REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 48, Número 3, 2007, Páginas 53–56

NON-HOMOGENEOUS N-KOSZUL ALGEBRAS

ROLAND BERGER

ABSTRACT. This is a joint work with Victor Ginzburg [4] in which we study a class of associative algebras associated to finite groups acting on a vector space. These algebras are non-homogeneous N-Koszul algebra generalizations of symplectic reflection algebras. We realize the extension of the N-Koszul property to non-homogeneous algebras through a Poincaré-Birkhoff-Witt property.

PART I - HOMOGENEOUS N-KOSZUL ALGEBRAS

I introduced these algebras in [2]. These algebras extend classic Koszul algebras (Priddy, 1970) corresponding to $N = 2$. A natural question is: why higher N's? I list below four answers.

1. There are some relevant examples coming from

- noncommutative projective algebraic geometry: cubic Artin-Schelter regular algebras [1] of global dimension 3, as

$$A = \frac{\mathbb{C}\langle x, y\rangle}{(ay^2x + byxy + axy^2 + cx^3, x \leftrightarrow y)},$$

where the second relation is obtained from the first one by exchanging x and y. The two generators x and y have degree one, and the two relations are cubic. Artin-Schelter regular algebras are noncommutative analogues of polynomial rings which are used to make noncommutative projective algebraic geometry in sense of M. Artin and J. Zhang.

- representation theory: skew-symmetrizer killing algebras (introduced in [2]):

$$A = \frac{\mathbb{C}\langle x_1, \dots, x_n\rangle}{(\sum_\sigma \operatorname{sgn}(\sigma)\ x_{i_{\sigma(1)}} \dots x_{i_{\sigma(p)}})}$$

for $2 \leq p \leq n$. The sum runs over all the permutations of $1, 2, \dots, p$. There are n generators of degree one, and the relations have degree p. The number of relations is the binomial coefficient $\binom{n}{p}$. I will go back to this example in Part III.

- theoretical physics: Yang-Mills algebras (A. Connes and M. Dubois-Violette [7]):

$$A = \frac{\mathbb{C}\langle \nabla_0, \dots, \nabla_s\rangle}{(\sum_{\lambda\mu} g^{\lambda\mu}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]])}$$

where $g^{\lambda\mu}$ are entries of an invertible symmetric real $(s+1)\times(s+1)$ matrix. There are $s+1$ generators of degree one, and $s+1$ cubic relations.

2. Poincaré duality in Hochschild (co)homology (R.B. and N. Marconnet [5]): if A is N-Koszul and AS-Gorenstein, then

$$HH^i(A, M) \cong HH_{d-i}(A, {}_{\varepsilon^{d+1}\phi}M),$$

where d is the global dimension of A, and $\varepsilon^{d+1}\phi$ is a certain automorphism of the algebra A twisting the left action on M.
3. Extension of N-Koszulity to quiver algebras with relations by E. Green, E. Marcos, R. Martínez-Villa, P. Zhang [10].
4. Extension of Koszul duality in terms of A_∞-algebras by J.-W. He and D.-M. Lu [11].

PART II - SYMPLECTIC REFLECTION ALGEBRAS

These algebras were introduced by P. Etingof and V. Ginzburg [8], and play an important role in representation theory and algebraic geometry (desingularization). Let V be a finite dimensional complex vector space which is endowed with a symplectic 2-form ω. Let Γ be a finite subgroup of $\mathrm{Sp}(V)$ and $T(V)\#\Gamma$ be the smash product of the tensor algebra $T(V)$ of V with the group algebra $\mathbb{C}\Gamma$ of Γ. From these data, a Γ-invariant linear map

$$\psi = \sum_{g\in\Gamma} \psi_g \cdot g : \Lambda^2(V) \to \mathbb{C}\Gamma$$

is defined, and the *symplectic reflection algebra* is the $\mathbb{C}\Gamma$-algebra

$$H_\psi = \frac{T(V)\#\Gamma}{(x\otimes y - y\otimes x - \psi(x,y)\,;\ x,\, y \in V)}\ .$$

The algebra H_ψ is *filtered* and there is a natural graded algebra morphism $H_0 = S(V)\#\Gamma \to gr(H_\psi)$.

Theorem (P.E.-V.G. [8]) This morphism is an isomorphism, i.e., the Poincaré-Birkhoff-Witt (PBW) property holds for H_ψ.

Ginzburg and I are able to provide an N-version of this theorem [4]. First we define an N-version of H_ψ with $N = p$ (the notation p is more convenient as far as symplectic reflection algebras are concerned). These generalized H_ψ's are called higher symplectic reflection algebras [4].

PART III - HIGHER SYMPLECTIC REFLECTION ALGEBRAS

Fix p, $2 \leq p \leq \dim V$. We have generalizations

$$\psi = \sum_{g\in\Gamma} \psi_g \cdot g : \Lambda^p(V) \to \mathbb{C}\Gamma,$$

$$H_\psi = \frac{T(V)\#\Gamma}{(\mathrm{Alt}(v_1,\ldots,v_p) - \psi(v_1,\ldots,v_p)\,;\ v_i \in V)}\ ,$$

where $\mathrm{Alt}(v_1, \ldots, v_p) = \sum_\sigma \mathrm{sgn}(\sigma)\ v_{\sigma(1)} \ldots v_{\sigma(p)}$.

Theorem (R.B.-V.G. [4]) The PBW property holds for generalized H_ψ.

The undeformed algebra H_0 is the skew-symmetrizer killing algebra of Part I (up to the change of rings $\mathbb{C} \to \mathbb{C}\Gamma$) which is still p-Koszul for the new ground ring. In order to prove the previous theorem, we state and prove the following.

N**-PBW Theorem** (R.B.-V.G. [4]) Assume that k is a von Neumann regular ring, V is a k-k-bimodule, $N \geq 2$, and P is a sub-k-k-bimodule of F^N, where $F^n = \oplus_{0 \leq i \leq n} V^{\otimes i}$ for any $n \geq 0$. Set $U = T(V)/I(P)$ and $A = T(V)/I(R)$, where $R = \pi(P)$ and π is the projection of F^N onto $V^{\otimes N}$ modulo F^{N-1}.
Assume that A is N-Koszul (this assumption can be weakened). Then the combination of the two conditions

$$P \cap F^{N-1} = 0, \tag{0.1}$$

$$(P \otimes V + V \otimes P) \cap F^N \subseteq P, \tag{0.2}$$

is equivalent to the PBW property for U.

Next, we check conditions (0.1) and (0.2) for generalized H_ψ. Condition (0.1) is easily drawn from the Γ-invariance of ψ, while condition (0.2) (which can be viewed as an N-version of the Jacobi identity) is obtained by a close analysis of a standard Koszul complex.

Comments on the N-PBW Theorem
- For $N = 2$ and k field, this theorem is due to A. Braverman-D. Gaitsgory [6], and A. Polishchuk-L. Positselski [12] (during the 1990's).
- The N-PBW theorem for k field and V finite-dimensional is independently stated and proved by G. Fløystad and J. Vatne [9].

Definitions Let us keep notations and assumptions of the N-PBW theorem. If the PBW property holds for U, one says that U is *Koszul* (R.B.-V.G.), or that U is a *PBW-deformation* of A (G.F.-J.V.).

The first definition extends nicely the definition of homogeneous N-Koszul algebras. A historical argument in favour of this terminology is given by the first Lie theory use by J. L. Koszul of his complex (mentioned in Cartan-Eilenberg's book, p. 281): working in the *filtered* context of the enveloping algebra of a Lie algebra, J. L. Koszul used the classical PBW property as a tool to carry over the exactness of his complex to the standard complex. The second definition is useful when one wants to find all the U's corresponding to a given A.
There are already some various applications of the N-PBW theorem:

1. G. Fløystad and J. Vatne have found [9] all the PBW-deformations of
- any cubic Artin-Schelter regular algebra of global dimension 3,
- any skew-symmetrizer killing algebra for $p < n - 1$ between (note that, for this second point, the intersection of their result and our result is very small since it corresponds to a trivial group Γ).
2. The PBW-deformations of Yang-Mills algebras have been determined by M. Dubois-Violette and R.B. [3].
In these applications, the N-PBW theorem of G.F.-J.V. suffices. However, our general setting for the N-PBW theorem allows us to include significant examples (as higher symplectic reflection algebras) for which the ground field $\mathbb{C}$ is enlarged to group algebras $\mathbb{C}\Gamma$ with non trivial Γ.

References

[1] M. Artin, W. F. Schelter, Graded algebras of global dimension 3, *Adv. Math.* **66** (1987), 171-216.

[2] R. Berger, Koszulity for nonquadratic algebras, *J. Algebra* **239** (2001), 705-734.

[3] R. Berger, M. Dubois-Violette, Inhomogeneous Yang-Mills Algebras, *Lett. Math. Phys.* **76** (2006), 65-75.

[4] R. Berger, V. Ginzburg, Symplectic reflection algebras and non-homogeneous N-Koszul property, *J. Algebra* **304** (2006), 577-601.

[5] R. Berger, N. Marconnet, Koszul and Gorenstein properties for homogeneous algebras, *Alg. and Rep. Theory* **9** (2006), 67-97.

[6] A. Braverman, D. Gaitsgory, Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type, *J. Algebra* **181** (1996), 315-328.

[7] A. Connes, M. Dubois-Violette, Yang-Mills algebra, *Lett. Math. Phys.* **61** (2002), 149-158.

[8] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, *Invent. math.* **147** (2002), 243-348.

[9] G. Fløystad, J.E. Vatne, PBW-deformations of N-Koszul algebras, *J. Algebra* **302** (2006), 116 -155.

[10] E.L. Green, E.N. Marcos, R. Martínez-Villa, P. Zhang, D-Koszul algebras, *J. Pure and Applied Algebra* **193** (2004), 141-162.

[11] J.-W. He, D.-M. Lu, Higher Koszul algebras and A-infinity algebras. *J. Algebra* **293** (2005), no. 2, 335–362.

[12] A. Polishchuk, L. Positselski, Quadratic algebras. University Lecture Series **37**, American Mathematical Society, Providence, RI, 2005.

Roland Berger
LaMUSE,
Faculté des Sciences et Techniques,
23, Rue P. Michelon,
42023 Saint-Etienne Cedex 2, France
Roland.Berger@univ-st-etienne.fr

Recibido: 6 de abril de 2006
Aceptado: 10 de octubre de 2006