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GRÖBNER BASIS IN ALGEBRAS EXTENDED BY LOOPS

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We dedicate this work to the 60th birthday of Maria Ines Platzeck and the 70th birthday of Hector Merklen.

ABSTRACT. In this work we extend, to the path algebras context, some results obtained in the commutative context, [2]. The main result is that one can extend the Gröbner bases of an ungraded ideal to one possible definition of homogenization for the non commutative case.

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We will introduce very briefly the homogenization process in the commutative case, just to explain the main motivation of our work. In the commutative context, the Buchberger Algorithm give us a very direct strategy for computing Gröbner Basis for a given ideal $I \in k[x_1, x_2, \ldots, x_n]$: we consider a finite set $\{f_1, f_2, ...f_k\}$ of generators of I, compute the S polynomials, for any pair i,j, reduce them, and if the remainder is non zero, add this remainder to the list of the given polynomials, to make all the S polynomials reduce to zero.

Although this process always finish, in the commutative case, it can be very inefficient and time consuming, by instance getting S polynomials of much higher degree that the ones we begin with. It is easy to see (see [1]) that if we begin with a set of homogeneos polynomials this problem does not occur and the S polynomials we obtain are again homogeneous.

So, lets define this process for $\Lambda = k[x_1, x_2, \ldots, x_n]$: Let $f \in \Lambda$ and w a new variable. If f has total degree d then the polynomial given by $f^* = w^d f(x_1/w, x_2/w, \ldots, x_n/w) \in k[x_1, x_2, \ldots, x_n, w]$ is a homogeneous polynomial in the extended polynomial algebra, called the homogenization of f. For an ideal $I \in k[x_1, x_2, \ldots, x_n]$ define I^* to be the ideal of $k[x_1, x_2, \ldots, x_n, w]$ given by $I^* =< f^* | f \in I >$. For any $h \in k[x_1, x_2, \ldots, x_n, w]$, define $h_* = h(x_1, x_2, \ldots, x_n, 1) \in k[x_1, x_2, \ldots, x_n]$.

As we will prove, in the last section, if G is a Gröbner basis for I with respect to a certain order, then the set $G^* = \{g^* | g \in G\}$ is a Gröbner Basis for the ideal I^* with respect to the extended order.

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For the non commutative case, this process has been extended in many contexts, and most computer programs devoted to non commutative Grobner Basis work only with homogeneous ideals [4].

2. Preliminaries

In this section, we define some concepts that will be used in the following sections. All these concepts can be found in [3], with a detailed description of the theory of Gröbner basis.

In order to have a Gröebner basis theory in an algebra we need a multiplicative basis with an admissible order. We define, in the sequence, these concepts.

A $\mathcal{K}$-basis is called a *multiplicative basis* of Λ. if for every $b, b' \in \mathcal{B}$ we have $bb' \in \mathcal{B}$ or $bb' = 0$.

We also will need the multiplicative basis to be completely ordered. We stress that we are not interested in an arbitrary order in $\mathcal{B}$, but we want an order that preserves the multiplicative structure of $\mathcal{B}$.

Definition. 2.1. [3] *We will say that a well order in $\mathcal{B}$, is admissible, if it satisfies the following conditions, for every $p, q, r, s \in \mathcal{B}$:*

(i) *If $p < q$ then $pr < qr$, if both are non zero;*
(ii) *If $p < q$ then $sp < sq$, if both are non zero;*
(iii) *If $p = sqr$ then $p \geq q$.*

Let $\mathcal{K}$ be a field and Λ a $\mathcal{K}$-algebra with a fixed $\mathcal{K}$-basis $\mathcal{B} = \{b_i\}_{i\in\mathcal{I}}$.

Since $\mathcal{B}$ is a $\mathcal{K}$-basis of Λ, for each $a \in \Lambda$, there is a unique family $(\lambda_i)_{i\in\mathcal{I}}$ such that $a = \sum_{i\in\mathcal{I}} \lambda_i b_i$, where $\lambda_i = 0$, except for a finite number of indices.

If $a = \sum_{i\in\mathcal{I}} \lambda_i b_i$, we will say that b_i *occurs in* a if $\lambda_i \neq 0$. We define now the notion of tip, which is also called in the literature by leading term.

Definition. 2.2. [3] *If $\mathcal{B} = \{b_i\}_{i\in\mathcal{I}}$ is a $\mathcal{K}$-basis of Λ, as a vector space, well ordered by $>$ in $\mathcal{B}$, and if $a = \sum_{i\in\mathcal{I}} \lambda_i b_i$ is non zero, we will call tip of a and denote by $Tip(a)$ the largest basis element in the support of a and its coefficient λ_i is denoted by $CTip(a)$.*

If X is a subset of Λ, we define

(i) $Tip(X) = \{b \in \mathcal{B} : b = Tip(x) \text{ for some } 0 \neq x \in X\}$
(ii) $NonTip(X) = \mathcal{B} \setminus Tip(X)$

So, both $Tip(X)$ and $NonTip(X)$ are subsets of $\mathcal{B}$ depending on the choice of the well order of $\mathcal{B}$.

Definition. 2.3. [3] *Let I be a two sided ideal of Λ, we will say that a set $\mathcal{G} \subset I$ is a Gröbner basis for I with respect to the order $>$, if*

$$\langle Tip(\mathcal{G})\rangle = \langle Tip(I)\rangle$$

as two-sided ideals.

Definition. 2.4. [3] *Let $b_1, b_2 \in X \subset \Lambda$, we will say that b_1 divides b_2 (in X) if there exist $c, d \in X$ such that $b_2 = b_1 d$, $b_2 = cb_1$ or $b_2 = cb_1 d$.*

We will say that a $\mathcal{K}$-algebra Λ has *Gröbner basis theory* if Λ has a multiplicative basis $\mathcal{B}$ with an admissible order $>$ in this basis.
From this point on, we assume that the $\mathcal{K}$-algebra Λ has a Gröbner basis theory. Moreover I will always denote a two-sided ideal in Λ.

Given a multiplicative basis B of an arbitrary algebra and U and a fixed minimal set of generators of B, as a semigroup, we define $\ell(b)$ the *length* of $b \in \mathcal{B}$ as the smaller $n \in \mathbb{N}$ such that $b = b_1 b_2 \cdots b_n$ with $b_i \in U$. If $f \in \Lambda$ $f \neq 0$ define the *length* of f by $\ell(f) = \max\{\ell(b) : b \in \mathcal{B}$ *occurs in* $f\}$.
We say that an element $f = \sum_{i=1}^{n} \lambda_i b_i$, with $\lambda_i \in \mathcal{K}$ and $b_i \in \mathcal{B}$, is *homogeneous* if $\ell(f) = \ell(b_i)$ for every $1 \leq i \leq n$. An ideal J is *homogeneous* if can be generated by homogeneous elements.

3. Homogenization

In this section, we present our main results, which extend the algorithms used in commutative algebra, and also some results obtained in [2].

Since the polynomial ring on n commutative variables is a special case of a quotient of a path algebra and, as it was proved in [2], any algebra with 1 that admits Gröbner basis theory is isomorphic to a quotient of a path algebra, we asked ourselves if the same process (that is, the homogenization process) can be extended, and which results remain true in the general case of quotient path algebras. In this work, we consider the non commutative version of the homogenization process, for path algebras $\mathcal{KQ}/I$, where I is a two-sided ideal in $\mathcal{KQ}$.

In [2], Green used a similar technic of the extension by loops, to construct Gröbner basis to some indecomposable projectives in Mod-$\mathcal{KQ}$, based on a special admissible order, where the loops where always maximal elements.

In our work, we start with a quotient of a path algebra with an admissible order, and we define another quotient of path algebra, also with an admissible order, which we will call the extended by loops algebra.

Let $\mathcal{K}$ be a field and Q a finite quiver. Let $\Lambda = \mathcal{K}Q/I$ be the path algebra associated to Q and I a two-sided ideal of $\mathcal{K}Q$. Consider in Λ the multiplicative basis $\mathcal{B}$ and $>$ an admissible order in $\mathcal{B}$.

Definition. 3.1. *Let $\Lambda = KQ/I$, as above, we define $\tilde{Q}$, where $\tilde{Q}$ has the same vertices of Q and for each vertex i of Q, we add a loop l_i in $\tilde{Q}_1$, and we consider the $\mathcal{K}$-algebra $\Lambda' = \mathcal{K}\tilde{Q}/\tilde{I}$, where $\tilde{I} = \langle I, \alpha Z - Z\alpha\rangle$ as a two-sided ideal of $\mathcal{K}\tilde{Q}$, with $\alpha \in \mathcal{Q}_1$ and $Z = \sum_{i\in|Q_0|} l_i$. We call Λ' the* extended by loops algebra *of Λ.*

Observe that Z is the sum of all the new loops that were added to the quiver. For Λ' we consider the following basis $\tilde{\mathcal{B}} = \{Z^n b : b \in \mathcal{B} \text{ and } n \geq 0\}$. Both Λ and Λ' are finitely generated as $\mathcal{K}$-algebras, moreover $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are finitely generated as semigroups.
For each generator set U of B, as a semigroup, we associate the following generator for $\tilde{\mathcal{B}}$, $\tilde{U} = U \cup \{l_i : i \in |\mathcal{Q}_0|\}$. It is not hard to see that U is minimal if and only if $\tilde{U}$ is minimal.

Define in $\tilde{\mathcal{B}}$ the order

$$e_i \prec l_j \prec b, \text{ for every } i,j \in |\mathcal{Q}_0| \text{ and } b \in \mathcal{B} \text{ and}$$

$$Z^n b_1 \prec Z^m b_2 \text{ if} : \begin{cases} \text{if } b_1 < b_2 \text{ in } \mathcal{B} \text{ or} \\ \text{if } b_1 = b_2 \text{ in } \mathcal{B} \text{ and } n < m \end{cases}$$

We show now that this is, in fact, an admissible order.
Let $Z^n b_1, Z^m b_2, Z^r b_3, Z^s b_4 \in \tilde{\mathcal{B}}$, then:

(1) if $Z^n b_1 \prec Z^m b_2$ and $Z^n b_1 Z^r b_3$ and $Z^m b_2 Z^r b_3$ are non zero, we have that, if $b_1 < b_2$ then $b_1 b_3 < b_2 b_3$. Now, if $b_1 = b_2$, $n < m$ and so $b_1 b_3 = b_2 b_3$ and $n + r < m + r$. Then, $Z^n b_1 Z^r b_3 \prec Z^m b_2 Z^r b_3$.
(2) in the same way, if $Z^n b_1 \prec Z^m b_2$, then $Z^r b_3 Z^n b_1 \prec Z^r b_3 Z^m b_2$, if the products are non zero.
(3) if $Z^n b_1 = Z^m b_2 Z^r b_3 Z^s b_4 = Z^{m+r+s} b_2 b_3 b_4$, we have that $r \leq n$ and $b_3 \leq b_1$ so $Z^r b_3 \preceq Z^n b_1$.

Therefore, the order $\prec$ given above is an admissible order.

Definition. 3.2. *For $f = \sum_{i=1}^m \lambda_i b_i \in \Lambda$, we define the homogenization of f in Λ' by*
$f^* = \sum_{i=1}^m \lambda_i Z^{\ell(f)-\ell(b_i)} b_i$.

Observe that, for every $f \in \Lambda$, the homogenization of f is an homogeneous element.

Lemma 3.3. *For every $f, g \in \Lambda$, we have $Z^k (fg)^* = f^* g^*$, with $k = \ell(f) + \ell(g) - \ell(fg)$.*

Proof. Let $f = \sum_{i=1}^n \lambda_i b_i$ and $g = \sum_{j=1}^m \beta_j b_j$, with $\lambda_i, \beta_j \in \Lambda$ and $b_i, b_j \in B$. Since $\ell(f) + \ell(g) \geq \ell(fg)$, consider the natural number $k = \ell(f) + \ell(g) - \ell(fg)$.

Then,

$$\begin{aligned}
f^*g^* &= (\sum_{i=1}^{n}\lambda_i b_i)^*(\sum_{j=1}^{m}\beta_j b_j)^* \\
&= (\sum_{i=1}^{n}\lambda_i Z^{\ell(f)-\ell(b_i)}b_i)(\sum_{j=1}^{m}\beta_j Z^{\ell(g)-\ell(b_j)}b_j) \\
&= \sum_{i,j}\lambda_i\beta_j Z^{(\ell(f)+\ell(g))-(\ell(b_i)+\ell(b_j))}b_ib_j \\
&= \sum_{i,j}\lambda_i\beta_j Z^{\ell(fg)-\ell(b_ib_j)}Z^k b_ib_j \\
&= Z^k(\sum_{i,j}\lambda_i\beta_j b_ib_j)^* \\
&= Z^k((\sum_{i=1}^{n}\lambda_i b_i)(\sum_{j=1}^{m}\beta_j b_j))^* \\
&= Z^k(fg)^*
\end{aligned}$$

∎

Now, we define the following application, between the algebras Λ' and Λ:

$$\varphi : \Lambda' \to \Lambda$$

that associates to each element $Z^n b \in \tilde{\mathcal{B}}$ the element $b \in \mathcal{B}$, for every $n \in \mathbb{N}$. Observe that, for every $b \in \mathcal{B}$, there exists $Zb \in \tilde{\mathcal{B}}$ such that $\varphi(Zb) = b$. In this way, we have that φ extended by linearity to every element in Λ' is, in fact, an epimorphism of algebras. Also, observe that $ker(\varphi) = \langle Z - 1\rangle$.

To simplify the notation, we call $g_* = \varphi(g)$ for every $g \in \Lambda'$.

Lemma 3.4. *For every $f \in \Lambda$ we have $(f^*)_* = f$.*

Proof. Let $f = \sum_{i=1}^{n}\lambda_i b_i$, with $\lambda_i \in \Lambda$ and $b_i \in \mathcal{B}$. Observe that

$$
\begin{aligned}
(f^*)_* &= (\sum_{i=1}^{n} \lambda_i Z^{\ell(f)-\ell(b_i)} b_i)_* \\
&= \sum_{i=1}^{n} (\lambda_i Z^{\ell(f)-\ell(b_i)} b_i)_* \\
&= \sum_{i=1}^{n} \lambda_i (Z^{\ell(f)-\ell(b_i)}))_* (b_i)_* \\
&= \sum_{i=1}^{n} \lambda_i b_i \\
&= f
\end{aligned}
$$

∎

Lemma 3.5. *Let $g \in \Lambda'$ homogeneous of length d and let $d' = \ell(g_*)$. Then $d' \leq d$ and $g = Z^{d-d'}(g_*)^*$.*

Proof. The inequality $d' \leq d$ follows from the definition of g_*. Let $m \in supp_{\tilde{\mathcal{B}}}(g)$, $m = Z^i t$, with $t \in \mathcal{B}$. Then the monomial $m_* \in supp_{\mathcal{B}}(g_*)$ correspondent to m is t. As $\ell(t) = d - i$ the monomial in $supp((g_*)^*)$ correspondent to m_* is $tZ^{d'-(d-i)}$. Then $Z^{d-d'}(g_*)^* = g$. ∎

Definition. 3.6. *Let $F \subset \Lambda$ and $\mathcal{G} \subset \Lambda'$, we define by*

$$F^* = \{f^* : f \in F\}$$

$$\mathcal{G}_* = \{g_* : g \in \mathcal{G}\}$$

Lemma 3.7. *Let $f \in \Lambda$. Then $\ell(f) = \ell(f^*)$.*

Proof. Consider $f = \sum_{i=1}^{m} \lambda_i b_i$ with $\lambda_i \in \mathcal{K}$ and $b_i \in \mathcal{B}$, where $\mathcal{B}$ is a basis of Λ, $1 \leq i \leq m$.

By definition we have that $f^* = \sum_{i=1}^{m} \lambda_i Z^{l(f)-l(b_i)} b_i$. For every summand of f^* we have:

$$\ell(Z^{\ell(f)-\ell(b_i)} b_i) = \ell(Z^{\ell(f)-\ell(b_i)}) + \ell(b_i) = (\ell(f) - \ell(b_i)) + \ell(b_i) = \ell(f)$$

Then, $\ell(f^*) = \max\{\ell(Z^{\ell(f)-\ell(b_i)} b_i) : 1 \leq i \leq m\} = \ell(f)$ ∎

Lemma 3.8. *Let $F = \{f_i\}_{i \in \mathcal{I}}$ be a subset of Λ, not necessarily finite, and $f = \sum_{i=1}^{m} r_i f_i s_i \in \langle F \rangle$. If $d = \max\{\ell((r_i)^*(f_i)^*(s_i)^*) : 1 \leq i \leq m\}$ and $d' = \ell(f)$. Then $Z^{d-d'} f^* \in \langle F^* \rangle$.*

Proof. Consider $k_i = \ell(r_i) + \ell(f_i) + \ell(s_i) - \ell(r_i f_i s_i)$, $1 \leq i \leq m$, by 3.3 we have $z^{k_i}(r_i f_i s_i)^* = (r_i)^*(f_i)^*(s_i)^*$

Let $\overline{f} = (\sum_{i=1}^m (r_i)^*(f_i)^*(s_i)^*)^* = \sum_{i=1}^m Z^{d-\ell(r_i)+l(f_i)+l(s_i)}(r_i)^*(f_i)^*(s_i)^* =$ $(\sum_{i=1}^m Z^{d-\ell(r_i)+l(f_i)+l(s_i)} Z^{k_i}(r_i f_i s_i)^*$, by lemma 3.3. So, $\overline{f} \in \langle F^* \rangle$ and is homogeneous (by construction) with $d'' = \ell(\overline{f}) \leq d$. Moreover, using lemma 3.3 and lemma 3.4, we have

$$\begin{aligned} \overline{f}_* &= (\sum_{i=1}^m Z^{d-\ell(r_i)+\ell(f_i)+\ell(s_i)} Z^{k_i}(r_i f_i s_i)^*)_* \\ &= \sum_{i=1}^m Z_*^{d-\ell(r_i)+\ell(f_i)+\ell(s_i)}(r_i^*)_*(f_i^*)_*(s_i^*)_* \\ &= \sum_{i=1}^m r_i f_i s_i = f \end{aligned}$$

Using Lemma 3.5, we can conclude that

$$\overline{f} = Z^{d''-d'}(\overline{f}_*)^* = Z^{d''-d'} f^*$$

As $d'' \leq d$, finally we have:

$$Z^{d-d'} f^* = Z^{d-d''} Z^{d''-d'} f^* = Z^{d-d''} \overline{f} \in \langle F^* \rangle$$

∎

Lemma 3.9. *Let F be a subset of Λ. Then $(\langle F^* \rangle)_* = \langle F \rangle$.*

Proof. Let $f \in \langle F \rangle$. By Lemma 3.8, $Z^k f^* \in \langle F^* \rangle$, for some $k \in \mathbb{N}$, and then

$$f = (f^*)_* = (Z^k f^*)_* \in (\langle F^* \rangle)_*$$

By the other hand, if $g \in \langle F^* \rangle$, say $g = \sum_{i=1}^m r_i (f_i)^* s_i$ with $f_i \in F$ and $r_i, s_i \in \Lambda'$ for $1 \leq i \leq m$, we have

$$\begin{aligned} g_* &= (\sum_{i=1}^m r_i (f_i)^* s_i)_* \\ &= \sum_{i=1}^m (r_i)_* [(f_i)^*]_* (s_i)_* \\ &= \sum_{i=1}^m (r_i)_* f_i (s_i)_* \end{aligned}$$

So, $g_* \in \langle F \rangle$. ∎

We reproduce here the Elimination Theorem, found in [2], to discuss and compare the two results. For that, we define some new concepts.

Let $\mathcal{Q}$ be a quiver and $<_{ll}$ a length-lexicographic order defined in the basis of paths $\mathcal{B}$ of $\mathcal{Q}$. Let α be a maximal arrow with respect to $<_{ll}$ in $\mathcal{B}$.

We define the quiver $\mathcal{Q}_\alpha$ in the following way: $(\mathcal{Q}_\alpha)_0 = \mathcal{Q}_0$ and $(\mathcal{Q}_\alpha)_1 = \mathcal{Q}_1 \backslash \{\alpha\}$.

For $\mathcal{T}$ a set of indices, we define the following application $V : \mathcal{T} \longrightarrow \mathcal{Q}_0$. Let $P = \coprod_{i\in\mathcal{T}} V(i)\mathcal{KQ}$ a (right) projective in $\mathcal{KQ}$-Mod.

We define $P_\alpha = \coprod_{i\in\mathcal{T}} V(i)\mathcal{KQ}_\alpha$ a right projective module in

$\mathcal{KQ}_\alpha$-Mod.

Let $\mathcal{B}_P$ be a $\mathcal{K}$-basis of P with order $\prec$ such that:

(1) For every $m_1, m_2 \in \mathcal{B}_P$ and every $b \in \mathcal{B}$, if $m_1 \prec m_2$, then $m_1 b \prec m_2 b$, if $m_1 b$ and $m_2 b$ are non zero.
(2) For every $m \in \mathcal{B}_P$ and every $b_1, b_2 \in \mathcal{B}$, if $b_1 <_{ll} b_2$, then $mb_1 \prec mb_2$, if both are non zero.
(3) For every $m \in \mathcal{B}_P$ and every $b \in \mathcal{B}$, $mb = 0$ or $mb \in \mathcal{B}_P$.

Let $m \in P$, $m = \sum_{i\in\mathcal{T}} \lambda_i m_i b_i$, with $m_i \in \mathcal{B}_P$, $b_i \in \mathcal{B}$ e $\lambda_i \in \mathcal{K}$. We call $tip(m) =$ the m_i such that $m_i \preceq m_j$ for every $j \in \mathcal{T}$. For $X \subset P$, we will call by $tip(X) = \{tip(x) : x \neq 0, x \in X\}$.

Following Green, we say that $\mathcal{G} \subset P$ is right a Gröbner basis for P, with respect to the order $\prec$, if $tip(\mathcal{G})$ generates $tip(P)$ as a right module.

Here is Green's Elimination Theorem, found in [2].

Theorem 3.10. [2] *Let $\mathcal{Q}$ be a quiver and let $<_{ll}$ be a length-lexicographic order in $\mathcal{B}$, where $\mathcal{B}$ is the set of paths in $\mathcal{Q}$. Let α be a maximal arrow with respect to $<_{ll}$ in $\mathcal{Q}$ and $P = \coprod_{i\in\mathcal{T}} V(i)\mathcal{KQ}$ a projective in $\mathcal{KQ}$-Mod. Let $\mathcal{B}_P$ be an ordered basis (as defined above) for P. If $\mathcal{G}$ is a right uniform (reduced) Gröbner basis for P, then $\mathcal{G}_\alpha = \mathcal{G} \cap P_\alpha$ is a right uniform (reduced) Gröbner basis for P_α.*

As a consequence of the Elimination Theory, Green find a new algebra $\mathcal{KQ}[T]$, that we will call added by loops.

This algebra $\mathcal{KQ}[T]$ is an hereditary algebra, obtained adding loops to Q, as above, but without adding any relation. Observe that both are hereditary algebras and the basis of $\mathcal{KQ}[T]$ is ordered in such a way that the new loops are maximal elements. In this situation, given two ideals and generators sets (Gröbner basis), we can find, as described in [2], a generators set (Gröbner basis), of the intersection of these ideals, constructed by the Elimination Theorem (that can be found, with more details, in [2], section 8).

In our work, there are no additional hypothesis over the given order, the only additional assumption is that the extra loops must be between the vertices and the arrows.

Moreover, we consider the more general case, where $\Lambda = \mathcal{K}\mathcal{Q}/I$, is not necessarily hereditary.

Theorem 3.11. *Let F be a subset of Λ and let $\mathcal{G} \subset \Lambda'$ be homogeneous. If $\mathcal{G}$ is a Gröbner basis for $\langle F^* \rangle$, then $\mathcal{G}_*$ is a Gröbner basis for $\langle F \rangle$.*

Proof. Suppose that $\mathcal{G}$ is a Gröbner basis for $\langle F^* \rangle$. We will prove the theorem, using the definition of Gröbner basis.

As $\mathcal{G}_* \subset \langle F \rangle$, then $\langle Tip(\mathcal{G}_*) \rangle \subseteq \langle Tip(\langle F \rangle) \rangle$, and we only need to verify that $\langle Tip(\langle F \rangle) \rangle \subseteq \langle Tip(\mathcal{G}_*) \rangle$, that is, if given $f \in \langle F \rangle$ there exists $g_* \in \mathcal{G}_*$ such that $Tip(g_*)$ divides $Tip(f)$.

Let $f \in \langle F \rangle$, we can write $f = \sum_{i=1}^{m} \lambda_i b_i$, where $\lambda_i \in \mathcal{K}$ and $b_i \in \mathcal{B}$ for $1 \leq i \leq m$.

Without lost of generality, assume that $Tip(f) = b_1$, then

$$f^* = \lambda_1 Z^{\ell(f)-\ell(b_1)} b_1 + \sum_{i=2}^{m} \lambda_i Z^{\ell(f)-\ell(b_i)} b_i$$

it follows by the given order that $Tip(f^*) = Z^{\ell(f)-\ell(b_1)} b_1 = Z^n Tip(f)$.

By lemma 3.8, there exists $k \in \mathbb{N}$ such that $h = Z^k f^* \in \langle F^* \rangle$. By the above observation, $Tip(h) = Z^k Z^n Tip(f)$.

As $\mathcal{G}$ is a Gröbner basis for $\langle F^* \rangle$, there exists $g \in \mathcal{G}$ such that $Tip(h) = Z^{k_r} r Tip(g) Z^{k_s} s$, for some $Z^{k_r} r, Z^{k_s} s \in \tilde{\mathcal{B}}$. $Tip(g) \in \tilde{\mathcal{B}}$, so $Tip(g) = Z^{k_g} b$ for some $b \in \mathcal{B}$ and $k_g \in \mathbb{N}$. By the definition of order in Λ', for every $Z^t b_t \neq Tip(g)$ that occurs in g, $Z^{k_g} b > Z^t b_t$, then $b > b_t$, or $b = b_t$ and $k_g > t$, but this cannot occur, because g is homogeneous, so $Tip(g)_* = b$.

Then, $Tip(h) = Z^k Z^n b_1 = Z^{k_r} r Tip(g) Z^{k_s} s =$
$Z^{k_r} r Z^{k_g} b Z^{k_s} s = Z^{k_r + k_g + k_s} rbs$. So, we have
$b_1 = (Tip(h))_* = (Z^k Z^n b_1)_* = (Z^{k_r} r Tip(g) Z^{k_s} s)_* = (Z^{k_r} r Z^{k_g} b Z^{k_s} s)_* =$

$$(Z^{k_r + k_g + k_s} rbs)_* = rbs.$$

Then $Tip(f) = rTip(g_*)s$, and s $\mathcal{G}_*$ is a Gröbner basis for $\langle F \rangle$. ∎

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