REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 48, Número 3, 2007, Páginas 57–71

ON POINTED HOPF ALGEBRAS ASSOCIATED WITH ALTERNATING AND DIHEDRAL GROUPS

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Abstract. We classify finite-dimensional complex pointed Hopf algebras with group of group-like elements isomorphic to $\mathbb{A}_5$. We show that any pointed Hopf algebra with infinitesimal braiding associated with the conjugacy class of $\pi \in \mathbb{A}_n$ is infinite-dimensional if the order of π is odd except for $\pi = (1\,2\,3)$ in $\mathbb{A}_4$. We also study pointed Hopf algebras over the dihedral groups.

Dedicado a María Inés Platzeck en sus # $\mathbb{A}_5$ años.

Introduction

In this article, we continue the work of [AZ, AF] on the classification of finite-dimensional complex pointed Hopf algebras H with $G = G(H)$ non-abelian. We follow the Lifting Method – see [AS2] for a general reference; in particular, we focus on the problem of determining when the dimension of the Nichols algebra associated with conjugacy classes of G is infinite. The paper is organized as follows. In Section 1, we review some general facts on Nichols algebras corresponding to finite groups. We discuss the notion of *absolutely real* element of a finite group in subsection 1.2. We then provide generalizations of [AZ, Lemma 1.3], a basic tool in [AZ, AF], see Lemmata 1.8 and 1.9. Section 2 is devoted to pointed Hopf algebras with coradical $\mathbb{C}\mathbb{A}_n$. We prove that any finite-dimensional complex pointed Hopf algebra H with $G(H) \simeq \mathbb{A}_5$ is isomorphic to the group algebra of $\mathbb{A}_5$; see Theorem 2.6. This is the first finite non-abelian group G such that all pointed Hopf algebras H with $G(H) = G$ are known. We also prove that $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, for any π in $\mathbb{A}_n$ of odd order, except for $\pi = (1\,2\,3)$ or $\pi = (1\,3\,2)$ in $\mathbb{A}_4$ – see Theorem 2.3. This last case is particularly interesting. It corresponds to a "tetrahedron" rack with constant cocycle $\omega \in \mathbb{G}_3 - 1$. The technique in the present paper does not provide information on the corresponding Nichols algebra. We also give partial results on pointed Hopf algebras with groups $\mathbb{A}_4$ and $\mathbb{A}_6$, and on Nichols algebras $\mathfrak{B}(\mathcal{O}_\pi, \rho)$, with $|\pi|$ even. In Section 3, we apply the technique to conjugacy classes in dihedral groups. It turns out that it is possible to decide when the associated Nichols algebra is finite-dimensional in all cases except for $M(\mathcal{O}_x, \text{sgn})$ (if n is odd), or $M(\mathcal{O}_x, \text{sgn} \otimes \text{sgn})$ or $M(\mathcal{O}_x, \text{sgn} \otimes \varepsilon)$ or $M(\mathcal{O}_{xy}, \text{sgn} \otimes \text{sgn})$ or $M(\mathcal{O}_{xy}, \text{sgn} \otimes \varepsilon)$ (if n is even). See below for undefined notations. We finally observe in Section 4 that

2000 *Mathematics Subject Classification.* 16W30; 17B37.

This work was partially supported by Agencia Córdoba Ciencia, ANPCyT-Foncyt, CONICET and Secyt (UNC).

there is no finite-dimensional Hopf algebra with coradical isomorphic to the Hopf algebra $(\mathbb{C}\mathbb{A}_5)^J$ discovered in [Ni], except for $(\mathbb{C}\mathbb{A}_5)^J$ itself.

1. Generalities

For $s \in G$ we denote by G^s the centralizer of s in G. If H is a subgroup of G and $s \in H$ we will denote $\mathcal{O}_s^H$ the conjugacy class of s in H. Sometimes we will write in rack notations $x \triangleright y = xyx^{-1}$, x, $y \in G$. Also, if (V, c) is a braided vector space, that is $c \in GL(V \otimes V)$ is a solution of the braid equation, then $\mathfrak{B}(V)$ denotes its Nichols algebra.

We denote by $\mathbb{G}_n$ the group of n-th roots of 1 in $\mathbb{C}$.

1.1. **Preliminaries.** Let G be a finite group, $\mathcal{O}$ a conjugacy class of G, $s \in \mathcal{O}$ fixed, ρ an irreducible representation of G^s, $M(\mathcal{O}, \rho)$ the corresponding irreducible Yetter-Drinfeld module. Let $t_1 = s, \dots, t_M$ be a numeration of $\mathcal{O}$ and let $g_i \in G$ such that $g_i \triangleright s = t_i$ for all $1 \leq i \leq M$. Then $M(\mathcal{O}, \rho) = \oplus_{1 \leq i \leq M} g_i \otimes V$. Let $g_i v := g_i \otimes v \in M(\mathcal{O}, \rho)$, $1 \leq i \leq M$, $v \in V$. If $v \in V$ and $1 \leq i \leq M$, then the coaction and the action of $g \in G$ are given by

$$\delta(g_i v) = t_i \otimes g_i v, \qquad g \cdot (g_i v) = g_j(\gamma \cdot v),$$

where $gg_i = g_j\gamma$, for some $1 \leq j \leq M$ and $\gamma \in G^s$. The Yetter-Drinfeld module $M(\mathcal{O}, \rho)$ is a braided vector space with braiding given by

$$c(g_i v \otimes g_j w) = t_i \cdot (g_j w) \otimes g_i v = g_h(\gamma \cdot v) \otimes g_i v \tag{1}$$

for any $1 \leq i, j \leq M$, $v, w \in V$, where $t_i g_j = g_h \gamma$ for unique h, $1 \leq h \leq M$ and $\gamma \in G^s$. Since $s \in Z(G^s)$, the Schur Lemma says that

$$s \text{ acts by a scalar } q_{ss} \text{ on } V. \tag{2}$$

Let G be a finite non-abelian group. Let $\mathcal{O}$ be a conjugacy class of G and let ρ be an irreducible representation of the centralizer G^s of a fixed $s \in \mathcal{O}$. Let $M(\mathcal{O}, \rho)$ be the irreducible Yetter-Drinfeld module corresponding to $(\mathcal{O}, \rho)$ and let $\mathfrak{B}(\mathcal{O}, \rho)$ be its Nichols algebra. As explained in [AZ, AF, Gñ], we look for a braided subspace U of $M(\mathcal{O}, \rho)$ of diagonal type such that the dimension of the Nichols algebra $\mathfrak{B}(U)$ is infinite. This implies that the dimension of $\mathfrak{B}(\mathcal{O}, \rho)$ is infinite too.

Lemma 1.1. *If W is a subspace of V such that $c(W \otimes W) = W \otimes W$ and $\dim \mathfrak{B}(W) = \infty$, then $\dim \mathfrak{B}(V) = \infty$.* □

Recall that a braided vector space (V, c) is of *diagonal type* if there exists a basis $v_1, \dots, v_\theta$ of V and non-zero scalars q_{ij}, $1 \leq i, j \leq \theta$, such that $c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$, for all $1 \leq i, j \leq \theta$. A braided vector space (V, c) is of *Cartan type* if it is of diagonal type and there exists $a_{ij} \in \mathbb{Z}$, $-|q_{ii}| < a_{ij} \leq 0$ such that $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ for all $1 \leq i \neq j \leq \theta$; by $|q_{ii}|$ we mean ∞ if q_{ii} is not a root of 1, otherwise it means the order of q_{ii} in the multiplicative group of the units in $\mathbb{C}$. Set $a_{ii} = 2$ for all $1 \leq i \leq \theta$. Then $(a_{ij})_{1 \leq i,j \leq \theta}$ is a generalized Cartan matrix.

Theorem 1.2. *(*[H, Th. 4]*, see also* [AS1, Th. 1.1]*). Let (V,c) be a braided vector space of Cartan type. Then $\dim \mathfrak{B}(V) < \infty$ if and only if the Cartan matrix is of finite type.* □

We say that $s \in G$ is *real* if it is conjugate to s^{-1}; if s is real, then the conjugacy class of s is also said to be *real*. We say that G is *real* if any $s \in G$ is real.

The next application of Theorem 1.2 was given in [AZ]. Let G be a finite group, $s \in G$, $\mathcal{O}$ the conjugacy class of s, $\rho : G^s \to GL(V)$ irreducible; $q_{ss} \in \mathbb{C}^\times$ was defined in (2).

Lemma 1.3. *Assume that s is real. If $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$ then $q_{ss} = -1$ and s has even order.* □

If $s^{-1} \neq s$, this is [AZ, Lemma 2.2]; if $s^2 = \mathrm{id}$ then $q_{ss} = \pm 1$ but $q_{ss} = 1$ is excluded by Lemma 1.1.

The class of real groups includes finite Coxeter groups. Indeed, all the characters of a finite Coxeter group are real valued, see subsection 1.2 below, and [BG] for H_4. Therefore, we have:

Theorem 1.4. *Let G be a finite Coxeter group. If $s \in G$ has odd order, then $\dim \mathfrak{B}(\mathcal{O}_s, \rho) = \infty$, for any $\rho \in \widehat{G^s}$.* □

1.2. **Absolutely real groups.** Let G be a finite group. We say that $s \in G$ is *absolutely real* if there exists an *involution* σ in G such that $\sigma s \sigma = s^{-1}$. If this happens, any element in the conjugacy class of s is absolutely real and we will say that the conjugacy class of s is *absolutely real*. We say that G is absolutely real if any $s \in G$ is so. The finite Coxeter groups are absolutely real. Indeed,

(i) the dihedral groups are absolutely real, by straightforward computations.
(ii) the Weyl groups of semisimple finite dimensional Lie algebras are absolutely real, by [C, Th. C (iii), p. 45].
(iii) H_3 is absolutely real, by Proposition 1.7 below.
(iv) H_4 is absolutely real, we have checked it using GAP3, [S].

Remark 1.5. Let G, H be finite groups. We note:

- $(s,t) \in G \times H$ is absolutely real iff both $s \in G$ and $t \in H$ are absolutely real.
- $G \times H$ is absolutely real iff both G and H are absolutely real.
- Assume H abelian. Then H is absolutely real iff H has exponent 2, i. e. $H \simeq \mathbb{Z}_2^n$ for some integer n.
- If G is absolutely real and H is abelian of exponent 2 then $G \times H$ is absolutely real.

We first discuss when an element of $\mathbb{A}_n$ is absolutely real. Assume that $\pi \in \mathbb{S}_n$ is of type $(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$. Then $\pi \in \mathbb{A}_n$ iff $\sum_{j \text{ even}} m_j$ is even.

Lemma 1.6. *(a). If $m_1 \geq 2$, then π is absolutely real in $\mathbb{A}_n$.*
(b). If $\sum_{h \in \mathbb{N}} (m_{4h} + m_{4h+3})$ is even then π is absolutely real in $\mathbb{A}_n$.

Proof. Let $\tau_j := (1\,2\,\dots\,j)$ for some j and take

$$g_j = \begin{cases} (1\ \ j-1)(2\ \ j-2)\cdots(k-1\ \ k+1), & \text{if } j = 2k \text{ is even,} \\ (1\ \ j-1)(2\ \ j-2)\cdots(k\ \ k+1), & \text{if } j = 2k+1 \text{ is odd.} \end{cases}$$

It is easy to see that $g_j\tau_j g_j = \tau_j^{-1}$, $g_j^2 = \mathrm{id}$ and

$$\mathrm{sgn}(g_j) = \begin{cases} (-1)^{k-1}, & \text{if } j = 2k \text{ is even,} \\ (-1)^k, & \text{if } j = 2k+1 \text{ is odd.} \end{cases}$$

To prove (b), observe that there exists an involution $\sigma \in \mathbb{S}_n$ such that $\sigma\pi\sigma = \pi^{-1}$, which is a product of "translations" of the g_j's. Since the sign of σ is $(-1)^{\sum_{h\in\mathbb{N}}(m_{4h}+m_{4h+3})}$, $\sigma \in \mathbb{A}_n$ iff $\sum_{h\in\mathbb{N}}(m_{4h} + m_{4h+3})$ is even; (b) follows. We prove (a). By assumption there are at least two points fixed by π, say $n-1$, n. By the preceding there exists an involution $\sigma \in \mathbb{S}_{n-2}$ such that $\sigma\pi\sigma = \pi^{-1}$. If $\sigma \in \mathbb{A}_{n-2} \subset \mathbb{A}_n$ we are done, otherwise take $\widetilde{\sigma} = \sigma(n-1\,n) \in \mathbb{A}_n$; $\widetilde{\sigma}$ is an involution and $\widetilde{\sigma}\pi\widetilde{\sigma} = \pi^{-1}$. □

Proposition 1.7. *The groups $\mathbb{A}_5$ and H_3 are absolutely real.*

Proof. The type of $\pi \in \mathbb{A}_5$ is either (1^5), $(1^2, 3^1)$, $(1, 2^2)$ or (5^1); in the first two cases π is absolutely real by Lemma 1.6 part (a), in the last two by part (b). Since $H_3 \simeq \mathbb{A}_5 \times \mathbb{Z}_2$ (see [Hu, Section 2.13]), then the Coxeter group is absolutely real by Remark 1.5. □

1.3. **Generalizations of Lemma 1.3.** The next two Lemmata are variations of [AZ, Lemma 2.2]. A result in the same spirit appears in [FGV]. We deal with elements s having a power in $\mathcal{O}$, the conjugacy class of s. Clearly, if $s^j = \sigma s\sigma^{-1}$ is in $\mathcal{O}$, then $s^{j^l} = \sigma^l s\sigma^{-l}$ is in $\mathcal{O}$, for every l. So, $s^{j^{|\sigma|}} = s$; this implies that $|s|$ divides $j^{|\sigma|} - 1$. Hence

$$N \quad \text{divides} \quad j^{|\sigma|} - 1, \tag{3}$$

with $N := |q_{ss}|$, recall (2).

Lemma 1.8. *Let G be a finite group, $s \in G$, $\mathcal{O}$ the conjugacy class of s and $\rho \in \widehat{G^s}$. Assume that there exists an integer j such that s, s^j and s^{j^2} are distinct elements and s^j is in $\mathcal{O}$. If $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$, then s has even order and $q_{ss} = -1$.*

Proof. We assume that $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$, thus $N > 1$. It is easy to see that

$$\sigma^{-h} s^{j^l} \sigma^h = (\sigma^{-h} s\sigma^h)^{j^l} = (s^{j^{|\sigma|-h}})^{j^l} = s^{j^{|\sigma|-h+l}}, \tag{4}$$

for every l, h. We will call $t_l := s^{j^l}$, $g_l := \sigma^l$, $l = 0, 1, 2$; so $t_l = g_l s g_l^{-1}$, for $l = 0, 1, 2$. The other relations between t_l's and g_h's are obtained from (4). For $v \in V - 0$ and $l = 1$ or 2, we define $W_l := \mathbb{C} - \text{span of } \{g_0 v, g_l v\}$. Hence, W_l is a braided vector subspace of $M(\mathcal{O}, \rho)$ of Cartan type with

$$\mathcal{Q}_l = \begin{pmatrix} q_{ss} & q_{ss}^{j^{|\sigma|-l}} \\ q_{ss}^{j^l} & q_{ss} \end{pmatrix}, \qquad \mathcal{A}_l = \begin{pmatrix} 2 & a_{12}(l) \\ a_{21}(l) & 2 \end{pmatrix},$$

where $a_{12}(l) = a_{21}(l) \equiv j^{|\sigma|-l} + j^l \mod (N)$. Since $\dim \mathfrak{B}(\mathcal{O},\rho) < \infty$, we have that $a_{12}(l) = a_{21}(l) = 0$ or -1. We consider now two cases.

(i) Let us suppose that $a_{12}(1) = a_{21}(1) = 0$. This implies that $j^{|\sigma|-1} + j \equiv 0 \mod (N)$. Since N divides $j^{|\sigma|} - 1$, we have that N divides $j^2 + 1$. We consider now two possibilities.

- Assume that $a_{12}(2) = a_{21}(2) = 0$. Then $j^{|\sigma|-2} + j^2 \equiv 0 \mod (N)$. Since N divides $j^{|\sigma|} - 1$, we have that N divides $j^4 + 1$. So, $-1 \equiv 1 \mod (N)$; hence the result follows.
- Assume that $a_{12}(2) = a_{21}(2) = -1$. Then $j^{|\sigma|-2} + j^2 \equiv -1 \mod (N)$. We can see that N divides $j^4 + j^2 + 1$. This implies that N divides 1, a contradiction.

(ii) Let us suppose that $a_{12}(1) = a_{21}(1) = -1$. This implies that $j^{|\sigma|-1} + j \equiv -1 \mod (N)$. Since N divides $j^{|\sigma|} - 1$, we have that N divides $j^2 + j + 1$. We consider now two possibilities.

- Assume that $a_{12}(2) = a_{21}(2) = 0$. Then $j^{|\sigma|-2} + j^2 \equiv 0 \mod (N)$. So, N divides $j^4 + 1$. It is easy to see that N divides j^2. Since j and $|s|$ are relatively prime, N must be 1, a contradiction.
- Assume that $a_{12}(2) = a_{21}(2) = -1$. This means that the subspace $\widetilde{W} := \mathbb{C}-$ span of $\{g_0v, g_1v, g_2v\}$ of $M(\mathcal{O},\rho)$ is of Cartan type with

$$\mathcal{Q} = \begin{pmatrix} q_{ss} & q_{ss}^{j^{|\sigma|-1}} & q_{ss}^{j^{|\sigma|-2}} \\ q_{ss}^{j} & q_{ss} & q_{ss}^{j^{|\sigma|-1}} \\ q_{ss}^{j^2} & q_{ss}^{j} & q_{ss} \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

By Theorem 1.2, we have that $\dim \mathfrak{B}(\mathcal{O},\rho) = \infty$, a contradiction.

This concludes the proof. □

Lemma 1.9. *Let G be a finite group, $s \in G$, $\mathcal{O}$ the conjugacy class of s and $\rho = (\rho, V) \in \widehat{G^s}$ such that $\dim \mathfrak{B}(\mathcal{O},\rho) < \infty$. Assume that there exists an integer j such that $s^j \neq s$ and s^j is in $\mathcal{O}$.*

(a) *If $\deg \rho > 1$, then s has even order and $q_{ss} = -1$.*
(b) *If $\deg \rho = 1$, then either s has even order and $q_{ss} = -1$, or $q_{ss} \in \mathbb{G}_3 - 1$.*

Proof. We will proceed and use the notation as in the proof of Lemma 1.8. If $s^{j^2} \neq s$, then the result follows by Lemma 1.8. Assume that $s^{j^2} = s$. This implies that $|s|$ divides $j^2 - 1$, so N divides $j^2 - 1$.

(a) Let v_1 and v_2 in V linearly independent and let $W = \mathbb{C}-$ span of $\{g_0v_1, g_0v_2, g_1v_1, g_1v_2\}$, with $g_0 := \mathrm{id}$ and $g_1 := \sigma$. Thus W is a braided vector subspace of $M(\mathcal{O},\rho)$ of Cartan type with

$$\mathcal{Q} = \begin{pmatrix} q_{ss} & q_{ss} & q_{ss}^{j^{|\sigma|-1}} & q_{ss}^{j^{|\sigma|-1}} \\ q_{ss} & q_{ss} & q_{ss}^{j^{|\sigma|-1}} & q_{ss}^{j^{|\sigma|-1}} \\ q_{ss}^{j} & q_{ss}^{j} & q_{ss} & q_{ss} \\ q_{ss}^{j} & q_{ss}^{j} & q_{ss} & q_{ss} \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 2 & a_{12} & a_{13} & a_{14} \\ a_{21} & 2 & a_{23} & a_{24} \\ a_{31} & a_{32} & 2 & a_{34} \\ a_{41} & a_{42} & a_{43} & 2 \end{pmatrix},$$

where $a_{ij} = a_{ji}$, $i \neq j$, $a_{12} \equiv 2 \equiv a_{34} \mod (N)$, $a_{13} = a_{14} = a_{23} = a_{24}$ and

$$a_{13} \equiv j^{|\sigma|-1} + j \mod (N).$$

If $a_{12} = 0$ or $a_{34} = 0$, then N divides 2 and the result follows. Besides, if $a_{13} = 0$, then $j^{|\sigma|-1} + j \equiv 0 \mod (N)$; this implies that N divides $j^2 + 1$, so N divides 2 and the result follows. On the other hand, if $a_{ij} = -1$, for all i, j, we have that the matrix $\mathcal{A}$ is not of finite type; hence $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, from Theorem 1.2. This is a contradiction by hypothesis. Therefore, (a) is proved.

(b) For $v \in V - 0$ we define $W := \mathbb{C} -$ span of $\{g_0 v, g_1 v\}$, with $g_0 :=$ id and $g_1 := \sigma$. Hence, W is a braided vector subspace of $M(\mathcal{O}, \rho)$ of Cartan type with

$$\mathcal{Q} = \begin{pmatrix} q_{ss} & q_{ss}^{j^{|\sigma|-1}} \\ q_{ss}^{j} & q_{ss} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 2 & a_{12} \\ a_{21} & 2 \end{pmatrix},$$

where $a_{12} \equiv j^{|\sigma|-1} + j \mod (N)$. Since $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$, we have that $a_{12} = 0$ or -1. We consider now two possibilities.
(i) Assume that $a_{12} = 0$. This implies that $j^{|\sigma|-1} + j \equiv 0 \mod (N)$. Since N divides $j^{|\sigma|} - 1$, we have that N divides $j^2 + 1$. Thus, N divides 2; hence $N = 2$, and the result follows.
(ii) Assume that $a_{12} = -1$. This implies that $j^{|\sigma|-1} + j \equiv -1 \mod (N)$. Since N divides $j^{|\sigma|} - 1$, we have that N divides $j^2 + j + 1$. Hence, N divides $j + 2$. If p is a prime divisor of N, then p divides $j - 1$ or $j + 1$, because N divides $j^2 - 1$. If p divides $j + 1$, then p divides 1, a contradiction. So, N divides $j - 1$. Hence, N divides 3, i.e. $N = 3$ and the result follows. □

2. On Nichols algebras over $\mathbb{A}_n$

We recall that we will denote $\mathcal{O}$ or $\mathcal{O}_\pi$ the conjugacy class of an element π in $\mathbb{A}_n$, and ρ in $\widehat{\mathbb{A}_n^\pi}$, a representative of an isomorphism class of irreducible representations of $\mathbb{A}_n^\pi$. We want to determine pairs $(\mathcal{O}, \rho)$, for which $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, following the strategy given in [AZ, AF]; see also [Gñ].

The following is a helpful criterion to decide when a conjugacy class of an even permutation π in $\mathbb{S}_n$ splits in $\mathbb{A}_n$.

Proposition 2.1. [JL, Proposition 12.17] *Let $\pi \in \mathbb{A}_n$, with $n > 1$.*

(1) *If π commutes with some odd permutation in $\mathbb{S}_n$, then $\mathcal{O}_\pi^{\mathbb{A}_n} = \mathcal{O}_\pi^{\mathbb{S}_n}$ and $[\mathbb{S}_n^\pi : \mathbb{A}_n^\pi] = 2$.*
(2) *If π does not commute with any odd permutation in $\mathbb{S}_n$, then $\mathcal{O}_\pi^{\mathbb{S}_n}$ splits into two conjugacy classes in $\mathbb{A}_n$ of equal size, with representatives π and $(1\,2)\pi(1\,2)$, and $\mathbb{S}_n^\pi = \mathbb{A}_n^\pi$.* □

Remarks 2.2. (i) Notice that if π satisfies (1) of Proposition 2.1, then π is real. The reciprocal is not true, e.g. consider $\tau_5 = (1\,2\,3\,4\,5)$ in $\mathbb{A}_5$.

(ii) One can see that if π in $\mathbb{A}_n$ is of type $(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$, then π satisfies (2) of Proposition 2.1 if and only if $m_1 = 0$ or 1, $m_{2h} = 0$ and $m_{2h+1} \leq 1$, for all $h \geq 1$. Thus, if $\pi \in \mathbb{A}_n$ has even order, then π is real.

We state the main Theorem of the section.

Theorem 2.3. *Let* $\pi \in \mathbb{A}_n$ *and* $\rho \in \widehat{\mathbb{A}_n^\pi}$. *Assume that* π *is neither* $(1\,2\,3)$ *nor* $(1\,3\,2)$ *in* $\mathbb{A}_4$. *If* $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) < \infty$, *then* π *has even order and* $q_{\pi\pi} = -1$.

Proof. If $|\pi|$ is even the result follows by Lemma 1.3 and Remark 2.2 (ii). Let us suppose that $|\pi| \geq 5$ and odd . If π^{-1} is in $\mathcal{O}_\pi$, then the result follows by Lemma 1.3. Assume that $\pi^{-1} \notin \mathcal{O}_\pi$. We consider two cases.

(i) If $\pi^2 \in \mathcal{O}_\pi$, then π^4 is in $\mathcal{O}_\pi$, and $\pi^4 \neq \pi^2$ because $|\pi| \geq 5$. Hence, the result follows from Lemma 1.8.

(ii) Assume that $\pi^2 \notin \mathcal{O}_\pi$. We know that there exist σ and σ' in $\mathbb{S}_n$, necessarily odd permutations, such that $\pi^{-1} = \sigma\pi\sigma^{-1}$ and $\pi^2 = \sigma'\pi\sigma'^{-1}$. Then $\sigma'' = \sigma\sigma' \in \mathbb{A}_n$ and $\pi^{-2} = \sigma''\pi\sigma''^{-1}$; so π^{-2} is in $\mathcal{O}_\pi$. This implies that π^4 is in $\mathcal{O}_\pi$, and $\pi^4 \neq \pi^{-2}$ because $5 \leq |\pi|$ is odd. Now, the result follows from Lemma 1.8.

Finally, let us suppose that $|\pi| = 3$, with type $(1^a, 3^b)$. If $a \geq 2$ or $b \geq 2$, then π is real, by Lemma 1.6 (a) and Remark 2.2, respectively. Hence, the result follows by Lemma 1.3. This concludes the proof. □

2.1. **Case** $\mathbb{A}_3$**.** Obviously, $\mathbb{A}_3 \simeq \mathbb{Z}_3$; thus $\mathbb{A}_3$ is not real. This case was considered in [AS1, Theorem 1.3].

2.2. **Case** $\mathbb{A}_4$**.** It is straightforward to check that $\mathbb{A}_4$ is not real, since $(1\,2\,3)$ is not real in $\mathbb{A}_4$. Let π in $\mathbb{A}_4$; then the type of π may be (1^4), (2^2) or $(1,3)$. If the type of π is (1^4), then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, for any ρ in $\widehat{\mathbb{A}_4}$, by Lemma 1.1. If the type of π is $(1,3)$, then π is not real; moreover we have

$$\mathcal{O}_{(1\ 2\ 3)} = \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\},$$

$$\mathcal{O}_{(1\ 3\ 2)} = \{(1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 3), (2\ 3\ 4)\},$$

and $\mathbb{A}_4^\pi = \langle \pi \rangle \simeq \mathbb{Z}_3$. If $\rho \in \widehat{\mathbb{A}_4^\pi}$ is trivial, then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$; otherwise it is not known.

The following result is a variation of [AZ, Theorem 2.7].

Proposition 2.4. *Let* π *in* $\mathbb{A}_4$ *of type* (2^2). *Then* $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, *for every* ρ *in* $\widehat{\mathbb{A}_4^\pi}$.

Proof. We can assume that $\pi = (1\,2)(3\,4)$. If we call $t_1 := \pi$, $t_2 := (1\,3)(2\,4)$ and $t_3 := (1\,4)(2\,3)$, then $\mathcal{O}_\pi = \{t_1, t_2, t_3\}$ and $\mathbb{A}_4^\pi = \langle t_1 \rangle \times \langle t_2 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. If $g_1 = \text{id}$, $g_2 = (1\,3\,2)$ and $g_3 = (1\,2\,3)$, then $t_j = g_j \pi g_j^{-1}$, $j = 1, 2, 3$, and

$$t_1 g_2 = g_2 t_3, \qquad t_2 g_1 = g_1 t_2, \qquad t_3 g_1 = g_1 t_3,$$
$$t_1 g_3 = g_3 t_2, \qquad t_2 g_3 = g_3 t_3, \qquad t_3 g_2 = g_2 t_2.$$

Let ρ in $\widehat{\mathbb{A}_4^\pi}$ and $M(\mathcal{O}_\pi, \rho) := g_1 v \oplus g_2 v \oplus g_3 v$, where $\langle v \rangle$ is the vector space affording ρ. Thus $M(\mathcal{O}_\pi, \rho)$ is a braided vector space with braiding given by – see (1)– $c(g_j v \otimes g_j v) = g_j t_1 \cdot v \otimes g_j v$ and $c(g_j v \otimes g_1 v) = g_1 t_j \cdot v \otimes g_j v$, $j = 1, 2, 3$ and

$$c(g_1 v \otimes g_2 v) = g_2 t_3 \cdot v \otimes g_1 v, \qquad c(g_1 v \otimes g_3 v) = g_3 t_2 \cdot v \otimes g_1 v,$$
$$c(g_2 v \otimes g_3 v) = g_3 t_3 \cdot v \otimes g_2 v, \qquad c(g_3 v \otimes g_2 v) = g_2 t_2 \cdot v \otimes g_3 v.$$

Clearly, $\dim \mathfrak{B}(\mathcal{O}_\pi, \varepsilon\otimes\varepsilon) = \dim \mathfrak{B}(\mathcal{O}_\pi, \varepsilon\otimes \operatorname{sgn}) = \infty$, by Lemma 1.1. If we consider $\rho = \operatorname{sgn}\otimes\varepsilon$ (resp. $\operatorname{sgn}\otimes\operatorname{sgn}$), then $M(\mathcal{O}_\pi,\rho)$ is of Cartan type with matrix of coefficients $(q_{ij})_{ij}$ given by

$$\mathcal{Q} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}, \qquad (\text{ resp. } \mathcal{Q} = \begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}).$$

In both cases the Cartan matrix is $\mathcal{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Therefore, $\dim \mathfrak{B}(\mathcal{O}_\pi,\rho) = \infty$, by Theorem 1.2. □

2.3. **Case $\mathbb{A}_5$.** Here is the key step in the consideration of this case.

Lemma 2.5. *Let $\pi\in\mathbb{A}_5$. Then $\dim \mathfrak{B}(\mathcal{O}_\pi,\rho) = \infty$, for every ρ in $\widehat{\mathbb{A}_5^\pi}$.*

Proof. Let $\pi\in\mathbb{A}_5$. If the type of π is either (1^5), $(1^2,3)$ or (5), we have that $\dim \mathfrak{B}(\mathcal{O}_\pi,\rho) = \infty$, by Lemma 1.3 and Proposition 1.7. Let us assume that the type of π is (2^2). For $j=1,2,3$, let t_j and g_j be as in the proof of Proposition 2.4. By Proposition 2.1 and straightforward computations, we have that $\mathcal{O}_\pi^{\mathbb{A}_5} = \mathcal{O}_\pi^{\mathbb{S}_5}$ and $\mathbb{A}_5^\pi = \langle t_1\rangle\times\langle t_2\rangle \simeq \mathbb{Z}_2\times\mathbb{Z}_2$. Notice that $t_j\in\mathcal{O}_\pi^{\mathbb{A}_5}$, $j=1,2,3$. Let $\rho\in\widehat{\mathbb{A}_5^\pi}$ and $W := g_1v\oplus g_2v\oplus g_3v$, where $\langle v\rangle$ is the vector space affording ρ; then W is a braided vector subspace of $M(\mathcal{O}_\pi,\rho)$. Therefore, $\dim \mathfrak{B}(\mathcal{O}_\pi,\rho) = \infty$, by the same argument as in the proof of Proposition 2.4. □

As an immediate consequence of Lemma 2.5 we have the following result.

Theorem 2.6. *Any finite-dimensional complex pointed Hopf algebra H with $G(H)\simeq \mathbb{A}_5$ is necessarily isomorphic to the group algebra of $\mathbb{A}_5$.*

Proof. Let H be a complex pointed Hopf algebra with $G(H)\simeq\mathbb{A}_5$. Let $M\in {}^{\mathbb{C}\mathbb{A}_5}_{\mathbb{C}\mathbb{A}_5}\mathcal{YD}$ be the infinitesimal braiding of H -see [AS2]. Assume that $H\neq\mathbb{C}\mathbb{A}_5$; thus $M\neq 0$. Let $N\subset M$ be an irreducible submodule. Then $\dim\mathfrak{B}(N)=\infty$, by Lemma 2.5. Hence, $\dim\mathfrak{B}(M)=\infty$ and $\dim H=\infty$. □

2.4. **Case $\mathbb{A}_6$.** Let π be in $\mathbb{A}_6$. If the type of π is (1^6), $(1^2,2^2)$, $(1^3,3)$, (3^2) or $(1,5)$, then π is absolutely real by Lemma 1.6, and if the type of π is $(2,4)$, then π is real because it has even order – see Remark 2.2 (ii). Hence, $\mathbb{A}_6$ is a real group. We summarize our results in the following statement.

Theorem 2.7. *Let $M(\mathcal{O},\rho)$ be an irreducible Yetter-Drinfeld module over $\mathbb{C}\mathbb{A}_6$, corresponding to a pair $(\mathcal{O},\rho)$. If $\dim\mathfrak{B}(\mathcal{O},\rho)<\infty$, then $\mathcal{O}=\mathcal{O}_\pi$, with $\pi = (1\,2)(3\,4\,5\,6)$, and $\rho=\operatorname{sgn}\in\widehat{\mathbb{Z}_4}$.*

Remark 2.8. In this Theorem we do not claim that the condition is sufficient.

Proof. Let π be in $\mathbb{A}_6$. If the type of π is

- (1^6), then $\dim\mathfrak{B}(\mathcal{O}_\pi,\rho)=\infty$, for any ρ in $\widehat{\mathbb{A}_6^\pi}$, by Lemma 1.1.

- $(1^3, 3)$, (3^2) or $(1, 5)$, then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, for any ρ in $\widehat{\mathbb{A}_6^\pi}$, by Lemma 1.3.

Let us suppose that the type of π is $(1^2, 2^2)$; we can assume that $\pi = (1\,2)(3\,4)$. It is easy to check that

$$\mathbb{A}_6^\pi = \langle a := (3\,4)(5\,6),\, b := (1\,3\,2\,4)(5\,6)\rangle \simeq \mathbb{D}_4.$$

Notice that $\pi = b^2$. It is known that $\widehat{\mathbb{D}_4} = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$, where ρ_j, $j = 1, 2, 3$ and 4, are the following characters

$$\begin{aligned}
&\rho_1(a) = 1, && \rho_2(a) = -1, && \rho_3(a) = 1, && \rho_4(a) = -1,\\
&\rho_1(b) = 1, && \rho_2(b) = 1, && \rho_3(b) = -1, && \rho_4(b) = -1,
\end{aligned}$$

and ρ_5 is the 2-dimensional representation given by

$$\rho_5(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_5(b) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It is clear that $\rho_j(\pi) = 1$, $j = 1, 2, 3$ and 4. Then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, by Lemma 1.1. Let us consider now that $\rho = \rho_5$. We define $t_1 := (1\,2)(3\,4)$, $t_2 := (1\,3)(2\,4)$, $t_3 := (1\,4)(2\,3)$, $g_1 :=$ id, $g_2 := (1\,3\,2)$ and $g_3 := (1\,2\,3)$. It is clear that

$$\rho(t_1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(t_2) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\rho(t_3).$$

If $v_1 := \begin{pmatrix} i \\ 1 \end{pmatrix}$ we have that $\rho(t_1)(v_1) = -v_1$ and $\rho(t_2)(v_1) = v_1 = -\rho(t_3)(v_1)$. We define $W := \mathbb{C}$ – span of $\{g_1v_1, g_2v_1, g_3v_1\}$. Then W is a braiding subspace of $M(\mathcal{O}_\pi, \rho)$ of Cartan type with

$$\mathcal{Q} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Since $\mathcal{A}$ is not of finite type we have that $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, by Theorem 1.2.

Finally, let us assume that the type of π is $(2, 4)$. Then $\mathcal{O}_\pi$ has 90 elements and $\mathbb{A}_6^\pi = \langle \pi \rangle \simeq \mathbb{Z}_4$. We call $\widehat{\mathbb{Z}_4} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$, where $\chi_l(\pi) = i^l$, $l = 0, 1, 2, 3$. It is clear that if $\rho = \chi_l$, with $l = 0$, 1 or 3, then $\rho(\pi) \neq -1$. This implies that $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, by Lemma 1.3. □

Remark 2.9. We can see that every maximal abelian subrack of $\mathcal{O}_{(12)(3456)}$ has two elements. Hence, $M(\mathcal{O}_{(12)(3456)}, \rho)$ is a negative braided space in the sense of [AF].

2.5. **Case $\mathbb{A}_m$, $m \geq 7$.** Let $\pi \in \mathbb{A}_m$, with $|\pi|$ even. We now investigate the Nichols algebras associated with π by reduction to the analogous study for the orbit of π in $\mathbb{S}_m$, [AF]. By Remark 2.2 (ii), $\mathcal{O}_\pi^{\mathbb{A}_m} = \mathcal{O}_\pi^{\mathbb{S}_m}$ and $[\mathbb{S}_m^\pi : \mathbb{A}_m^\pi] = 2$. So, we can determinate the irreducible representations of $\mathbb{A}_m^\pi$ from those of $\mathbb{S}_m^\pi$. We know that if the type of π is $(1^{b_1}, 2^{b_2}, \dots, m^{b_m})$, then $\mathbb{S}_m^\pi = T_1 \cdots T_m$, with $T_i \simeq \mathbb{Z}_i^{b_i} \rtimes \mathbb{S}_{b_i}$, $1 \leq i \leq m$.

Some generalities and notation. Let G be a finite group, H a subgroup of G of index two, and η a representation of G. It is easy to see that

$$\eta'(g) := \begin{cases} \eta(g) & \text{, if } g \in H, \\ -\eta(g) & \text{, if } g \in G \setminus H, \end{cases}$$

defines a new representation of G. Notice that $\operatorname{Res}^G_H \eta = \operatorname{Res}^G_H \eta'$. On the other hand, any representation ρ of H defines a representation $\overline{\rho}$ of H, call it the *conjugate representation of* ρ, given by $\overline{\rho}(h) := \rho(ghg^{-1})$, for every $h \in H$, where g is an arbitrary fixed element in $G \setminus H$. Since g is unique up to multiplication by an element of H, the conjugate representation is unique up to isomorphism.

Let $s \in H$ such that $\mathcal{O}^H_s = \mathcal{O}^G_s$; thus $[G^s : H^s] = 2$. Let η in $\widehat{G^s}$. Then we have two cases:

(i) $\eta \not\simeq \eta'$. If $\rho := \operatorname{Res}^{G^s}_{H^s} \eta$, then $\rho \in \widehat{H^s}$, $\rho \simeq \overline{\rho}$ and $\operatorname{Ind}^{G^s}_{H^s} \rho \simeq \eta \oplus \eta'$.

(ii) $\eta \simeq \eta'$. We have that $\operatorname{Res}^{G^s}_{H^s} \eta \simeq \rho \oplus \overline{\rho}$ and $\operatorname{Ind}^{G^s}_{H^s} \rho \simeq \eta \simeq \operatorname{Ind}^{G^s}_{H^s} \overline{\rho}$.

Moreover, if ρ is an irreducible representation of H^s, then ρ is a restriction of some $\eta \in \widehat{G^s}$ or is a direct summand of $\operatorname{Res}^{G^s}_{H^s} \eta$ as in (ii), see [FH, Ch. 5].

Remark 2.10. If $\eta \in \widehat{G^s}$ and $\rho := \operatorname{Res}^{G^s}_{H^s} \eta$, it is easy to check that

$$M(\mathcal{O}^G_s, \eta) \simeq M(\mathcal{O}^H_s, \rho), \qquad \text{for the case (i),} \tag{5}$$

$$M(\mathcal{O}^G_s, \eta) \simeq M(\mathcal{O}^H_s, \rho) \oplus M(\mathcal{O}^H_s, \overline{\rho}), \qquad \text{for the case (ii),} \tag{6}$$

as braided vector spaces.

We apply these observations to the case $G = \mathbb{S}_m$ and $H = \mathbb{A}_m$. We use some notations given in [AF, Section II.D].

Lemma 2.11. *Assume that the type of* π *is* $((2r)^n)$, *with* $r \geq 1$ *and* n *even. Let* ρ *in* $\widehat{\mathbb{A}^\pi_m}$, *with* $m = 2rn$.

(a) *If* $q_{\pi\pi} \neq -1$, *then* $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$.

(b) *If* $\rho \simeq \overline{\rho}$ *and* $q_{\pi\pi} = -1$, *then*

  (I) *if* $r = 1$, *then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

  (II) *Assume that* $r > 1$. *If* $\deg \rho > 1$, *then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. *Assume that* $\deg \rho = 1$. *If* $\rho = \chi_{r,\dots,r} \otimes \mu$, *with* r *even or odd, or if* $\rho = \chi_{c,\dots,c} \otimes \mu$, *with* r *even and* $c = \frac{r}{2}$ *or* $\frac{3r}{2}$, *where* $\mu = \varepsilon$ *or* sgn, *then the braiding is negative; otherwise,* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. (a) follows by Remark 2.2 (ii) and Lemma 1.3. (b). Since $\rho \simeq \overline{\rho}$, $\rho = \operatorname{Res}^{\mathbb{S}^\pi_m}_{\mathbb{A}^\pi_m}(\eta)$, with $\eta \in \widehat{\mathbb{S}^\pi_m}$, $\eta \not\simeq \eta'$ and $\operatorname{Ind}^{\mathbb{S}^\pi_m}_{\mathbb{A}^\pi_m} \rho \simeq \eta \oplus \eta'$. Notice that $\eta(\pi) = -\operatorname{Id}$ because $\rho(\pi) = -\operatorname{Id}$. Now, as the racks are the same, i.e. $\mathcal{O}^{\mathbb{A}_m}_\pi = \mathcal{O}^{\mathbb{S}_m}_\pi$, we can apply [AF, Theorem 1]. □

Remark 2.12. Keep the notation of the Lemma. If ρ is not isomorphic to its conjugate representation $\overline{\rho}$, then there exists $\eta \in \widehat{\mathbb{S}^\pi_m}$ such that $\operatorname{Res}^{\mathbb{S}^\pi_m}_{\mathbb{A}^\pi_m}(\eta) = \rho \oplus \overline{\rho}$ and $\operatorname{Ind}^{\mathbb{S}^\pi_m}_{\mathbb{A}^\pi_m} \rho \simeq \eta \simeq \eta' \simeq \operatorname{Ind}^{\mathbb{S}^\pi_m}_{\mathbb{A}^\pi_m} \overline{\rho}$. Clearly, $\eta(\pi)$ and $\overline{\rho}(\pi)$ act by scalar -1, and we have that $M(\mathcal{O}_\pi^{\mathbb{S}_m}, \eta) \simeq M(\mathcal{O}_\pi^{\mathbb{A}_m}, \rho) \oplus M(\mathcal{O}_\pi^{\mathbb{A}_m}, \overline{\rho})$ as braided vector spaces. We do not get new information with the techniques available today.

3. On Nichols algebras over $\mathbb{D}_n$

We fix the notation: the dihedral group $\mathbb{D}_n$ of order $2n$ is generated by x and y with defining relations $x^2 = e = y^n$ and $xyx = y^{-1}$. Let ω be a primitive n-th root of 1 and let χ be the character of $\langle y\rangle$, $\chi(y) = \omega$. If $s \in \mathbb{D}_n$ then we denote the conjugacy class by $\mathcal{O}_s^n$ or simply $\mathcal{O}_s$.

Theorem 3.1. *Let $M(\mathcal{O}, \rho)$ be the irreducible Yetter-Drinfeld module over $\mathbb{C}\mathbb{D}_n$ corresponding to a pair $(\mathcal{O}, \rho)$. Assume that its Nichols algebra $\mathfrak{B}(\mathcal{O}, \rho)$ is finite-dimensional.*

(a) *If n is odd, then $(\mathcal{O}, \rho) = (\mathcal{O}_x, \mathrm{sgn})$, where $\mathrm{sgn} \in \widehat{\mathbb{D}_n^x}$, $\mathbb{D}_n^x = \langle x\rangle \simeq \mathbb{Z}_2$.*

(b) *If $n = 2m$ is even, then $(\mathcal{O}, \rho)$ is one of the following:*

   (i) *$(\mathcal{O}_{y^m}, \rho)$ where $\rho \in \widehat{\mathbb{D}_n}$ satisfies $\rho(y^m) = -1$.*

   (ii) *$(\mathcal{O}_{y^h}, \chi^j)$ where $1 \le h \le n-1$, $h \ne m$ and $\omega^{hj} = -1$.*

   (iii) *$(\mathcal{O}_x, \mathrm{sgn} \otimes \mathrm{sgn})$ or $(\mathcal{O}_x, \mathrm{sgn} \otimes \varepsilon)$, where $\mathrm{sgn} \otimes \mathrm{sgn}$, $\mathrm{sgn} \otimes \varepsilon \in \widehat{\mathbb{D}_n^x}$, $\mathbb{D}_n^x = \langle x\rangle \oplus \langle y^m\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.*

   (iv) *$(\mathcal{O}_{xy}, \mathrm{sgn} \otimes \mathrm{sgn})$ or $(\mathcal{O}_{xy}, \mathrm{sgn} \otimes \varepsilon)$, where $\mathrm{sgn} \otimes \mathrm{sgn}$, $\mathrm{sgn} \otimes \varepsilon \in \widehat{\mathbb{D}_n^{xy}}$, $\mathbb{D}_n^{xy} = \langle xy\rangle \oplus \langle y^m\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.*

In the cases (i) and (ii) the dimension is finite. In the cases (iii) and (iv), the braiding is negative in the sense of [AF].

Remark 3.2. There are isomorphisms of braided vector spaces

$$M(\mathcal{O}_x, \mathrm{sgn} \otimes \mathrm{sgn}) \simeq M(\mathcal{O}_{xy}, \mathrm{sgn} \otimes \mathrm{sgn}),$$
$$M(\mathcal{O}_x, \mathrm{sgn} \otimes \varepsilon) \simeq M(\mathcal{O}_{xy}, \mathrm{sgn} \otimes \varepsilon).$$

Remark 3.3. Assume for simplicity that n is odd and that $n = de$, where d, e are integers ≥ 2. Then the (indecomposable) rack $\mathcal{O}_x^n$ is a disjoint union of e racks isomorphic to $\mathcal{O}_x^d$; in other words, $\mathcal{O}_x^n$ is an extension of $\mathcal{O}_x^e$ by $\mathcal{O}_x^d$ (and vice versa), see [AG, Section 2]. Thus, there is an epimorphism of braided vector spaces $M(\mathcal{O}_x^n, \mathrm{sgn}) \to M(\mathcal{O}_x^e, \mathrm{sgn})$, as well as an inclusion $M(\mathcal{O}_x^d, \mathrm{sgn}) \to M(\mathcal{O}_x^n, \mathrm{sgn})$. The techniques available today do not allow to compute the Nichols algebra $\mathfrak{B}(\mathcal{O}_x^n, \mathrm{sgn})$ from the knowledge of the Nichols algebra $\mathfrak{B}(\mathcal{O}_x^e, \mathrm{sgn})$.

Remark 3.4. In Theorem 3.1 we do *not* claim that the conditions are sufficient. See Tables 1, 2. For instance, it is known that $\dim \mathfrak{B}(\mathcal{O}_x^n, \mathrm{sgn}) < \infty$ when $n = 3$ – see [MS]; for other odd n, this is open.

Let us now proceed with the proof of Theorem 3.1.

Proof. If $s = \mathrm{id}$, then $q_{ss} = 1$ and $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, from Lemma 1.1.

We consider now two cases.

CASE 1: n odd.

Orbit	**Isotropy group**	**Rep.**	$\dim \mathfrak{B}(V)$
e	$\mathbb{D}_n$	any	∞
$\mathcal{O}_{y^h} = \{y^{\pm h}\}$, $h \neq 0$, $\mid \mathcal{O}_{y^h} \mid = 2$	$\mathbb{Z}_n \simeq \langle y \rangle$	any	∞
$\mathcal{O}_x = \{xy^h : 0 \leq h \leq n-1\}$, $\mid \mathcal{O}_x \mid = n$	$\mathbb{Z}_2 \simeq \langle x \rangle$	ε	∞
		sgn	negative braiding

TABLE 1. Nichols algebras of irreducible Yetter-Drinfeld modules over $\mathbb{D}_n$, n odd.

(I) If $s = y^h$, with $1 \leq h \leq n$, it is easy to see that $\mathcal{O}_{y^h} = \{y^h, y^{-h}\}$ and $\mathbb{D}_n^{y^h} = \langle y \rangle \simeq \mathbb{Z}_n$. Then $\widehat{\mathbb{Z}_n} = \{\chi_l\}_{l=1}^n$, where $\chi_l(y) = \omega^l$, with $\omega = \exp(\frac{i2\pi}{n}) \in \mathbb{G}_n$ a primitive n-th root of 1. Let us consider $M(\mathcal{O}_{y^h}, \chi_l)$; it is a braided vector space of diagonal type. If $q_{ss} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{y^h}, \chi_l) = \infty$, from Lemma 1.3. Assume $q_{ss} = -1$; so we have $-1 = \chi_l(s) = \chi_l(y^h) = \omega^{lh}$. This is a contradiction because n is odd.

(II) If $s = x$, then $\mathcal{O}_x = \{x, xy, \dots, xy^{n-1}\}$ and $\mathbb{D}_n^x = \langle x \rangle \simeq \mathbb{Z}_2$. Clearly, $\dim \mathfrak{B}(\mathcal{O}_x, \varepsilon) = \infty$. On the other hand, $M(\mathcal{O}_x, \mathrm{sgn})$ is a negative braided vector space, since every abelian subrack of $\mathcal{O}_x$ has one element; indeed $xy^j xy^k = xy^k xy^j$, $0 \leq j, k \leq n-1$, if and only if $j = k$.

Therefore, the part (a) of the Theorem is proved.

CASE 2: n even. Let us say $n = 2m$.

(I) If $s = y^m$, then $\mathcal{O}_{y^m} = \{y^m\}$ and $\mathbb{D}_n^{y^m} = \mathbb{D}_n$. Clearly, $\dim \mathfrak{B}(\mathcal{O}_{y^m}, \rho) = \infty$, for every $\rho \in \widehat{\mathbb{D}_n}$ with $\rho(s) = \mathrm{Id}$. On the other hand, if $(\rho, V) \in \widehat{\mathbb{D}_n}$ is such that $\rho(s) = -\mathrm{Id}$, then it is straightforward to prove that $\mathfrak{B}(\mathcal{O}_{y^m}, \rho) = \bigwedge(V)$, the exterior algebra of V; hence $\dim \mathfrak{B}(\mathcal{O}_{y^m}, \rho) = 2^{\dim V}$.

(II) If $s = y^h$, $h \neq 0, m$; then $\mathcal{O}_{y^h} = \{y^h, y^{-h}\}$ and $\mathbb{D}_n^{y^h} = \langle y \rangle \simeq \mathbb{Z}_n$. From Lemma 1.3, it is clear that $\dim \mathfrak{B}(\mathcal{O}_{y^h}, \chi_l) = \infty$, for every l such that $\chi_l(y^h) \neq -1$, i.e. $\omega^{hl} \neq -1$. On the other hand, it is easy to see that $\mathfrak{B}(\mathcal{O}_{y^h}, \chi_l) = \bigwedge(M(\mathcal{O}_{y^h}, \chi_l))$, hence $\dim \mathfrak{B}(\mathcal{O}_{y^h}, \chi_l) = 4$, for every χ_l with $\chi_l(y^h) = -1$.

(III) If $s = x$, then $\mathcal{O}_x = \{xy^{2h} : 0 \leq h \leq m-1\}$ and $\mathbb{D}_n^x = \langle x \rangle \oplus \langle y^m \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. From Lemma 1.1, $\dim \mathfrak{B}(\mathcal{O}_x, \varepsilon \otimes \varepsilon) = \dim \mathfrak{B}(\mathcal{O}_x, \varepsilon \otimes \mathrm{sgn}) = \infty$.

For the cases $\rho = \mathrm{sgn} \otimes \varepsilon$ or $\mathrm{sgn} \otimes \mathrm{sgn}$, we note the following fact.

(i) If m is odd and $0 \leq j, k \leq m-1$, we have that

$$xy^{2j}xy^{2k} = xy^{2k}xy^{2j} \quad \text{if and only if} \quad j = k.$$

(ii) If m is even and $0 \leq j \leq k \leq m-1$, we have that

$$xy^{2j}xy^{2k} = xy^{2k}xy^{2j} \quad \text{if and only if} \quad k = j \text{ or } j + \frac{m}{2}.$$

Orbit	**Isotropy group**	**Rep.**	$\dim \mathfrak{B}(V)$
e	$\mathbb{D}_n$	any	∞
$\mathcal{O}_{y^m} = \{y^m\}$, $\mid \mathcal{O}_{y^m} \mid = 1$	$\mathbb{D}_n$	$(V,\rho) \in \widehat{\mathbb{D}_n}$, $\rho(y^m) = 1$	∞
		$(V,\rho) \in \widehat{\mathbb{D}_n}$, $\rho(y^m) = -1$	$2^{\dim V}$
$\mathcal{O}_{y^h} = \{y^{\pm h}\}$, $h \neq 0, m$, $\mid \mathcal{O}_{y^h} \mid = 2$	$\mathbb{Z}_n \simeq \langle y \rangle$	χ^j, $\omega^{hj} = -1$	4
		χ^j, $\omega^{hj} \neq -1$	∞
$\mathcal{O}_x = \{xy^{2h} : 0 \leq h \leq m-1\}$ $\mid \mathcal{O}_x \mid = m$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \langle x \rangle \oplus \langle y^m \rangle$	$\varepsilon \otimes \varepsilon$, $\varepsilon \otimes \operatorname{sgn}$	∞
		$\operatorname{sgn} \otimes \operatorname{sgn}$, $\operatorname{sgn} \otimes \varepsilon$	negative braiding
$\mathcal{O}_{xy} = \{xy^{2h+1} : 0 \leq h \leq m-1\}$ $\mid \mathcal{O}_x \mid = m$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \langle xy \rangle \oplus \langle y^m \rangle$	$\varepsilon \otimes \varepsilon$, $\varepsilon \otimes \operatorname{sgn}$	∞
		$\operatorname{sgn} \otimes \operatorname{sgn}$, $\operatorname{sgn} \otimes \varepsilon$	negative braiding

Table 2. Nichols algebras of irreducible Yetter-Drinfeld modules over $\mathbb{D}_n$, $n = 2m$ even.

The cases (i) and (ii) say that every maximal abelian subrack of $\mathcal{O}_x$ has one and two elements, respectively. Hence, in both cases the braiding is negative. Indeed, the result is obvious for the case (i), while in the case (ii) we have that if $t_j := xy^{2j} = xy^j\, x\, (xy^j)^{-1}$ and $t_k := xy^{2k} = xy^k\, x\, (xy^k)^{-1}$ commute in $\mathcal{O}_x$, then $q_{jj} = -1 = q_{kk}$ and $q_{jk}q_{kj} = 1$; thus the braiding is negative.

(IV) If $s = xy$, then $\mathcal{O}_{xy} = \{xy^{2h+1} : 0 \leq h \leq m-1\}$ and $\mathbb{D}_n^{xy} = \langle xy \rangle \oplus \langle y^m \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. The result follows as in (III) using the isomorphism $\mathbb{D}_n \to \mathbb{D}_n$, $x \mapsto xy$, $y \mapsto y$. □

4. On Nichols algebras over semisimple Hopf algebras

Let A be a Hopf algebra. Let $J \in A \otimes A$ be a twist and let A^J be the corresponding twisted Hopf algebra. If A is a Hopf subalgebra of a Hopf algebra H, then J is a twist for H and A^J is a Hopf subalgebra of H^J. Now, if A is semisimple, then this induces a bijection

$$\{\text{isoclasses of Hopf algebras with coradical} \simeq A\} \xrightarrow{\sim} \{\text{isoclasses of Hopf algebras with coradical} \simeq A^J\}, \quad (7)$$

that preserves standard invariants like dimension, Gelfand-Kirillov dimension, etc. Let now $A = \mathbb{C}\mathbb{A}_5$ and let $J \in A \otimes A$ be the non-trivial twist defined in [Ni]. By (7), we conclude immediately from Theorem 2.6.

Theorem 4.1. *Let H be a finite-dimensional Hopf algebra with coradical isomorphic to $(\mathbb{C}\mathbb{A}_5)^J$. Then $H \simeq (\mathbb{C}\mathbb{A}_5)^J$.* □

Again, this is the first classification result we are aware of, for finite-dimensional Hopf algebras with coradical isomorphic to a fixed non-trivial semisimple Hopf algebra. Recently, a semisimple Hopf algebra $B \simeq (\mathbb{C}\,\mathbb{D}_3 \times \mathbb{D}_3)^{J'}$ was discovered in [GN]. This Hopf algebra B is simple, that is it has no non-trivial normal Hopf subalgebra. Since there are finite-dimensional non-semisimple pointed Hopf algebras with group $\mathbb{D}_3 \simeq \mathbb{S}_3$, there is a finite-dimensional non-semisimple Hopf algebra with coradical isomorphic to B.

Acknowledgement. We are grateful to Professor John Stembridge for information on Coxeter groups, in particular reference [BG]. We thank Matías Graña, Sebastián Freyre and Leandro Vendramín for interesting discussions.

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Recibido: 16 de febrero de 2007
Aceptado: 2 de agosto de 2007