geq 0$ the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega A \cap J^{t+1}P_0^A \arrow[r] \arrow[d] & \Omega B \cap J^{t+1}P_0^B \arrow[r] \arrow[d] & \Omega C \cap J^{t+1}P_0^C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^{t+1}P_0^A \arrow[r] \arrow[d] & J^{t+1}P_0^B \arrow[r] \arrow[d] & J^{t+1}P_0^C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^{t+1}A \arrow[r] \arrow[d] & J^{t+1}B \arrow[r] \arrow[d] & J^{t+1}C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and also

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J^t\Omega A \arrow[r] \arrow[d] & J^t\Omega B \arrow[r] \arrow[d] & J^t\Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega A \cap J^{t+1}P_0^A \arrow[r] & \Omega B \cap J^{t+1}P_0^B \arrow[r] & \Omega C \cap J^{t+1}P_0^C \arrow[r] & 0
\end{tikzcd}$$

where the top row is a complex acyclic in each term except possibly in $J^t\Omega A$. It is easy however to verify that the equality $J^t\Omega A = \Omega A \cap J^{t+1}P_0^A$ implies that the first row is in fact exact. This further implies that $J^t\Omega B = \Omega B \cap J^{t+1}P_0^B$ iff $J^t\Omega C = \Omega C \cap J^{t+1}P_0^C$. The results follow now from these observations and an easy induction. □

We have the following very useful consequence:

Corollary 2.6. *Let $R = K\mathcal{Q}/I$ be a length graded quiver algebra and let $0 \to A \to B \to C \to 0$ be an exact sequence of linear R-modules. Then there exist for each $t \geq 1$ exact sequences $0 \to \Omega^t A \to \Omega^t B \to \Omega^t C \to 0$ satisfying $J^l\Omega^t A = J^l\Omega^t B \cap \Omega^t A$ for each $l \geq 0$.* □

Using passage to the Yoneda algebra, we have the following characterization of quasi-Koszul modules:

Theorem 2.7. *Let $R = K\mathcal{Q}/I$ be a semiperfect quiver algebra and let M be a finitely generated R-module. Then M is quasi-Koszul if and only if it has a finitely generated minimal projective resolution, and for each $t \geq 1$ we have*

$$Ext_R^t(M, \Bbbk) = Ext_R^1(\Bbbk, \Bbbk) \cdot \ldots \cdot Ext_R^1(\Bbbk, \Bbbk) \cdot Hom_R(M, \Bbbk)$$

Proof. Suppose that the module M is quasi-Koszul. Then the short exact sequence

$$0 \to \Omega M \xrightarrow{\alpha} P_0 \xrightarrow{\pi} M \to 0$$

where P_0 is projective cover of M, induces a short exact sequence

$$0 \to \Omega M \xrightarrow{\bar{\alpha}} JP_0 \xrightarrow{\bar{\pi}} JM \to 0$$

and therefore we also have the folowing exact sequence of semisimple modules:

$$0 \to \frac{\Omega M}{J\Omega M} \to \frac{JP_0}{J^2P_0} \to \frac{JM}{J^2M} \to 0$$

Applying $Hom_{\mathbb{k}}(-,\mathbb{k})$ to the above sequence, we have that the induced sequence

$$0 \to Hom_{\mathbb{k}}(\frac{JM}{J^2M},\mathbb{k}) \to Hom_{\mathbb{k}}(\frac{JP_0}{J^2P_0},\mathbb{k}) \to Hom_{\mathbb{k}}(\frac{\Omega M}{J\Omega M},\mathbb{k}) \to 0$$

is also exact, and this implies that we have a short exact sequence

$$0 \to Hom_R(JM,\mathbb{k}) \to Hom_R(JP_0,\mathbb{k}) \to Hom_R(\Omega M,\mathbb{k}) \to 0$$

or equivalently every R-homomorphism $g\colon \Omega M \to \mathbb{k}$ extends to JP_0. We now use induction on t, so we will prove first that $Ext^1_R(M,\mathbb{k}) = Ext^1_R(\mathbb{k},\mathbb{k}) \cdot Hom_R(M,\mathbb{k})$. The inclusion $Ext^1_R(M,\mathbb{k}) \supseteq Ext^1_R(\mathbb{k},\mathbb{k}) \cdot Hom_R(M,\mathbb{k})$ holds always, so we must prove the reverse inclusion. Pick a nonzero element $x \in Ext^1_R(M,\mathbb{k})$ and write it as a nonsplit exact sequence

$$x\colon \quad 0 \to \mathbb{k} \xrightarrow{i} E \xrightarrow{p} M \to 0$$

We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "\bar{\alpha}"] \arrow[d, equal] & JP_0 \arrow[r, "\bar{\pi}"] \arrow[d, "\bar{j}"] & JM \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & \Omega M \arrow[r, "\alpha"] \arrow[d, "g"] & P_0 \arrow[r, "\pi"] \arrow[d, "h"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{k} \arrow[r, "i"] & E \arrow[r, "p"] & M \arrow[r] & 0
\end{tikzcd}$$

We have seen that g extends to JP_0 so there exists $s\colon JP_0 \to \mathbb{k}$ such that $s\bar{\alpha} = g$. Then we have $is\bar{\alpha} = ig = h\alpha = h\bar{j}\bar{\alpha}$. Continuing, we obtain that $(is - h\bar{j})\bar{\alpha} = 0$ so there exists a map $t\colon JM \to E$ such that $t\bar{\pi} = -is + h\bar{j}$. It follows that $pt = j$. We have now the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & JM \arrow[r, equal] \arrow[d, "t"] & JM \arrow[d, "j"] & \\
0 \arrow[r] & \Bbbk \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d] & M \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \Bbbk \arrow[r] & Z \arrow[r] \arrow[d] & M/JM \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

and denote by y the bottom exact sequence. Now M/JM decomposes into a direct sum of simple modules $M/JM = \bigoplus_{l=1}^{n} S_i$. Therefore we have

$$y \in Ext_R^1(\bigoplus_{l=1}^{n} S_i, \Bbbk) \subset \bigoplus_{l=1}^{m} Ext_R^1(\Bbbk, \Bbbk)$$

for some positive integer m. We can then write $y = \sum_{i=1}^{m} y_i$ where

$$y_i\colon \quad 0 \to \Bbbk \to Z_i \to \Bbbk \to 0$$

and letting q' be the induced map $q'\colon M \to \bigoplus_{l=1}^{n} S_i \hookrightarrow \bigoplus_{l=1}^{m} \Bbbk$ we have $q' = (q_1, q_2, \dots, q_m)$ whre the maps $q_i\colon M \to \Bbbk$ are defined in the obvious way. Let us consider the pullbacks

$$\begin{tikzcd}
0 \arrow[r] & \Bbbk \arrow[r] \arrow[d, equal] & E_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "q_i"] & 0 \\
0 \arrow[r] & \Bbbk \arrow[r] & Z_i \arrow[r] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

and denote by x_i the top exact sequences. We have $x = \sum x_i = \sum y_i q_i$ which proves the inclusion $Ext_R^1(M, \Bbbk) \subseteq Ext_R^1(\Bbbk, \Bbbk) \cdot Hom_R(M, \Bbbk)$. By induction and dimension shift it follows that $Ext_R^t(M, \Bbbk) = Ext_R^1(\Bbbk, \Bbbk) \cdot \ldots \cdot Ext_R^1(\Bbbk, \Bbbk) \cdot Hom_R(M, \Bbbk)$. For the reverse implication we will go backwards, so assume now that we have the equality $Ext_R^1(M, \Bbbk) = Ext_R^1(\Bbbk, \Bbbk) \cdot Hom_R(M, \Bbbk)$. We want to prove that every map $g\colon \Omega M \to \Bbbk$ extends to JP_0 where again P_0 denotes the projective cover of M. The map g induces the pushout x:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "i"] \arrow[d, "g"] & P_0 \arrow[r, "p"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Bbbk \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

By hypothesis we can write $x = \sum y_i h_i = \sum x_i$ where y_i are exact sequences

$$y_i\colon \quad 0 \to \Bbbk \xrightarrow{u_i} Z_i \xrightarrow{v_i} \Bbbk \to 0$$

and x_i are the pullbacks

$$\begin{tikzcd}
0 \arrow[r] & \Bbbk \arrow[r] \arrow[d, equal] & E_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "h_i"] & 0 \\
0 \arrow[r] & \Bbbk \arrow[r] & Z_i \arrow[r] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

Each x_i induces a map $g_i\colon \Omega M \to \Bbbk$ such that $g = \sum g_i$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "i"] \arrow[d, "g_i"] & P_0 \arrow[r, "p"] \arrow[d, "\bar{g}_i"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Bbbk \arrow[r] & E_i \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

It is enough to prove that each g_i extends to JP_0. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "\bar{\alpha}"] \arrow[d, equal] & JP_0 \arrow[r, "\bar{\pi}"] \arrow[d, "\bar{j}"] & JM \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & \Omega M \arrow[r, "\alpha"] \arrow[d, "g_i"] & P_0 \arrow[r, "\pi"] \arrow[d, "\bar{h}_i"] & M \arrow[r] \arrow[d, "h_i"] & 0 \\
0 \arrow[r] & \Bbbk \arrow[r, "u_i"] & Z_i \arrow[r, "v_i"] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

The composition $h_i j = 0$ yields the existence of a map $s_i\colon JP_0 \to \Bbbk$ such that $u_i s_i = \bar{h}_i \bar{j}$. It is an immediate exercise now to show that $g_i = s_i \bar{\alpha}$ as claimed and that this implies that $J^2 P_0 \cap \Omega M = J\Omega M$. Again an induction proves that M is quasi-Koszul □

We return now to the situation where our algebra is a Koszul algebra. We have the following characterization:

Theorem 2.8. *Let $R = KQ/I$ be a graded quiver algebra. Then R is a Koszul algebra if and only if the graded semisimple part $\Bbbk = R/J$ is a linear module.* □

We consider the situation where our algebra is a Koszul algebra. Using the usual duality and the fact: $Ext^k_R(M,N) \cong Ext^k_{R^{op}}(D(N), D(M)$ for locally finite graded modules M, N it is easy to prove:

Theorem 2.9. *Let $R = KQ/I$ be a graded quiver algebra. Then R is a Koszul algebra if and only if the opposite algebra R^{op} is also a Koszul algebra .* □

First note that if M is a module over a K-algebra R, then its associated graded module is Gr$M = M/JM \oplus JM/J^2M \oplus \dots$ is a graded module over the associated graded algebra Gr$R = R/J \oplus J/J^2 \oplus \dots$. Assume now that R is a graded quiver algebra so we immediately have that $R \cong GrR$ as graded algebras. If the module M is graded and generated in degree zero, then M is isomorphic to its associated graded module.

¿From the previous lemmas it is easy to prove the main theorem on Koszul duality stated at the end of last lecture.

Theorem 2.10. *If R is a Koszul algebra, then the Yoneda algebra $E(R)$ is Koszul and the functor $F\colon Mod_R^{\mathbf{Z}} \to Mod_{E(R)}^{\mathbf{Z}}$ given by*

$$F(M) = \oplus_{t\geq 0} Ext_R^t(M, \Bbbk)$$

induces a "duality" between the categories of linear R-modules and the linear $E(R)^{op}$-modules over the Yoneda algebra $E(R)$; in particular

$$Ext^*_{E(R)}(Ext^*_R(M, \Bbbk), \Bbbk) \cong M$$

for every linear R-module M.

Proof. (sketch)

Assume M is a Koszul $R-$module, since M and M/JM have the same projective cover and M/JM is Koszul, the exact sequence:

$0 \to JM \to M \to M/JM \to 0$ induces an exact sequence:

$$0 \to \Omega(M)\,[1] \to \Omega(M/JM)\,[1] \to JM\,[1] \to 0$$

where,$\Omega(M)\,[1]$ and $\Omega(M/JM)\,[1]$ are Koszul.

It follows by previous lemmas that $JM\,[1]$ is Koszul and for each $k \geq 1$we get exact sequences:

$$0 \to \Omega^k(M) \to \Omega^k(M/JM) \to \Omega^{k-1}(JM) \to 0$$

Hence; by above lemma, exact sequences:

$$0 \to Hom_R(\Omega^{k-1}(JM), \Bbbk) \to Hom_R(\Omega^k(M/JM), \Bbbk) \to Hom_R(\Omega^k(M), \Bbbk) \to 0$$

It follows that the sequences:

$$0 \to Ext_R^{k-1}(JM, \Bbbk) \to Ext_R^k(M/JM, \Bbbk) \to Ext_R^k(M, \Bbbk) \to 0$$

Adding all this sequences we get an exact sequence:

$$0 \to F(JM)\,[-1] \to F(M/JM) \to F(M) \to 0$$

where $F(M/JM)$ is a projective generated in degree zero.

It follows: $\Omega F(M) = F(JM\,[1])\,[-1]$.

Since $JM\,[1]$ is Koszul, it follows by induction $F(M)$ is linear. In particular, $F(R) = \Bbbk$ is a linear $E(R)$-module, hence; $E(R)$ is Koszul.

Define $G(N) = \oplus_{t\geq 0} Ext_{E(R)}^t(N, \Bbbk)$.

Since $\Omega^k(F(M))$ has projective cover $F(J^kM/J^{k+1}M)$,

$$Hom_{E(R)}(\Omega^k(F(M)), \Bbbk) = Hom_{E(R)}(F(J^kM/J^{k+1}M), \Bbbk)$$

$$= Hom_{E(R)}(Hom_R(J^kM/J^{k+1}M, \Bbbk), \Bbbk) \cong J^kM/J^{k+1}M.$$

It follows:

$$GF(M) \cong \underset{k\geq 0}{\oplus} J^kM/J^{k+1}M \cong M$$

In particular:

$$GF(R) \cong R$$

where F(R) is the semisimple part of E(R).

Therefore:

$$GF(R) = E(E(R)) \cong R$$

The functors F, G are quasi inverse in the category of linear modules. □

Using this, one can prove another characterization of weakly Koszul modules:

Proposition 2.11. *Let R be a Koszul algebra and let M be a finitely generated graded R-module. Then M is weakly Koszul if and only if $F(M)$ is a linear $E(R)$-module.* □

Note also that if R is a Koszul algebra and $0 \to A \to B \to C \to 0$ is an exact sequence of linear R-modules, then $0 \to F(C) \to F(B) \to F(A) \to 0$ is an exact sequence of linear $E(R)$-modules. We can use some of the results of this lecture to construct new weakly Koszul modules from existing ones:

Proposition 2.12. *Let R be a Koszul algebra and M is a weakly Koszul module, then every graded shift of M is weakly Koszul, and so are JM and ΩM.* □

We also have the following interpretation of weakly Koszul modules. In this context we have the following:

Proposition 2.13. *Let R be a Koszul algebra and M a finitely generated graded R-module. Then M is a weakly Koszul module if and only if its associated graded module is a linear R-module.* □

Another result related with weakly Koszul modules is the following:

Proposition 2.14. *Let R be a Koszul algebra and let $M = M_0 \oplus M_1 \oplus ... M_n \oplus ...$ a graded weakly Koszul module with $M_0 \neq 0$. Let $K_M = \langle M_0 \rangle$ be the submodule generated by the degree zero part of M. Then the following statements hold:*

1) K_M is linear.

2) For each $k \geq 0$, $J^kM \cap K_M = J^kK_M$.

3) M/K_M is weakly Koszul. □

Not all modules over a Koszul algebra are Koszul, however we have the following approximation:

Lemma 2.15. $[AE]$ *Let R be a Koszul algebra and M a graded $R-$module with minimal projective resolution consisting of finitely generated projective and M of finite projective dimension. Then there exists an integer n such that $M_{\geq n}[n]$ is Koszul.* □

The lemma has the following partial dual:

Proposition 2.16. *Let R be a finite dimensional Koszul algebra with Yoneda algebra $E(R)$ noetherian and let M be a finitely generated $R-$module. Then there exists a non negative integer k such that $\Omega^k(M)$ is weakly Koszul.* □

Definition 2.17. *Given a finite dimensional K-algebra R and a finitely generated $R-$module M,we define de Poincare series of M as:*

$P_R^M(t) = \sum\limits_{m\geq 0} \dim_K Ext_R^m(M, \mathbb{k})t^m.$

We can prove the following:

Theorem 2.18. *Let R be a finite dimensional Koszul algebra with noetherian Yoneda algebra $E(R)$ and let M be a finitely generated R -module. Then the Poincare series $P_R^M(t)$ is rational.*

Proof. The proof consists in reducing to a module $\Omega^n M$ which is weakly Koszul and then to use Wilson's result $[W]$.$[GMRSZ]$ □

We know that every Koszul algebra is quadratic. In certain cases the converse also holds. We end this lecture with two more examples of Koszul algebras.

Proposition 2.19. *Let R be a finite dimensional algebra.*

(1) If R has global dimension 2, then R is a Koszul algebra if and only if it is quadratic.

(2) If R is a monomial algebra, then R is Koszul if and only if it is quadratic $[GZ]$. □

Note that there are examples of quadratic algebras that are not Koszul. For instance, let R be the hereditary K-algebra of Loewy length two

$$R = \begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{bmatrix}$$

and let $D = Hom_K(-, K)$ denote the usual duality. Then the trivial extension algebra $A = R \ltimes D(R)$ is quadratic but not Koszul.

3. SELFINJECTIVE KOSZUL ALGEBRAS

We will apply now the results of the previous two lectures to the study of selfinjective Koszul algebras. Recall first that if $R = K\mathcal{Q}/I$ is a graded quiver algebra, and if M is a finitely presented R-module, then its transpose TrM is a finitely presented R^{op} module and can prove that we have an Auslander-Reiten sequence

$$0 \to D\,TrM \to E \to M \to 0$$

in the category of locally finite graded R-modules. At the beginning of this section we will study an important class of modules over a graded algebra. Let M be an indecomposable finitely presented graded R-module, and assume also that M has a *linear presentation*, that is the graded projective presentation $P_1 \to P_0 \to M \to 0$ of M has the property that P_0 is generated in degree 0, and P_1 is generated in degree 1. Then the transpose TrM is linear; it has a presentation $P_0^* \to P_1^* \to TrM \to 0$ where P_0^* is generated in degree 0 and P_1^* is generated in degree -1. Then the truncation $(D\,TrM)_{\geq 0} = soc^2 D\,TrM$ and we will prove in a minute that $soc^2 D\,TrM$ is indecomposable. In this way we obtain a non-split exact sequence

$$0 \to soc^2 D\,TrM \to F \to M \to 0$$

with $F = E_{\geq 0}$ which is an Auslander-Reiten sequence in the category $\operatorname{mod}_0^{\mathbf{Z}} R$ of finitely generated graded modules generated in degree zero. The proof that $soc^2 D\,TrM$ is indecomposable follows from more general considerations.

Definition 3.1. *Denote by $\mathcal{L}_R$ the full subcategory of $\operatorname{mod}_R^{\mathbf{Z}}$ consisting of those module having a linear presentation.*

The following is a reformulation of the above discussion.

Corollary 3.2. *Let M be an indecomposable nonprojective graded R-module having a linear presentation. Then $TrM[1]$ has a linear presentation.* □

We have the following $[GMRSZ]$:

Proposition 3.3. *There exists an equivalence between $\mathcal{L}_R$ and $\operatorname{mod}_0^{\mathbf{Z}} R/J^2$. In particular a graded R-module M having a linear presentation is indecomposable if and only if M/J^2M is indecomposable.*

Proof. We only sketch a proof of the fact that the functor

$$R/J^2 \otimes_R - \colon \mathcal{L}_R \longrightarrow \operatorname{mod}_0^{\mathbf{Z}} R/J^2$$

is full and faithful.

(1) "Fullness": Let $\bar{f}\colon M/J^2M \to N/J^2N$ be a nonzero morphism and let

$$0 \to K \xrightarrow{j} P \xrightarrow{g} M \to 0$$

be a projective cover of M in $\operatorname{mod}_R^{\mathbf{Z}}$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "j"] & P \arrow[r, "\pi_P"] \arrow[d, "g"] \arrow[dd, dashed, bend right, "h"'] & P/J^2P \arrow[r] \arrow[d, "\bar{g}"] & 0 \\
& & M \arrow[r, "\pi_M"] & M/J^2M \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
& & N \arrow[r, "\pi_N"] & N/J^2N \arrow[r] & 0
\end{tikzcd}$$

It is easy to see that the composition $\bar{f}\bar{g}\pi_P$ lifts to N using the projectivity of P, hence there is a homomorphism $h\colon P \to N$ such that $\pi_N hj = \bar{f}\bar{g}\pi_P j = \bar{f}\pi_M gj = 0$. Therefore hj factors through J^2N. But K is generated in degree 1 and J^2N is generated in degree 2, hence $hj = 0$ and we can use the universal property of the cokernel to get a morphism $f\colon M \to N$ with $fg = h$. From here it is immediate to see that $R/J^2 \otimes f = \bar{f}$.

(2) "Faithfulness": A nonzero graded homomorphism $f\colon M \to N$ is given by a family of maps $\{f_i\}$ where for each i, $f_i\colon M_i \to N_i$. Since M is generated in degree zero, $f_0 \neq 0$ and this implies that the induced map $M/J^2 \to N/J^2$ is nonzero. $\square$

We can show now, as promised that the module $soc^2 D\,TrM$ is indecomposable if M has a linear presentation: indeed, $TrM/J^2_{R^{op}}TrM$ is indecomposable by the preceding arguments and its dual is isomorphic to $soc^2 D\,TrM$.

Let R be now a selfinjective Koszul algebra, and let us assume from now on that R is indecomposable as an algebra. Recall that the Nakayama functor $\nu = DHom_R(-, R)$ is an autoequivalence of $\operatorname{mod} R$ that restricts to an autoequivalence of $\operatorname{mod}^{\mathbf{Z}}_R$ taking projective modules into projective modules. It is also known that if M is a nonprojective indecomposable module over any finite dimensional graded algebra, then there exists an Auslander-Reiten sequence ending at M in the category $\operatorname{mod}^{\mathbf{Z}}_R$ of graded modules. Moreover, if we ignore the grading, this sequence is in fact an Auslander-Reiten sequence in the (ungraded) module category $\operatorname{mod} R$. It turns out that all the predecessors of a weakly Koszul module in the graded Auslander-Reiten quiver of R are weakly Koszul:

Proposition 3.4. *Let R be a selfinjective Koszul algebra with $J^2 \neq 0$ and let M be an indecomposable nonprojective weakly Koszul module. Let*

$$0 \to D\,TrM \to E \to M \to 0$$

be an Auslander-Reiten sequence in $\operatorname{mod}^{\mathbf{Z}}_R$. Then both $D\,TrM$ and E are weakly Koszul. Consequently, the category of weakly Koszul R-modules has left Auslander-Reiten sequences.

Proof. We only sketch the idea of the proof. First one proves that we apply the Nakayama equivalence functor ν to a weakly Koszul module we obtain a weakly Koszul module and we use this fact to infer that if M is weakly Koszul then $D\,TrM = \nu\Omega^2 M$ is also weakly Koszul.. Then one shows that if an indecomposable module is weakly Koszul then its second syzygy is not simple. Finally, we use these facts to show that the Auslander-Reiten sequence Let

$$0 \to D\,TrM \to E \to M \to 0$$

satisfies the conditions of Lemma 2.6. (1) of the previous lecture to conclude that the middle term E is also weakly Koszul. □

We will now give a characterization of the selfinjective Koszul algebra of radical cube zero. We start with the following general results:

Lemma 3.5. *Let $R = K\mathcal{Q}/I$ be a quiver algebra, with radical $J = L/I$ where L is the two-sided ideal of $K\mathcal{Q}$ generated by the arrows of $\mathcal{Q}$ Assume that $J^3 = 0$. Then the ideal I is a length-homogeneous ideal of the path algebra, hence R is a graded algebra with the grading induced by the path lengths.*

Proof. We have $0 = J^3 = L^3 + I/I$ and so $L^3 \subseteq I$, and since I is an admissible ideal we get $L^3 \subseteq I \subseteq L^2$. For each n, let $I_n = (K\mathcal{Q})_n \cap I$. To show that I is a homogeneous ideal of the path algebra K we must show that $I = \sum_n I_n$. Note that for each $n \geq 3$ we have $(K\mathcal{Q})_n \subseteq L^3 \subseteq I$ so $I_n = (K\mathcal{Q})_n$ for all $n \geq 3$. Let $\rho \in I$ and write it as a linear combination of paths γ in : $\rho = \sum_{l(\gamma_i)=2} c_i\gamma_i + \sum_{l(\gamma_j)\geq 3} c_j\gamma_j$ where $l(\gamma)$ denotes the length of the path γ. Therefore we have $\sum_{l(\gamma_i)=2} c_i\gamma_i = \rho - \sum_{l(\gamma_j)\geq 3} c_j\gamma_j \in I$ and so $I = \sum_n I_n$. This proves the homogeneity of I. □

Using the indecomposability of R as an algebra, the following result is not hard to prove.

Lemma 3.6. *Let R be a selfinjective graded quiver algebra. Then all the indecomposable projective R-modules have the same Loewy length.* □

Theorem 3.7. [$M2$] *Let $R = K\mathcal{Q}/I$ be a selfinjective quiver algebra, with radical J where $J^2 \neq 0$ but $J^3 = 0$. Then R is a Koszul algebra if and only if R is of infinite representation type.*

Proof. "$\Rightarrow$" Assume that R is Koszul of finite representation type and let S be a simple R-module. Since R is selfinjective every syzygy of S is indecomposable and since there are only finitely many nonisomorphic indecomposable modules, S must be isomorphic to the graded shift of some syzygy. It cannot be the first syzygy since the Loewy length of R is 3. But then after shifting we may assume that there exists a linear module whose syzygy is S, again contradicting our assumption on the Loewy length. Therefore R is of infinite representation type.

"$\Leftarrow$" For this direction, we will only sketch the proof. We observe first that if M is an indecomposable non projective R-module then M, having Loewy length two is generated in a single degree. Let i be that degree. Then it is easy to see that its first syzygy ΩM is generated in degree $i+1$ or is simple and then it is generated in degree $i+2$. Therefore, in order to prove that R is a Koszul algebra it is enough to show that for every simple modules S and T and for every nonzero integer n, $T \not\cong \Omega^n S$, and then use the preceding argument. To prove then that no simple module can occur as some syzygy of another simple module one uses the fact that every radical square zero algebra is stably equivalent to a hereditary algebra. In our case, the indecomposable non projective R-modules are all R/J^2-modules and R/J^2 is stably equivalent to a hereditary algebra H also of infinite representation type. Then, using this stable equivalence we translate our problem over to this H where we prove the equivalent statement that no simple H-module can occur as some power of $D\,Tr$ of some other simple H-module. □

Using some general considerations about the quiver and relations of trivial extension algebras, the following is an immediate consequence of Theorems 3.7. and 1.7.

Corollary 3.8. *Let $\mathcal{Q}$ be a connected bipartite graph. Then the trivial ex- tension algebra $R = K\mathcal{Q} \ltimes D(K\mathcal{Q})$ is a selfinjective Koszul algebra if and only if the underlying graph $\mathcal{Q}$ is not a Dynkin diagram. In that case, the Koszul dual $E(R) \cong R^!$ of R is the preprojective algebra of $\mathcal{Q}$.* □

We want to look now at the Auslander-Reiten sequences, and also at the graded Auslander-Reiten quiver of a selfinjective Koszul algebra of radical cube zero. Let R be such an algebra. Then as we mentioned above R/J^2 is stably equivalent to a hereditary algebra H that is we have an equivalence of categories [Re]

$$G\colon \underline{\mathrm{mod}}R/J^2 \to \underline{\mathrm{mod}}H$$

We use this equivalence and the structure of the Auslander-Reiten quiver of H to determine the shape of the connected components of the Auslander-Reiten quiver of R. From the observations above one can prove that up to shifts, the graded components of the Auslander-Reiten quiver are of the following types: connecting components that are obtained by taking preprojective and preinjective components over H and pasting together their images in $\mathrm{mod}\,R$, and regular components that are of type $\mathbf{Z}A_\infty$ or tubes.

Over a Koszul algebra we also have the dual notion of *colinear modules.* These are the duals of the linear R^{op}-modules, or equivalently those graded modules having a finitely generated colinear injective resolution. Then the "preinjective" part of the connecting components consists of linear modules and the preprojective part of these components consists of colinear modules. Finally one can show that the regular components consist of modules that are both linear and colinear. Note that this means that over a radical cube zero selfinjective Koszul algebra every indecomposable regular module is up to shift, both linear and colinear.

We have seen that the exterior algebra is Koszul. It is of interest to characterize the selfinjective Koszul algebras, we can do this in terms of their Yoneda algebras.

Theorem 3.9. $[S], [M1]$

Let $R = K\mathcal{Q}/I$ be a finite dimensional indecomposable Koszul algebra with Yoneda algebra $E(R)$. Then R is selfinjective if and only if the following conditions hold:

1) The algebra $E(R)$ has global dimension n.

2) If S is a graded simple then we have:

3)$Ext^k_R(S, R) = 0$ for $0 \leq k < n$.

4) $Ext^n_R(S,R)$ *is a simple module and* $Ext^n_R(S,R)$ *gives a bijection between the simple graded R-modules and the simple graded* $R^{op}-$*modules.* □

We will call Artin Schelter regular algebras to the algebras satisfying conditions 1-4 of the thoerem

Artin-Schelter algebras play an important role in non commutative algebraic geometry. Using our preceding results, it follows that the global dimension two quadratic Artin-Schelter regular algebras are the Koszul dual of selfinjective algebras on infinite representation type with radical cube zero.

We will look now at the Gelfand-Kirillov dimension of these algebras. Therefore let $R = K\mathcal{Q}/I$ be a selfinjective algebra of infinite representation type with radical J such that $J^3 = 0$ and $J^2 \neq 0$. We know that R/J^2 is stably equivalent to a hereditary algebra $H = K\bar{\mathcal{Q}}$ and the quiver $\bar{\mathcal{Q}}$ of H, called the *separated quiver* of $\mathcal{Q}$ is obtained in a prescribed way from the original quiver $\mathcal{Q}$. It also follows from our previous discussions that the quiver $\bar{\mathcal{Q}}$ is not a union of Dynkin diagrams. Keeping in mind that the quiver of a Koszul algebra is the opposite quiver of its Koszul dual, we have the following:

Theorem 3.10. $[GMT]$

Let $S = K\mathcal{Q}/L$ *be a quadratic Artin-Schelter regular algebra of global dimension two. Let* M *be an indecomposable linear* S*-module. Then:*

(a) If a connected component of the separated quiver of $\mathcal{Q}$ *is a Euclidean diagram, then* $GKdimM = 1$ *or* $GKdimM = 2$.

(b) If there is a non Euclidean component of the separated quiver of $\mathcal{Q}$*, then* $GKdimM = \infty$. □

We obtain the following corollary:

Corollary 3.11. *Let* R *be a quadratic Artin-Schelter algebra of global dimension two with associated bipartite graph* $\bar{\mathcal{Q}}$*. If there exists a connected component of* $\bar{\mathcal{Q}}$ *of Euclidean type, then* $GKdimR = 2$*, and if there exists a component that is neither Euclidean nor Dynkin then* $GKdimR = \infty$*. In particular* R *is noetherian if and only if it has finite GK-dimension.* □

4. APPLICATIONS

We start recalling the main results we have so far proved:

In the first lecture we saw various definitions concerning graded algebras and modules, the definition of Koszul algebras and Koszul modules, we stated the main theorem on Koszul algebras and we looked to some examples.

We dedicated the second lecture to the study of quasi Koszul, weakly Koszul and Koszul modules and their relations. We also sketched a proof of the main theorem on Koszul algebras.

In the third lecture we initiated the study of selfinjective Koszul algebras, looking in detail to selfinjective algebras of radical cube zero. For such algebras we studied the existence of almost split sequences in the category of Koszul modules and we described the shape of the Auslander Reiten components.

The aim of the last lecture is: to continue the study of selfinjective Koszul algebras, to find the shape of the Auslander Reiten components of graded modules, to study the existence of left almost split sequences $[AR1], [AR2], [ARS]$ in the category of Koszul modules and to describe the shape of the A-R sequences. In the particular case of the exterior algebra, we will apply these results to the study of the category of coherent sheaves on projective space and we will investigate the existence of A-R components in the subcategory of locally free sheaves. We will end the lectures giving some theorems concerning the growth of the ranks of the locally free sheaves. $[MVZ], [MVZ2]$

We will start with a series of propositions which will culminate in a theorem about the graded stable A-R components of a selfinjective Koszul algebra.

Theorem 4.1. *Let $R = KQ/I$ be a finite dimensional indecomposable selfinjective Koszul algebra with noetherian Yoneda algebra and assume $J^3 \neq 0$. Then the following statements hold:*

1. For any indecomposable non projective R-module M there exists an integer n such that $\tau^n M$ is weakly Koszul, where τ denotes the Auslander Reiten translation.

2. If M is an indecomposable weakly Koszul module generated in degrees $m_0 < m_1 < ...m_k$ and

$$0 \to \tau M \to E \to M \to 0$$

is the almost split sequence, then τM is generated in degrees $\ell_0 < \ell_1 < ...\ell_t$ with $\ell_i = -n + 2 + m_{i_j}$,where n is the Loewy length of R and $\{m_{i_j}\}$ is a subset of $\{m_0, m_1, ...m_k\}$ □

Corollary 4.2. *Let R be as in the theorem, M an indecomposable non projective $R-$module with almost split sequence:*

$$0 \to \tau M \stackrel{(g_1,g_2)^t}{\to} E_1 \oplus E_2 \stackrel{(f_1,f_2)}{\to} M \to 0$$

Assume f_1 is an epimorphism. Then g_1 is not an epimorphism. □

Lemma 4.3. *Let $R = KQ/I$ be a finite dimensional indecomposable selfinjective Koszul algebra with noetherian Yoneda algebra and assume $J^3 \neq 0$.Let M is an indecomposable non projective weakly Koszul module generated in degrees $m_0 < m_1 < ...m_k$ and*

$$0 \to \tau M \stackrel{(g_1,g_2)^t}{\to} E_1 \oplus E_2 \stackrel{(f_1,f_2)}{\to} M \to 0$$

the almost split sequence. If f_1 is a monomorphism, then g_1 is not a monomorphism. □

Lemma 4.4. *Let $R = KQ/I$ be a finite dimensional indecomposable selfinjective Koszul algebra with noetherian Yoneda algebra and assume $J^3 \neq 0$. Let M is an indecomposable non projective weakly Koszul module with almost split sequence*

$$0 \to \tau M \overset{(g_1,g_2,..g_k)^t}{\to} E_1 \oplus E_2 \oplus ...E_k \overset{(f_1,f_2,...f_k)}{\to} M \to 0$$

Then exactly one map f_1 is an epimorphism and exactly one map g_1 is a monomorphism. □

Lemma 4.5. *Let $R = KQ/I$ be a finite dimensional indecomposable selfinjective Koszul algebra and assume $J^3 \neq 0$. Let M be an indecomposable non projective weakly Koszul module and $f : E' \to M$ an irreducible epimorphism. Then $Ker\ f$ is not simple.* □

Corollary 4.6. *Let R and M as in the lemma and $f : E' \to M$ an irreducible epimorphism. Then the map: $\tau(f) : \tau(E') \to \tau(M)$ is an irreducible epimorphism.* □

Putting together the previous lemmas we can prove as in $[ABRPS]$ the following:

Theorem 4.7. *Let $R = KQ/I$ be a finite dimensional indecomposable selfinjective Koszul algebra and assume $J^3 \neq 0$. Let C be an Auslander Reiten component containing an indecomposable non projective graded weakly Koszul module M. Let C_M be the cone consisting of the predecessor of M. Then each module in C_M has the property that the middle term of the almost split sequence has at most two terms.* □

From this result we get our first important theorem, which generalizes a result by Ringel $[R1]$:

Theorem 4.8. *Let $R = KQ/I$ be a finite dimensional indecomposable selfinjective Koszul algebra and assume $J^3 \neq 0$. Let C be an Auslander Reiten graded component. Then the stable part of C is of the form ZA_∞.* □

We will see now how this result applies to the coherent sheaves on projective space:

If we consider the exterior algebra R in $n+1$ variables, then the Yoneda algebra $K[x_0, x_1, x_2, ...x_n]$ is noetherian and our results will apply, so all graded Auslander Reiten components of R are of type ZA_∞.

By a theorem of Bernstein-Gelfand-Gelfand [BGG]there exists an equivalence of triangulated categories: $\underline{mod}_R^Z \cong D^b(CohP^n)$, where $D^b(CohP^n)$ denotes the derived category of bounded complexes of coherent sheaves on projective space. As a consequence we have:

Theorem 4.9. *$D^b(CohP^n)$ has Auslander Reiten triangles and the Auslander Reiten quivers are of type ZA_∞.* $\square$

We will recall some well known results on quotient categories:

We will consider graded quiver algebras $R = KQ/I$ and define the torsion part of a graded module M as $t(M) = \sum N_\alpha$ such that N_α is a submodule of M of finite length, for any module M the torsion part satisfies: $t(M/tM)) = 0$. We say M is torsion if $t(M) = M$ and torsion free if $t(M) = 0$. The category $T = \{M \mid M \text{ is torsion}\}$ is a Serre subcategory of Mod_R^Z. In this situation there exists a quotient category $QMod_R^Z$ with the same objects as Mod_R^Z, and $Hom_{QMod_R^Z}(M, N)$ is a direct limit of $Hom_{Mod_R^Z}(M', N/N')$ where the limit is taken over all the pairs (M', N') such that M/M' and N' are torsion. In particular, if X and Y are two finitely generated graded S-modules, it is easy to see that for any integers k, l, we have

$$Hom_{QMod_R^Z}(\pi(Y)[l], \pi(X)[k]) = \bigcup_{t\geq 0} Hom_{Mod_R^Z}(Y_{\geq t}, X[k-l])$$

The category $QMod_R^Z$ is abelian and there is an exact functor: $\pi : Mod_R^Z \to QMod_R^Z$ such that a map $f : M \to N$ goes to an isomorphism under π if and only if $Kerf$ and $Cokerf$ are in T.

We have the following:

Theorem 4.10. *(Serre) [H2] Let S be the polynomial algebra in $n+1$ variables. Then there exists an equivalence of categories $\Phi : Qmod_S^Z \overset{\sim}{\to} coh\mathbf{P^n}$.* $\square$

We have a sequence of functors:

$$mod_R^Z \xrightarrow{F} mod_S^Z \xrightarrow{\pi} Qmod_S^Z \xrightarrow{\Phi} coh(\mathbf{P}^n)$$

Since S is noetherian any finitely generated graded module M has a truncation $M_{\geq k}$ with $M_{\geq k}[k]$ Koszul, but $\pi(M_{\geq k}) = \pi(M)$ implies $Qmod_S^Z$ is a union of categories:

$Qmod_S^Z = \underset{i\in Z}{\cup} \pi(\mathcal{K}_S)[i]$. If we denote by $\widehat{M}$ the sheaf corresponding to $\Phi\pi(M)$, then $coh(\mathbf{P}^n) = \underset{i\in Z}{\cup} \widehat{\mathcal{K}_S}[i]$.

We have seen before that there is an equivalence of categories: $R/J^2 \otimes_R - : \mathcal{L}_R \to mod_{R/J^2}^Z$, where $\mathcal{L}_R$ denotes the category of finitely related graded $R-$modules with linear presentations.

It is clear $\mathcal{K}_R \subseteq \mathcal{L}_R$ and for any indecomposable non projective Koszul module M we have an almost split sequence:

$$0 \to \sigma M \to E \to M \to 0$$

in $\mathcal{K}_R$, where $\sigma M = soc^2\tau M$. The sequence induces an almost split sequence:

$$0 \to \sigma M \to E/J^2E \to M/J^2M \to 0$$

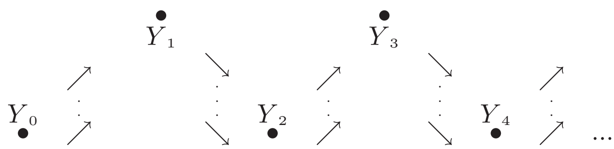
in mod^Z_{R/J^2}.

Assume E decomposes in sum of indecomposables: $E = E_1 \oplus E_2 \oplus ...E_k$. Then E decomposes in sum of indecomposables: $E/J^2E = E_1/J^2E_1 \oplus E_2/J^2E_2 \oplus ...E_k/J^2E_k$ and $E_i \cong E_j$ if and only if $E_i/J^2E_i \cong E_j\ /J^2E_j$.

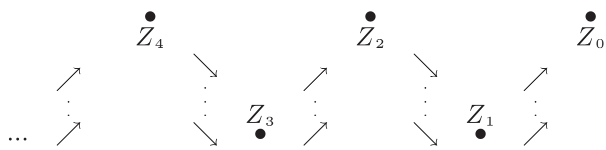
We know [Re], that R/J^2 is stably equivalent to the Kronecker algebra with $n+1$ arrows: $\cdot \overset{\rightrightarrows}{\underset{\rightrightarrows}{\vdots}} \cdot$.

The module M/J^2M belongs either a preprojective or preinjective component in R/J^2 or to a component of type ZA_∞.

The preprojective components are of the form:



The preinjective components are of the form:



Z_1 is the unique indecomposable injective R/J^2-module and $Z_0 = K$.

It is clear that M/J^2M is not in the preprojective component of mod^Z_{R/J^2}, otherwise there exists a Koszul module M with σM simple.

Proposition 4.11. *Let R be the exterior algebra in $n+1$ variables, $n > 1$. Then the Auslander Reiten quiver of $\mathcal{K}_R$ has a connected component thatmardelplata2.pdf coincides with the preinjective component of* $\cdot \overset{\rightrightarrows}{\underset{\rightrightarrows}{\vdots}} \cdot$ *and all other components are subquivers of* ZA_∞. □

Theorem 4.12. *Let S be the polynomial algebra in $n+1$, variables, $n>1$. For any integer i the subcategory $\widehat{\mathcal{K}_S}[i]$ of coh($\mathbf{P}^n$) has left Auslander Reiten sequences and the Auslander Reiten quiver of $\widehat{\mathcal{K}_S}[i]$ has one component that coincides with preprojective component of* $.\begin{matrix}\rightrightarrows\\ \vdots\\ \rightrightarrows\end{matrix}.$ *and all other components are full subquivers of a quiver of type ZA_∞.* □

Corollary 4.13. *coh($\mathbf{P}^1$) has Auslander Reiten sequences.* □

We look for the location of locally free sheaves on the A-R quiver and prove that their ranks are given by Chebysheff polynomials of the second kind.

Observe that if M is a Koszul S module corresponding to a locally free sheaf, so is $M_{\geq k}=J^kM$ and since the sequence:

$$0\to\Omega M\to P\to M\to 0$$

with P projective, splits when we localize at any maximal graded relevant ideal, then ΩM also corresponds with a locally free sheaf. Hence if N is a Koszul R-module such that $\widehat{F(N)}$ is locally free also JN and ΩNcorrespond to locally free sheaves, it follows also σN is locally free. The category of locally free sheaves is closed under extensions, therefore it has right almost split sequences.

Proposition 4.14. *The preprojective component of the A-R quiver of coh($\mathbf{P}^n$) consists entirely of locally free sheaves.*

Proof. The sheaves corresponding to K, and $soc^2R[2]$ are locally free since by applying the Koszul duality, Kcorresponds to the S-module S, and $soc^2R[2]$ to a syzygy of the trivial S-module K.

Since the preprojective component of coh($\mathbf{P}^n$) consists only of the orbits of K and $soc^2R[2]$, the result follows immediately. □

It is possible to compute the ranks of the locally free sheaves in coh($\mathbf{P}^n$) by doing an easy computation in the category of linear R-modules.

Let $0\to F_n\to F_{n-1}...F\to M\to 0$ be a free resolution of a finitely generated S-module M.Then the Euler number is $\chi(M)=\sum\limits_{i=0}^{n}$ (-1)^{i}rank F_i. It follows that the sheafication $\widehat{M}$ is locally free, then rank$\widehat{M}$ $=\chi(M)$.

If $N=N_0\oplus N_1\oplus\ldots\oplus N_p$ is a linear R-module such that $F(N)=M$, then we define the sheaf rank *of* N as the alternating sum $\sum_{i=0}^{n}(-1)^i dimN_i$. It is clear from Koszul duality, that sheaf rank *of* N is equal to the rank of $\widehat{M}$.

If $\mathcal{F}$ is a sheaf in the preprojective component, then we may assume $\mathcal{F}$ is obtained by the sheafication of a Koszul S-module M of projective dimension 1 which in turn corresponds under Koszul duality to a Koszul $R-$module N of Loewy length 2.

It follows rank$\mathcal{F}$= $dimN_0 - dimN_1$.

It is rather easy to compute dimension vectors for the module lying in preinjective component of $\mathcal{K}_R$, since we can reduce to the R/J^2 and using stable equivalence, to the Kronecker algebra and then compute the growth of the dimensions using the Coxeter transformation [R2].

We have the following:

Proposition 4.15. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \dots$ be the locally free sheaves lying in the pre-projective component of some subcategory $\widehat{\mathcal{K}_S}[i]$ of coh($\mathbf{P}^n$). Denote by*

$$T_k(x) = \sum_{m=0}^{[k/2]} (-1)^m \frac{(m-k)!}{m!(k-2m)!}(2x)^{k-2m}$$

the Chebysheff polynomials of the second kind. Then for each $k \geq 1$, we have

$$rank\mathcal{F}_k = T_k(\frac{n+1}{2}) - T_{k-1}(\frac{n+1}{2}).$$

In addition, if $n > 1$, then for each k, $rank\mathcal{F}_{k+1} > rank\mathcal{F}_k$. □

By specializing to the projective plane, we get the following corollary.

Corollary 4.16. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \dots$ be the locally free sheaves lying in the preprojective component of some subcategory $\widehat{\mathcal{K}_S}[i]$ of coh($\mathbf{P}^2$). Then, for each $k \geq 0$, $rank\mathcal{F}_k = f_{2k}$, where $f_0, f_1, f_2 \dots$ is the Fibonacci numbers sequence.* □

We get our main theorem:

Theorem 4.17. *Let $n > 1$ and $\mathcal{F} \in$ coh($\mathbf{P}^n$) is an indecomposable locally free sheaf in some $\widehat{\mathcal{K}_S}[i]$. Then*

1. Every successor of $\mathcal{F}$ in the A-R component $\mathcal{F}$ is locally free.

2. Let $0 \to \mathcal{F} \to \mathcal{B} \to \delta^{-1}\mathcal{F} \to 0$ be the Auslander-Reiten sequence in $\widehat{\mathcal{K}_S}[i]$ starting at $\mathcal{F}$. Then

$$rank\delta^{-2}\mathcal{F} > n\delta^{-1}\mathcal{F}$$

Consequently, the ranks increase exponentially in each Auslander-Reiten quiver. □

Hartshorne [H1] asked about the existence of locally free sheaves of small rank on projective space, in this direction we have:

Theorem 4.18. *Let $n > 1$. For each integer i each Auslander Reiten component of $\widehat{\mathcal{K}_S}[i]$ contains at most one locally free sheaf of rank less than n.* □

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