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CLASSIFICATION OF SPLIT TTF-TRIPLES IN MODULE CATEGORIES

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Abstract. In our work [9], we complete Jans' classification of TTF-triples [8] by giving a precise description of those two-sided ideals of a ring associated to one-sided split TTF-triples in the corresponding module category.

1. Introduction

Since the 1960s torsion theories have played an important role in algebra. They translate to general abelian categories (and so, significantly, to arbitrary module categories) many features of modules over a PID [3], they have been a fundamental tool for developing a general theory of noncommutative localization [10], they have had a great impact in the representation theory of Artin algebras [6, 5, 1], ...

In the context of module categories over arbitrary rings, one of the important concepts related to torsion theory is that of TTF-triple. This notion was introduced by J. P. Jans [8], who proved that TTF-triples in the category $\mathsf{Mod}\,A$ of modules over an arbitrary ring A are in bijection with idempotent two-sided ideals of the ring A. He also proved that this bijection restricts to a bijection between the socalled *centrally split* TTF-triples and the two-sided ideals generated by a single central idempotent. Then, a natural question arises: which are the idempotent ideals corresponding to the TTF-triples which are not centrally split but only *one-sided split*?

In section 2, we recall the notion of *torsion pair* and its basic properties. In section 3, we recall the notion of *TTF-triple*, its basic properties and Jans' parametrization by means of idempotent two-sided ideals. We also recall some deep results of G. Azumaya relating properties of a TTF-triple and properties of the associated idempotent ideal. These results have been crucial for our classification of one-sided split TTF-triples. In section 4, we give a precise description of those two-sided ideals corresponding to the socalled *left split* TTF-triples according to Jans' parametrization. The analogous description for *right split* TTF-triples is more complicated and it is explained in section 5: firstly for 'good' rings, and finally for arbitrary rings.

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2. Torsion theory: the axiomatic of Dickson

S. E. Dickson introduced *torsion theories* (also called *torsion pairs*) in arbitrary abelian categories [3]. If $\mathcal{A}$ is an abelian category, a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{A}$ is a *torsion pair* if it satisfies

1) $\mathcal{A}(T, F) = 0$ for all T in $\mathcal{T}$ and F in $\mathcal{F}$.
2) If $\mathcal{A}(M, F) = 0$ for all F in $\mathcal{F}$ then M is in $\mathcal{T}$.
3) If $\mathcal{A}(T, M) = 0$ for all T in $\mathcal{T}$ then M is in $\mathcal{F}$.

$\mathcal{T}$ is said to be the *torsion class*, and its objects are the *torsion objects*. Similarly, $\mathcal{F}$ is the *torsionfree class* and its objects are the *torsionfree objects*. A torsion pair $(\mathcal{T}, \mathcal{F})$ can be uniquely determined in different ways. For instance, it is uniquely determined by its torsion class, since $\mathcal{F}$ agrees with the class of objects M such that $\mathcal{A}(T, M) = 0$ for all torsion objects T. Also, torsion pairs in $\mathcal{A}$ are in bijection with the (isomorphism classes of) *idempotent radicals*, *i.e.* subfunctors $\mathsf{t} : \mathcal{A} \to \mathcal{A}$ of the identity functor such that $\mathsf{t}^2(M) = \mathsf{t}(M)$ and $\mathsf{t}(M/\,\mathsf{t}(M)) = 0$ for each object M of $\mathcal{A}$. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, its associated idempotent radical t is uniquely determined by the fact that for each object M of $\mathcal{A}$ the object $\mathsf{t}(M)$ is the largest torsion subobject of M. We say that a torsion pair with idempotent radical t is *split* if $\mathsf{t}(M)$ is a direct summand of M for each object M of $\mathcal{A}$.

3. TTF-triples

3.1. **Jans' classification.** Shortly after the axiomatic of S. E. Dickson appeared, J. P. Jans introduced and studied in [8] what he called *torsion torsionfree(=TTF) theories* (also called *TTF-triples*) in module categories. The definition still make sense for arbitrary abelian categories, and it is as follows. A triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ of classes of objects of an abelian category $\mathcal{A}$ is a *TTF-triple* if both $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are torsion pairs. A TTF-triple is uniquely determined by its central class, which is said to be a *TTF-class* since it is both a torsion class of $(\mathcal{T}, \mathcal{F})$ and a torsionfree class of $(\mathcal{C}, \mathcal{T})$. A TTF-triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is *left split* (resp. *right split*) if $(\mathcal{C}, \mathcal{T})$ (resp. $(\mathcal{T}, \mathcal{F})$) splits, and it is *centrally split* if it is both left and right split.

J. P. Jans proved [8, Corollary 2.2], by using a result of P. Gabriel [4], that TTF-triples in the category $\mathsf{Mod}\,A$ of right modules over an arbitrary ring A are in bijection with the idempotent ideals of that ring A. This is the most remarkable result result concerning TTF-triples, and this is why they are interesting mainly in the framework of module categories.

Theorem. *Let A be a ring. There exists a one-to-one correspondence between:*
1) Idempotent two-sided ideals $I^2 = I$ of A.
2) TTF-triples in $\mathsf{Mod}\,A$.

The bijection is as follows. Given an idempotent ideal I of A, the corresponding TTF-triple is the one whose TTF-class $\mathcal{T}$ is formed by the modules M such that $MI = 0$. Reciprocally, if $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is a TTF-triple and $\mathsf{c} : \mathsf{Mod}\,A \to \mathsf{Mod}\,A$ is the idempotent radical associated to the torsion class $\mathcal{C}$, the corresponding ideal is $\mathsf{c}(A)$ (where A is regarded as a right A-module with its regular structure).

J. P. Jans also studied some elementary properties of centrally split TTF-triples and he essentially proved in [8, Theorem 2.4] the following:

Corollary. *Let A be a ring. The one-to-one correspondence of Theorem 3.1 restricts to a one-to-one correspondence between:*

1) (Ideals of A generated by single) central idempotents of A.
2) Centrally split TTF-triples in $\mathsf{Mod}\,A$.

3.2. **The main question.** Let A be an arbitrary ring, and put $\mathfrak{L}$, $\mathfrak{C}$ and $\mathfrak{R}$ for the sets of left, centrally and right split TTF-triples in $\mathsf{Mod}\,A$, respectively. The existence of TTF-triples for which only one of the constituent torsion pairs split, which we shall call *one-sided split*, has been known for a long time [11] (see also the remark 5.2), and so we should have a diagram of the form:

$$\begin{tikzcd}
& \{\text{TTF-triples}\} \cong \{\text{idempotent ideals}\} & \\
\mathfrak{L} \simeq ? \arrow[ur, hook] & & \mathfrak{R} \simeq ? \arrow[ul, hook'] \\
& \mathfrak{C} \cong \{\text{central idempotents}\} \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

The main question tackled in our work is: What should replace the question marks in the diagram above?

3.3. **The work of Azumaya.** Some efforts have been made to answer this question (cf. [2], [7], ...). Specially useful for us has been the paper of G. Azumaya [2] in which he expresses some deep properties of TTF-triples in module categories in terms of the associated idempotent ideal. We present in the following theorem the results of G. Azumaya we have used [2, Theorem 3, Theorem 6 and Theorem 8].

Theorem. *Let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be a TTF-triple in a module category $\mathsf{Mod}\,A$ and let I be the corresponding idempotent ideal of A. The following properties hold:*

1) $\mathcal{F}$ is a TTF-class if and only if $I = Ae$ for some idempotent e of A.
2) $\mathcal{C}$ is a TTF-class if and only if $I = eA$ for some idempotent e of A.
3) $\mathcal{C}$ is closed under submodules if and only if A/I is a flat right A-module.
4) $\mathcal{C}$ is a TTF-class if and only if $\mathcal{C}$ is closed under submodules and the left A-module A/I has a projective cover.

4. Classification of left split TTF-triples over arbitrary rings

Let A be an arbitrary ring. Recall that an A-module M is *hereditary Σ-injective* if every quotient of a direct sum of copies of M is an injective A-module. Now we present Theorem 3.1 and Corollary 3.2 of [9].

Theorem. *Let A be a ring. The one-to-one correspondence of Theorem 3.1 restricts to a one-to-one correspondence between*

1) *Left split TTF-triples in* $\mathsf{Mod}\,A$.
2) *Two-sided ideals of A of the form $I = eA$ where e is an idempotent of A such that $eA(1-e)$ is hereditary Σ-injective as a right $(1-e)A(1-e)$-module.*

Proof. If the TTF-triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is left split, then $I := \mathsf{c}(A)$ is of the form eA for some idempotent e of A. The difficult part is to prove that $eA(1-e)$ is hereditary Σ-injective as a right $(1-e)A(1-e)$-module.

On the other hand, if e is an idempotent of A such that $I = eA$ is a two-sided ideal, then $(1-e)Ae = 0$. Then A is isomorphic to the *triangular matrix ring*

$$\begin{bmatrix} C & 0 \\ M & B \end{bmatrix} := \begin{bmatrix} (1-e)A(1-e) & 0 \\ eA(1-e) & eAe \end{bmatrix}$$

where the $B-C$-bimodule M is hereditary Σ-injective in $\mathsf{Mod}\,C$. This property of M allows us to prove that NeA is a direct summand of N for every A-module N. $\surd$

5. Classification of right split TTF-triples

The 'dual' of the Theorem 4 is not true in general but only for some classes of rings.

5.1. **Over 'good' rings.** Recall that if B is a ring, then a B-module P is *hereditary projective* (resp. *hereditary Π-projective*) in case every submodule of P (resp. every submodule of a direct product of copies of P) is projective. Recall also that a B-module N is called *FP-injective* if it is injective relative to the class of finitely presented modules, *i.e.* if $\mathsf{Ext}^1_B(?, N)$ vanishes on all the finitely presented B-modules. Now we present Proposition 4.5 of [9].

Proposition. *Let B be a ring and M be a left B-module. The following conditions are equivalent:*

1) *For every bimodule structure ${}_BM_C$ and every right C-module X, the right B-module $\mathsf{Hom}_C(M, X)$ is hereditary projective.*
2) *There exists a bimodule structure ${}_BM_C$ such that for every right C-module X the right B-module $\mathsf{Hom}_C(M, X)$ is hereditary projective.*
3) *Put $S = \mathsf{End}({}_BM)^{op}$. If Q is the minimal injective cogenerator of* $\mathsf{Mod}\,S$, *then the right B-module $\mathsf{Hom}_S(M, Q)$ is hereditary Π-projective.*
4) *The* character module $M^+ := \mathsf{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ *is a hereditary Π-projective right B-module.*

5) $\mathsf{ann}_B(M) = eB$ *for some idempotent* $e \in B$, $\overline{B} := B/\mathsf{ann}_B(M)$ *is a hereditary perfect ring and* M *is FP-injective as a left* $\overline{B}$*-module.*

When B *is an algebra over a commutative ring* k*, the above assertions are equivalent to:*

6) If Q *is a minimal injective cogenerator of* $\mathsf{Mod}\,k$*, then* $D(M) := \mathsf{Hom}_k(M, Q)$ *is a hereditary* Π*-projective right* B*-module.*

A left B-module M satisfying the equivalent conditions of the proposition above is said to *have a hereditary* Π*-projective dual.* Now we can formulate the classification of right split TTF-triples under particularly good circumstances (cf. section 4 of [9]):

Theorem. *Let* A *be a ring. The one-to-one correspondence of Theorem 3.1 restricts to a one-to-one correspondence between:*

1) *Right split TTF-triples in* $\mathsf{Mod}\,A$ *whose associated idempotent ideal* I *is finitely generated on the left.*
2) *Two-sided ideals of* A *of the form* $I = Ae$ *where* e *is an idempotent of* A *such that the left* $(1-e)A(1-e)$*-module* $(1-e)Ae$ *has hereditary* Π*-projective dual.*

In particular, when A *satisfies one of the following two conditions, all the TTF-triples in* $\mathsf{Mod}\,A$ *have the associated idempotent ideal finitely generated on the left:*

i) *A is semiperfect.*
ii) *Every idempotent ideal of* A *which is pure on the left is also finitely generated on the left (*e.g. *if* A *is left Nœtherian).*

Proof. If $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is right split, then $(\mathcal{C}, \mathcal{T})$ is hereditary. By Theorem 3.3, I is pure as a left ideal. Since I is finitely generated on the left, then $I = Ae$ for some idempotent e of A. The difficult part is to prove that the left $(1-e)A(1-e)$-module $(1-e)Ae$ has hereditary Π-projective dual.

On the other hand, let $I = Ae$ be an ideal like in (2). Since $eA(1-e) = 0$, then A is isomorphic to the triangular matrix ring

$$\begin{bmatrix} C & \\ M & B \end{bmatrix} = \begin{bmatrix} (1-e)A(1-e) & 0 \\ (1-e)Ae & eAe \end{bmatrix}.$$

The fact that the left $(1-e)A(1-e)$-module $(1-e)Ae$ has hereditary Π-projective dual allows us to prove that the TTF-triple associated to $I = Ae$ is right split.

We use Theorem 3.3 to prove that if A satisfies either condition (i) or (ii) then all the TTF-triples in $\mathsf{Mod}\,A$ have the associated idempotent ideal finitely generated on the left. $\surd$

5.2. **Over arbitrary rings.** The parametrization of the right split TTF-triples over arbitrary rings is more involved. The first observation is the following (cf. Proposition 5.1. of [9]):

Proposition. *The one-to-one correspondence of Theorem 3.1 restricts to a one-to-one correspondence between:*

1) *Right split TTF-triples in* $\mathsf{Mod}\,A$*.*

2) *Idempotent ideals I of A such that, for some idempotent ε of A, one has that* $\mathsf{lann}_A(I) = (1-\varepsilon)A$ *and the TTF-triple in* $\mathsf{Mod}\,\varepsilon A\varepsilon$ *associated to I is right split.*

In the situation of the proposition above one has that $\mathsf{lann}_{\varepsilon A\varepsilon}(I) = 0$, that is, the TTF-triple $(\mathcal{C}', \mathcal{T}', \mathcal{F}')$ in $\mathsf{Mod}\,\varepsilon A\varepsilon$ associated to I has the property that $\varepsilon A\varepsilon \in \mathcal{F}'$. Therefore the problem of classifying right split TTF-triples reduces to answering the following:

Question. Let I be an idempotent ideal of a ring B such that $\mathsf{lann}_B(I) = 0$, *i.e.* the $B_B \in \mathcal{F}$ where $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is the associated TTF-triple in $\mathsf{Mod}\,B$. Which conditions on I are equivalent to saying that $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is right split?

The elucidation of these conditions leads us to the following 'arithmetic' definition: Given a right B-module M and a submodule N , we shall say that N is *I-saturated* in M when $xI \subseteq N$, with $x \in M$, implies that $x \in N$. Equivalently, this occurs when $\frac{M}{N} \in \mathcal{F}$.

When X is a subset of B, we shall denote by $\mathcal{M}_{n\times n}(X)$ the subset of matrices of $\mathcal{M}_{n\times n}(B)$ with entries in X.

Definition. An idempotent ideal I of a ring B is called *right splitting* if:
1) it is pure as a left ideal,
2) $\mathsf{lann}_B(I) = 0$,
3) it satisfies one of the following two equivalent conditions:
   i) for every integer $n > 0$ and every $\mathcal{M}_{n\times n}(I)$-saturated right ideal $\mathfrak{a}$ of $\mathcal{M}_{n\times n}(B)$, there exists $x \in \mathfrak{a}$ such that $(1_n - x)\mathfrak{a} \subseteq \mathcal{M}_{n\times n}(I)$,
   ii) for every integer $n > 0$ and every I-saturated submodule K of $B^{(n)}$, the quotient $\frac{B^{(n)}}{K+I^{(n)}}$ is projective as a right $\frac{B}{I}$-module.

The fact that conditions (i) and (ii) are equivalent is proved in [9, Lemma 5.2].

Finally, we can formulate the general classification of right split TTF-triples (cf. Theorem 5.4 of [9]):

Theorem. *Let A be an arbitrary ring. The one-to-one correspondence of Theorem 3.1 restricts to a one-to-one correspondence between:*
1) *Right split TTF-triples in* $\mathsf{Mod}\,A$.
2) *Idempotent ideals I such that* $\mathsf{lann}_A(I) = (1-\varepsilon)A$ *for some idempotent ε of A and I is a right splitting ideal of $\varepsilon A\varepsilon$ with $\varepsilon A\varepsilon/I$ a hereditary perfect ring.*

Example. Let H, C be rings, the first one being hereditary perfect, and let ${}_HM_C$ be a bimodule such that ${}_HM$ is faithful. The idempotent ideal

$$I \xrightarrow{\sim} \begin{bmatrix} C & 0 \\ M & 0 \end{bmatrix}$$

of

$$A \xrightarrow{\sim} \begin{bmatrix} C & 0 \\ M & H \end{bmatrix}$$

is clearly pure on the left and $\mathsf{lann}_A(I) = 0$. One can see that I is right splitting if, and only if, ${}_HM$ is FP-injective (equivalently, ${}_HM$ has a hereditary Π-projective dual).

Remark. Let A be a commutative ring. Denote by $\mathfrak{L}$, $\mathfrak{C}$ and $\mathfrak{R}$ the sets of left, centrally and right split TTF-triples in $\mathsf{Mod}\,A$, respectively. Then $\mathfrak{L} = \mathfrak{C} \subset \mathfrak{R}$ and the last inclusion may be strict. Indeed, since all idempotents in A are central, the equality $\mathfrak{L} = \mathfrak{C}$ follows from Theorem 4. On the other hand, if k is a field and A is the ring of all the eventually constant sequences of elements of k, then the set of all the sequences of elements of k with finite support, $I = k^{(\mathbf{N})}$, is an idempotent ideal of A which is pure and satisfies that $\mathsf{ann}_A(I) = 0$. Moreover, one has $A/I \cong k$ and then I is right splitting. Then, by the Theorem 5.2 the TTF-triple in $\mathsf{Mod}\,A$ associated to I is right splitting, but it is not centrally split.

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