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POISSON-LIE T-DUALITY AND INTEGRABLE SYSTEMS

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Abstract. We describe a hamiltonian approach to Poisson-Lie T-duality based on the geometry of the underlying phases spaces of the dual sigma and WZW models. Duality is fully characterized by the existence of a hamiltonian action of a Drinfeld double Lie group on the cotangent bundle of its factors and the associated equivariant momentum maps. The duality transformations are explicitly constructed in terms of these actions. It is shown that compatible integrable dynamics arise in a general collective form.

Classical Poisson-Lie T-duality [1, 2, 3, 4] is a kind of canonical transformation relating $(1+1)$-dimensional σ-models with targets on the factors of a perfect Drinfeld double group [5], [6]. Each one describes the motion of a string on a Poisson-Lie groups, and they become linked by the self dual character of the Drinfeld double. Former studies relies on the lagrangian formulation where the lagrangians of the *T-dual* models are written in terms of the underlying bialgebra structure of the Lie groups. On the other hand, the hamiltonian approach allows to characterize T-duality transformation when restricted to *dualizable* subspaces of the sigma models phase-spaces, mapping solutions reciprocally. As $(1+1)$-dimensional field theory describing closed string models, a sigma model has as phase space the cotangent bundle T^*LG of a loop group LG. A hamiltonian description [13] reveals that there exists Poisson maps from the *T-dual* phase-spaces to the centrally extended loop algebra of the Drinfeld double, and this holds for any hamiltonian dynamics on this loop algebra lifted to the T-dual phase-spaces. There is a third model involved [1, 2, 3, 4] [14]: a WZW-type model with target on the Drinfeld double group D, and T-duality works by lifting solutions of a σ-model to the WZW model on D and then projecting it onto the dual one. It was also noted that PL T-duality just works on some subspaces satisfying some *dualizable* conditions expressed as monodromy constraints.

These notes are aimed to review the hamiltonian description of classical PL T-duality based on the symplectic geometry of the involved phase spaces as introduced in [11],[12], remarking the connection with integrable systems. There, PL T-duality is encoded in a diagram as this one

$$\begin{array}{ccccc} & & (L\mathfrak{d}^*_\Gamma;\ \{,\}_{KK}) & & \\ & \overset{\mu}{\nearrow} & \uparrow \hat{\Phi} & \overset{\tilde{\mu}}{\nwarrow} & \\ (T^*LG;\omega_o) & & (\Omega D;\omega_{\Omega D}) & & (T^*LG^*;\ \tilde{\omega}_o) \end{array} \qquad (1)$$

where the left and right vertices are the σ-models phase spaces, equipped with the canonical Poisson (symplectic) structures, $L\mathfrak{d}_{\Gamma}^{*}$ is the dual of the centrally extended Lie algebra of LD with the Kirillov-Kostant Poisson structure, and ΩD is the symplectic manifold of based loops. The arrows labeled by $\hat{\Phi}$, μ and $\tilde{\mu}$ are provided by momentum maps associated to hamiltonian actions of the centrally extended loop group $LD^{\wedge}$ on the WZW and σ models phase spaces, respectively. These actions split the tangent bundles of the pre-images under μ and $\tilde{\mu}$ of the pure central extension orbit, and the dualizable subspaces are identified as the orbits of ΩD which turn to be the symplectic foliation.

In the present work, we describe T-duality scheme on a generic Drinfeld double Lie group $\mathrm{H} = \mathrm{N} \times \mathrm{N}^{*}$ in a rather formal fashion, showing that the action $\hat{\mathfrak{d}}^{\mathrm{N}\times\mathfrak{n}^{*}} : \mathrm{H}_{C} \times (\mathrm{N} \times \mathfrak{n}^{*}) \longrightarrow (\mathrm{N} \times \mathfrak{n}^{*})$ (4), of the central extension H_{C} of H on the cotangent bundles $T^{*}\mathrm{N}$ and $T^{*}\mathrm{N}^{*}$, besides the reduction procedure, allows to characterize all its essential features making clear the connection with the master WZW-type model.

This notes are organized as follows: in Section I, we review the main features of the symplectic geometry of the WZW model; in Section II, we describe the phase spaces of the sigma model with dual targets and introduced a symmetry relevant for PL T-duality; in Section III, the contents of the diagram (1) are developed, presenting the geometric description of the PLT-duality. The dynamic is addressed in Section IV, describing hamiltonian compatible with the above scheme and linking with integrable systems.

1. WZW like models on $\mathcal{O}_c(0,1)$

WZW models are infinite dimensional systems whose phase space is the cotangent bundle of a loop group, endowed with the symplectic form obtained by adding a 2-cocycle to the canonical one [17],[15]. We review in this section the main features of its hamiltonian structure working on a generic Lie group.

Let H be a Lie group and consider the phase space $T^{*}\mathrm{H} \cong \mathrm{H} \times \mathfrak{h}^{*}$, trivialized by left translations, with the symplectic form $\omega_{\Gamma} = \omega_{o} + \Gamma_{R}$ where ω_{o} is the canonical symplectic form and

$$\Gamma_{R}(l) = c\left(dll^{-1} \overset{\otimes}{,} dll^{-1}\right)$$

ω_{Γ} is just invariant under right translation $\varrho_{m}^{\mathrm{H}\times\mathfrak{h}^{*}}(l,\lambda) = (lm^{-1}, Ad_{m^{-1}}^{*}\lambda)$, $m \in \mathrm{H}$. It has associated the non Ad-equivariant momentum map $J^{R} : \mathrm{H} \times \mathfrak{h}^{*} \longrightarrow \mathfrak{h}^{*}$.

Introducing the central extension H_{C} by the Ad^{*}-cocycle $C : \mathrm{H} \longrightarrow \mathfrak{h}^{*}$, satisfying $C(lk) = Ad_{l^{-1}}^{\mathrm{H}*} C(k) + C(l)$, then $\hat{c} \equiv -\left. dC \right|_{e} : \mathfrak{h} \longrightarrow \mathfrak{h}^{*}$ produces the 2-cocycle $c : \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{R}$, $c(\mathbf{X},\mathbf{Y}) \equiv \langle \hat{c}(\mathbf{X}), \mathbf{Y} \rangle$. Let $\mathfrak{h}_{c}$ be the central extension of the Lie algebra $\mathfrak{h}$ by the 2-cocycle c and $\mathfrak{h}_{c}^{*}$ its dual. The coadjoint actions of H_{C} on $\mathfrak{h}_{c}^{*}$ is

$$\widehat{Ad}_{l^{-1}}^{\mathrm{H}*}(\xi, b) = \left(Ad_{l^{-1}}^{*}\xi + bC(l)\ , b\right).$$

In this way, we define the extended momentum map $\hat{J}_{c}^{R} : \mathrm{H} \times \mathfrak{h}^{*} \longrightarrow \mathfrak{h}_{c}^{*}$,

$$\hat{J}_{c}^{R}(l,\eta) = \left(\eta - Ad_{l}^{*}C(l), 1\right)$$

which is now $\widehat{Ad}^{\mathrm{H}}$-equivariant, $\hat{J}_{c}^{R}\left(\varrho_{m}^{\mathrm{H}\times\mathfrak{h}^{*}}(l,\xi)\right) - \widehat{Ad}_{m^{-1}}^{\mathrm{H}*}\hat{J}_{c}^{R}(l,\xi) = (0,0)$.

Applying the Marsden-Weinstein reduction procedure [19] to the regular value $(0,1) \in \mathfrak{h}_c^*$ of $\operatorname{Im} \hat{J}_c^R$. Hence, the level set

$$\left[\hat{J}_c^R\right]^{-1}(0,1) = \{(l, Ad_l^* C(l)) \diagup l \in \mathrm{H}\} \subset \mathrm{H} \times \mathfrak{h}^*$$

turns into a presymplectic submanifold when equipped with the 2-form obtained by restricting the symplectic form ω_Γ

$$\tilde{\omega}(l) = \omega_\Gamma|_{\left[\hat{J}_c^R\right]^{-1}(0,1)} = c \circ (L_{l^{-1}})_*^{\otimes 2} \circ \Pi_{\mathrm{H}}^{\otimes 2}$$

and its null distribution is spanned by the infinitesimal generators of the action of the subgroup $\mathrm{H}_{(0,1)}$, the stabilizer of $(0,1) \in \mathfrak{h}_c^*$, that coincides with $\ker C$. Therefore, the reduced symplectic space is

$$M_c^{(0,1)} = \frac{\left[\hat{J}_c^R\right]^{-1}(0,1)}{\mathrm{H}_{(0,1)}} \cong \frac{\mathrm{H}}{\mathrm{H}_{(0,1)}} \tag{2}$$

with symplectic form ω_R defined by

$$\pi^*_{\mathrm{H}/\mathrm{H}_{(0,1)}} \omega_R = \tilde{\omega}$$

in the fiber bundle

$$\mathrm{H} \overset{\pi_{\mathrm{H}/\mathrm{H}_{(0,1)}}}{\longrightarrow} \mathrm{H}/\mathrm{H}_{(0,1)}.$$

ω_R is still invariant under the residual left action $\mathrm{H} \times \mathrm{H}/\mathrm{H}_{(0,1)} \longrightarrow \mathrm{H}/\mathrm{H}_{(0,1)}$, $\left(h, l \cdot \mathrm{H}_{(0,1)}\right) \longrightarrow hl \cdot \mathrm{H}_{(0,1)}$. The associated equivariant momentum map $\hat{\Phi}_c : \mathrm{H}/\mathrm{H}_{(0,1)} \longrightarrow \mathfrak{h}_c^*$ is

$$\hat{\Phi}_c\left(l \cdot \mathrm{H}_{(0,1)}\right) = (C(l), 1)$$

$\hat{\Phi}_c$ is a local symplectic diffeomorphism with the coadjoint orbit $\mathcal{O}_c(0,1) \subset \mathfrak{h}_c^*$ equipped with the Kirillov-Kostant symplectic structure ω_c^{KK}. so we get the commutative diagram

$$\begin{array}{ccc} M_c^{(0,1)} & \longrightarrow & \mathrm{H}/\mathrm{H}_{(0,1)} \\ \hat{J}_c^R \big\downarrow & \swarrow \hat{\Phi}_c & \\ \mathcal{O}_c(0,1) & & \end{array} \tag{3}$$

When H is loop group, $\mathrm{H}/\mathrm{H}_{(0,1)}$ is the group of *based loops*, and it can be regarded as the reduced space of a WZNW model as explained in [15].

2. Double Lie groups and sigma models phase spaces

From now on, H is the Drinfeld double of the Poisson Lie group N [5, 6], $\mathrm{H} = \mathrm{N} \bowtie \mathrm{N}^*$, with tangent Lie bialgebra $\mathfrak{h} = \mathfrak{n} \oplus \mathfrak{n}^*$. This bialgebra $\mathfrak{h}$ is naturally equipped with the non degenerate symmetric *Ad*-invariant bilinear form provided by the contraction between $\mathfrak{n}$ and $\mathfrak{n}^*$, and which turns them into isotropic subspaces. This bilinear form allows for the identification $\psi : \mathfrak{n} \longrightarrow \mathfrak{n}^*$.

We shall construct a couple of dual phase spaces on the factors N and N^*. T-duality take place on centrally extended coadjoint orbit and, as a matter of fact, we consider the *trivial* orbit $\mathcal{O}_c(0,1)$ as it was considered in ref. [11] in relation to Poisson-Lie T-duality for loop groups and trivial monodromies.

Our approach to PL T-duality is based on the existence of some hamiltonian H-actions on phase spaces $T^*\mathrm{N}$ and $T^*\mathrm{N}^*$ so that the arrows in the diagram (1) are provided by the corresponding equivariant momentum maps. The key ingredients here are the reciprocal actions between the factors N and N^* called *dressing actions* [18],[6]. Writing every element $l \in \mathrm{H}$ as $l = g\tilde{h}$, with $g \in \mathrm{N}$ and $\tilde{h} \in \mathrm{N}^*$, the product $\tilde{h}g$ in H can be expressed as $\tilde{h}g = g^{\tilde{h}}\tilde{h}^g$, with $g^{\tilde{h}} \in \mathrm{N}$ and $\tilde{h}^g \in \mathrm{N}^*$. The dressing action of N^* on N is then defined as

$$\mathsf{Dr} : \mathrm{N}^* \times \mathrm{N} \longrightarrow \mathrm{N} \qquad / \qquad \mathsf{Dr}\left(\tilde{h}, g\right) = \Pi_{\mathrm{N}}\left(\tilde{h}g\right) = g^{\tilde{h}}$$

where $\Pi_{\mathrm{N}} : \mathrm{H} \longrightarrow \mathrm{N}$ is the projector. For $\xi \in \mathfrak{n}^*$, the infinitesimal generator of this action at $g \in \mathrm{N}$ is

$$\xi \longrightarrow \mathsf{dr}\left(\xi\right)_g = -\left.\frac{d}{dt}\mathsf{Dr}\left(e^{t\xi}, g\right)\right|_{t=0}$$

such that, for $\eta \in \mathfrak{n}^*$, we have $\left[\mathsf{dr}\left(\xi\right)_g, \mathsf{dr}\left(\eta\right)_g\right] = \mathsf{dr}\left(\left[\xi,\eta\right]_{\mathfrak{n}^*}\right)_g$. It satisfies the relation $Ad^{\mathrm{H}}_{g^{-1}}\xi = -g^{-1}\mathsf{dr}\left(\xi\right)_g + Ad^*_g\xi$, where $Ad^{\mathrm{H}}_{g^{-1}} \in Aut\left(\mathfrak{h}\right)$ is the adjoint action of H on its Lie algebra. Then, using the $\Pi_{\mathfrak{n}} : \mathfrak{h} \longrightarrow \mathfrak{n}$, we can write $\mathsf{dr}\left(\xi\right)_g = -g\ \Pi_{\mathfrak{n}} Ad^{\mathrm{H}}_{g^{-1}}\xi$.

Let us now consider the action of H on itself by left translations $L_{a\tilde{b}}g\tilde{h} = a\tilde{b}g\tilde{h}$. Its projection on the one of the factors, N for instance, yields also an action of H on that factor

$$\Pi_{\mathrm{N}}\left(L_{a\tilde{b}}g\tilde{h}\right) = \Pi_{\mathrm{N}}\left(a\tilde{b}g\tilde{h}\right) = ag^{\tilde{b}}$$

The projection on the factor N^* is obtained by the reversed factorization of H, namely $\mathrm{N}^* \times \mathrm{N}$, such that

$$\Pi_{\mathrm{N}^*}\left(L_{\tilde{b}a}\tilde{h}g\right) = \Pi_{\mathrm{N}^*}\left(\tilde{b}a\tilde{h}g\right) = \tilde{b}\tilde{h}_a$$

Both these actions, lifted to the corresponding cotangent bundles $T^*\mathrm{N}$ and $T^*\mathrm{N}^*$, and then centrally extended, will furnish the arrows we are looking for.

2.0.1. *The* $(T^*\mathrm{N}, \omega_o)$ *phase space and dualizable subspaces.* We now consider a phase space $T^*\mathrm{N} \cong \mathrm{N} \times \mathfrak{n}^*$, trivialized by left translations and equipped with the canonical symplectic form ω_o. We shall realize the symmetry described above, as it was introduced in [11].

For duality purposes, we consider the adjoint N-cocycle $C^{\mathrm{N}^*}: \mathrm{N}^* \longrightarrow \mathfrak{n}^*$ as the restriction to the factor N of an H-coadjoint cocycle on $C^{\mathrm{H}}: \mathrm{H} \longrightarrow \mathfrak{h}^*$. The restrictions properties

$$\begin{array}{c} C^{\mathrm{H}}: \mathrm{N} \longrightarrow \mathfrak{n} \\ C^{\mathrm{H}}: \mathrm{N}^* \longrightarrow \mathfrak{n}^* \end{array}$$

make C^{H} *compatible with the factor decomposition.*

Thus, we promote this symmetry to a *centrally extended* one on $T^*\mathrm{N}$ by means of an $\mathfrak{n}^*$-valued cocycle $C^{\mathrm{N}^*}: \mathrm{N}^* \longrightarrow \mathfrak{n}^*$,

$$\begin{array}{l} \hat{\mathfrak{d}}^{\mathrm{N}\times\mathfrak{n}^*}: \mathrm{H}_C \times (\mathrm{N}\times\mathfrak{n}^*) \longrightarrow (\mathrm{N}\times\mathfrak{n}^*) \\ \hat{\mathfrak{d}}^{\mathrm{N}\times\mathfrak{n}^*}\left(a\tilde{b},(g,\lambda)\right) = \left(ag^{\tilde{b}}, Ad^{\mathrm{H}}_{\tilde{b}^g}\lambda + C^{\mathrm{N}^*}\left(\tilde{b}^g\right)\right) \end{array} \tag{4}$$

For duality purposes, we consider the adjoint N-cocycle $C^{\mathrm{N}^*}: \mathrm{N}^* \longrightarrow \mathfrak{n}^*$ as the restriction to the factor N of an H-coadjoint cocycle on $C^{\mathrm{H}}: \mathrm{H} \longrightarrow \mathfrak{h}^*$. The restrictions properties

$$\begin{array}{c} C^{\mathrm{H}}: \mathrm{N} \longrightarrow \mathfrak{n} \\ C^{\mathrm{H}}: \mathrm{N}^* \longrightarrow \mathfrak{n}^* \end{array} \tag{5}$$

make C^{H} *compatible with the factor decomposition.* One of the key ingredients for describing the resulting duality is that the above H-action is hamiltonian.

Proposition: *Let $T^*\mathrm{N}$ be identified with $\mathrm{N}\times\mathfrak{n}^*$ by left translations and endowed with the canonical symplectic structure. The action $\hat{\mathfrak{d}}^{\mathrm{N}\times\mathfrak{n}^*}$, defined in eq.(4), is hamiltonian and the momentum map $\mu: (\mathrm{N}\times\mathfrak{n}^*, \omega_o) \rightarrow (\mathfrak{h}_c^*, \{,\}_c)$*

$$\mu(g,\lambda) = \widehat{Ad}^{\mathrm{H}*}_{g^{-1}}\left(\psi(\lambda),1\right) = \left(\psi\left(Ad^{\mathrm{H}}_g\lambda + C^{\mathrm{N}^*}(g)\right),1\right) .$$

is Ad^{H}-equivariant.

T-duality works on some subspaces of the phase space $T^*\mathrm{N}$. As shown in [11], these *dualizable subspaces* can be identified as some symplectic submanifolds in the pre-images of the coadjoint orbit in $\mathcal{O}_c(0,1) \subset \mathfrak{h}_c^*$ by μ_0, namely $\mu_0^{-1}(\mathcal{O}_c(0,1))$ characterized as

$$\mu^{-1}(\mathcal{O}_c(0,1)) = \left\{\left(g, C\left(\tilde{b}\right)\right) \in \mathrm{N}\times\mathfrak{n}^*/g\in\mathrm{N},\ \tilde{b}\in\mathrm{N}^*\right\}$$

that coincides with the H_C-orbit through $(e,0)$ in $\mathrm{N}\times\mathfrak{n}^*$

$$\mathcal{O}_{\mathrm{N}\times\mathfrak{n}^*}(e,0) = \left\{\hat{\mathfrak{d}}^{\mathrm{N}\times\mathfrak{n}^*}\left(g\tilde{b},(e,0)\right) \in \mathrm{N}\times\mathfrak{n}^*/g\in\mathrm{N},\ \tilde{b}\in\mathrm{N}^*\right\}$$

since $\hat{\mathfrak{d}}^{\mathrm{N}\times\mathfrak{n}^*}\left(g\tilde{b},(e,0)\right) = \left(g, C\left(\tilde{b}\right)\right)$. Tangent vectors to $\mu^{-1}(\mathcal{O}_c(0,1))$ at the point (g,ξ) are

$$\mathbf{X}_{\mu^{-1}(\mathcal{O}_c(0,1))}\Big|_{(g,\xi)} = \hat{\mathfrak{d}}^{\mathrm{N}\times\mathfrak{n}^*}_*(\mathbf{X})\Big|_{(g,\xi)}$$

for $\mathbf{X}\in\mathfrak{h}$, and then we have the following statement.

Proposition: *$\mu^{-1}\left(\mathcal{O}_c\left(0,1\right)\right)$ is a presymplectic submanifold with the closed 2-form given by the restriction of the canonical form ω_o,*

$$\left\langle \omega_o, \hat{\mathsf{d}}_*^{\mathrm{N}\times\mathfrak{n}^*}\left(\mathbf{X}\right)\otimes\hat{\mathsf{d}}_*^{\mathrm{N}\times\mathfrak{n}^*}\left(\mathbf{Y}\right)\right\rangle_{(g,\xi)} = \left\langle \left(C\left(a\tilde{b}\right),1\right), \widehat{ad}_{\mathbf{X}}^{\mathfrak{h}}\mathbf{Y}\right\rangle_{\mathfrak{h}} \tag{6}$$

for $\mathbf{X},\mathbf{Y}\in\mathfrak{h}$, and $(g,\xi)=\hat{\mathsf{d}}_*^{\mathrm{N}\times\mathfrak{n}^*}\left(a\tilde{b},(e,0)\right)\in\mu^{-1}\left(\mathcal{O}_c\left(0,1\right)\right)$. Its null distribution is spanned by the infinitesimal generators of the right action by the stabilizer $\mathrm{H}_{(0,1)}:=\ker C$ of the point $(0,1)$

$$\mathsf{r}:\mathrm{H}_{(0,1)}\times\mu^{-1}\left(\mathcal{O}_c\left(0,1\right)\right)\longrightarrow\mu^{-1}\left(\mathcal{O}_c\left(0,1\right)\right)$$
$$\mathsf{r}\left(l_\circ,\hat{\mathsf{d}}_*^{\mathrm{N}\times\mathfrak{n}^*}\left(a\tilde{b},(e,0)\right)\right)=\hat{\mathsf{d}}_*^{\mathrm{N}\times\mathfrak{n}^*}\left(a\tilde{b}l_\circ^{-1},(e,0)\right)$$

and the null vectors at the point $\hat{\mathsf{d}}^{\mathrm{N}\times\mathfrak{n}^*}\left(a\tilde{b},(e,0)\right)\in\mu_0^{-1}\left(\mathcal{O}\left(0,1\right)\right)$ are

$$\mathsf{r}_{*(g,\xi)}Z_o=-\hat{\mathsf{d}}_*^{\mathrm{N}\times\mathfrak{n}^*}\left(Ad_{a\tilde{b}}^{\mathrm{H}}Z_\circ\right)_{(g,\xi)}$$

for all $Z_\circ\in\mathrm{Lie}\left(\ker C\right)\subset\mathfrak{h}$.

We may think of $\mu^{-1}\left(\mathcal{O}_c\left(0,1\right)\right)$ as a principal bundle on $\mathcal{O}_c\left(0,1\right)$ where the fibers are the orbits of $\mathrm{H}_{(0,1)}$ by the action r. Assuming that $\mathrm{H}_{(0,1)}=\ker C$ is a *normal subgroup*, as it happens in the case of $\mathrm{H}=LD$, the loop group of the double Lie group D [11], a flat connection is defined by the action of the group $\mathrm{H}/\mathrm{H}_{(0,1)}$ and each flat section is a symplectic submanifold. These are the *dualizable subspaces.*

2.0.2. *The dual factor $(T^*\mathrm{N}^*,\tilde{\omega}_o)$ factor phase space.* Let us consider H with the opposite factorization, denoted as $\mathrm{H}\to\mathrm{H}^\top=\mathrm{N}^*\bowtie\mathrm{N}$, so that every element is now written as $\tilde{h}g$ with $\tilde{h}\in\mathrm{N}^*$ and $g\in\mathrm{N}$. This factorization relates with the former one by $\tilde{b}_a=\left(\left(\tilde{b}^{-1}\right)^{a^{-1}}\right)^{-1}$ and $a_{\tilde{b}}=\left(\left(a^{-1}\right)^{\tilde{b}^{-1}}\right)^{-1}$. The dressing action $\widetilde{\mathsf{Dr}}:\mathrm{N}\times\mathrm{N}^*\longrightarrow\mathrm{N}^*$ is $\widetilde{\mathsf{Dr}}\left(g,\tilde{h}\right)=\tilde{h}_g$ and, by composing it with the right action of N^* on itself, we get the action $\mathsf{b}^{\mathrm{N}^*}:\mathrm{N}\times\mathrm{N}^*\longrightarrow\mathrm{N}^*$ defined as $\mathsf{b}^{\mathrm{N}}\left(\tilde{b}a,\tilde{h}\right)=\tilde{b}\tilde{h}_a$ with $a\in\mathrm{N}$ and $\tilde{h},\tilde{b}\in\mathrm{N}^*$.

As the dual partner for the phase spaces on $\mathrm{N}\times\mathfrak{n}^*$ we consider the symplectic manifold $(\mathrm{N}^*\times\mathfrak{n},\tilde{\omega}_o)$ where $\tilde{\omega}_o$ is the canonical 2-form in body coordinates.

Proposition: *Let us consider the symplectic manifold $(\mathrm{N}^*\times\mathfrak{n},\tilde{\omega}_o)$. The map $\hat{\mathsf{b}}:\mathrm{H}_c\times(\mathrm{N}^*\times\mathfrak{n})\longrightarrow(\mathrm{N}\times\mathfrak{n}^*)$ defined as*

$$\hat{\mathsf{b}}\left(\tilde{b}a,\left(\tilde{h},X\right)\right)=\left(\tilde{b}\tilde{h}_a, Ad_{a_{\tilde{h}}}^{\mathrm{H}}X+C\left(a_{\tilde{h}}\right)\right) \tag{7}$$

is a hamiltonian H-action and $\tilde{\mu}_0$ is the associated $\widehat{Ad}^{\mathrm{H}_c*}$-equivariant the momentum map

$$\tilde{\mu}\left(\tilde{h},Z\right)=\widehat{Ad}_{\tilde{h}^{-1}}^{\mathrm{H}_c*}\left(\psi\left(Z\right),1\right)=\left(\psi\left(Ad_{\tilde{h}}^{\mathrm{H}}Z+C\left(\tilde{h}\right)\right),1\right)$$

In terms of the orbit map $\hat{\mathsf{b}}_{(e,0)} : \mathrm{H} \longrightarrow \mathrm{N}^* \times \mathfrak{n}$ associated to the action $\hat{\mathsf{b}}$,

$$\hat{\mathsf{b}}_{(e,0)}\left(a\tilde{b}\right) = \hat{\mathsf{b}}\left(\tilde{b}a, (e,0)\right) = (\tilde{b}, C(a))$$

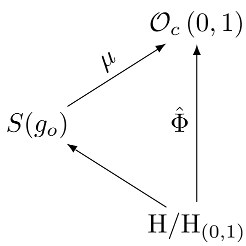
providing the identification

$$\tilde{\mu}^{-1}\left(\mathcal{O}_c(0,1)\right) = \mathrm{Im}\,\hat{\mathsf{b}}_{(e,0)}$$

Analogously, we may think of $\tilde{\mu}^{-1}\left(\mathcal{O}_c(0,1)\right)$ as a fiber bundle on $\mathcal{O}_c(0,1)$ and define a flat connection by the action of $\mathrm{H}/\mathrm{H}_{(0,1)}$ thus obtaining the corresponding *dualizable subspaces* as flat (global) sections.

3. The scheme for T duality

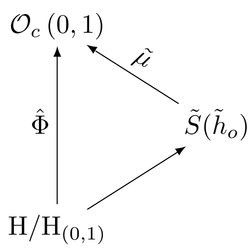
Let us name the dualizable subspaces in $\mathrm{N}\times\mathfrak{n}^*$ as $S(g_o)$, $(g_o, 0)$ in $\mu^{-1}\left(\mathcal{O}_c(0,1)\right)$ the point the orbit pass through, then $S(g_o)$ becomes a symplectic submanifold when equipped with the restriction of the presymplectic form (6). On the other side, each dualizable subspace coincides with an orbits of $\mathrm{H}/\mathrm{H}_{(0,1)}$. Regarding $\mathrm{H}/\mathrm{H}_{(0,1)}$ as the reduced space of the WZNW system $\left(M_c^{(0,1)}, \omega_R\right)$ (2), we may easily see that the orbit map $\mathrm{H}/\mathrm{H}_{(0,1)} \longrightarrow S(g_o)$ is a symplectic morphism, giving rises to a factorization of $\hat{\Phi}$ through $S(g_o) \subset T^*\mathrm{N}$,



(8)

with arrows being symplectic maps.

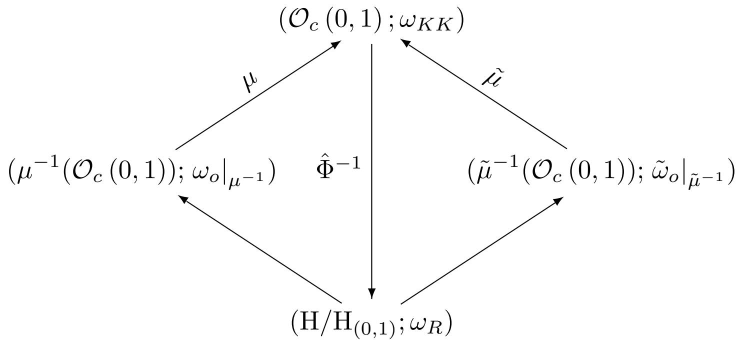
An analogous analysis can be performed for the mirror image on $T^*\mathrm{N}^*$ obtaining the factorization of $\hat{\Phi}$ through $\tilde{S}(\tilde{h}_o) \subset T^*\mathrm{N}^*$,



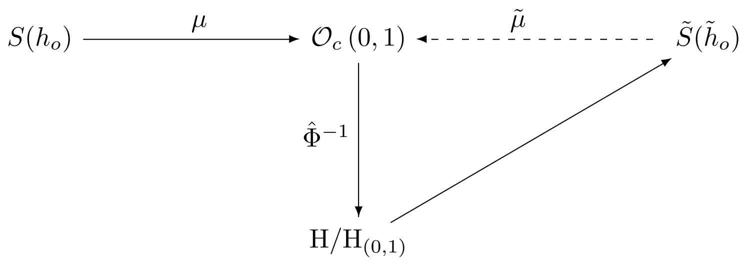
(9)

Joining together diagrams (8) and its mirror image, we obtain a commutative four vertex diagram relating dualizable subspaces in $T^*\mathrm{N}$ and $T^*\mathrm{N}^*$, through symplectic maps. Because μ and $\tilde{\mu}$ are equivariant momentum maps, the intersection region extends to the whole coadjoint orbit of the point $\mu(g,\eta) = \tilde{\mu}\left(\tilde{h}, Z\right)$ in $\mathfrak{h}_c^*$, establishing a connection between H_C-orbits in $T^*\mathrm{H}$ and $T^*\mathrm{N}^*$. It can be seen that this common region coincides with the pure central extension orbit

$\mathcal{O}_c(0,1) = \operatorname{Im}\mu \cap \operatorname{Im}\tilde{\mu}$ and it is a isomorphic image of the WZW reduced space, so that we may refine diagram (1) to get a connection between the phase spaces of sigma models on dual targets and WZW model on the associated Drinfeld double group



Hence, we define *Poisson-Lie T-duality* by restricting this diagram to the symplectic leaves in $\mu^{-1}(\mathcal{O}_c(0,1))$ and $\tilde{\mu}^{-1}(\mathcal{O}_c(0,1))$. In terms of the dualizable subspaces $S(g_o) \subset \mu^{-1}(\mathcal{O}_c(0,1))$ and $\tilde{S}\left(\tilde{h}_o\right) \subset \tilde{\mu}^{-1}(\mathcal{O}_c(0,1))$, *Poisson-Lie T-duality* is the symplectic map resulting from the composition of arrows



(10)

defines the *T-duality* transformation $\Psi_{\tilde{h}_o} : S(g_o) \longrightarrow \tilde{S}\left(\tilde{h}_o\right)$

$$\Psi_{\tilde{h}_o}(g,\lambda) = \left(\tilde{b}\left(\tilde{h}_o\right)_a, C\left(a_{\tilde{h}_o}\right)\right)$$

where $\left[a\tilde{b}\right] \in \mathrm{H}/\mathrm{H}_{(0,1)}$ is such that $(g,\lambda) = \hat{\mathsf{d}}\left(\left[a\tilde{b}\right],(g_o,0)\right)$. This are nothing but the duality transformations given in [1, 2, 3, 4] and other references (see references in [11]). Obviously, as a composition of symplectic maps, $\Psi_{\tilde{h}_o}$ is a canonical transformation and a hamiltonian vector field tangent to $S(g_o)$ is mapped onto a hamiltonian vector field tangent to $\tilde{S}\left(\tilde{h}_o\right)$.

4. T- dual dynamics and integrability

The diagram (10) provides the geometric links underlying Poisson Lie T-duality, and now we work out the compatible dynamics. To this end , we observe that $\mathrm{H}/\mathrm{H}_{(0,1)} \cong \Phi^{-1}(\mathcal{O}_c(0,1))$ and the symplectic leaves in $\mu^{-1}(\mathcal{O}_c(0,1))$, $\tilde{\mu}^{-1}(\mathcal{O}_c(0,1))$

are *replicas* of the coadjoint orbit $\mathcal{O}_c(0,1)$. The equivariance entails a local isomorphism of tangent bundles and since $\mathcal{O}_c(0,1)$ is in the vertex linking the three models, it is clear that T-duality transformation (10) exist at the level of hamiltonian vector fields. So, singling out a hamiltonian function on $\mathcal{O}_c(0,1)$, one gets the associated vector field in associated T-dual partners on symplectic leaves in $\mu^{-1}(\mathcal{O}_c(0,1))$ and $\tilde{\mu}^{-1}(\mathcal{O}_c(0,1))$ and, whenever they exist, a couple T-dual related solution curves belonging to some kind of dual sigma models.

In terms of hamiltonian functions, once a suitable function $\mathsf{h} \in C^\infty(\mathfrak{d}_c^*)$ is fixed, we have the corresponding hamiltonian function on $\mu^{-1}(\mathcal{O})$ and $\tilde{\mu}^{-1}(\mathcal{O})$ by pulling back it through the momentum maps μ and $\tilde{\mu}$, to get $\mathsf{h}\circ\mu$, $\mathsf{h}\circ\tilde{\mu}$ and $\mathsf{h}\circ\Phi$.These systems on $T^*\mathrm{N}$, $T^*\mathrm{N}^*$, $T^*\mathrm{H}^*$ are said to be in *collective hamiltonian form* [20], and the scheme of integrability is complemented by applying the Adler-Kostant-Symes theory, which provides a set of functions in $C^\infty(\mathfrak{d}_c^*)$ in involution.

The geometric meaning of collective dynamics can be understood by considering a generic hamiltonian system (M,ω,H), with an Ad-equivariant momentum map $J: M \to \mathfrak{g}^*$ associated to the symplectic action $\varphi: G\times M \longrightarrow M$ of a Lie group G. Then, a *collective hamiltonian* is a composition

$$H = \mathsf{h}\circ J$$

for some function $\mathsf{h}:\mathfrak{g}^*\to\mathbb{R}$.

In terms of the linear map $\mathcal{L}_\mathsf{h}:\mathfrak{g}^*\to\mathfrak{g}$, namely the *Legendre transformation of* h defined as $\langle\xi,\mathcal{L}_\mathsf{h}(\eta)\rangle_\mathfrak{g} = \langle\, d\mathsf{h}|_\eta, \xi\rangle$, for any $\xi\in\mathfrak{g}^*$, we have the relation $(\varphi_m)_*[\mathcal{L}_\mathsf{h}\circ J](m) = [\mathcal{L}_\mathsf{h}(J(m))]_M|_m$, for $m\in M$ and where $\varphi_m: G\longrightarrow M$ is the orbit map through $m\in M$.

Proposition: *The hamiltonian vector field associated to $H=\mathsf{h}\circ J$ is*

$$V_H|_m = (\varphi_m)_*[\mathcal{L}_\mathsf{h}\circ J](m)$$

and its image by J is tangent to the coadjoint orbit through $J(m)$

$$J_*|_m V_H = -ad^*_{\mathcal{L}_\mathsf{h}(J(m))}J(m)$$

That means, the hamiltonian vector fields V_H is mapped into a coadjoint orbits in $\mathfrak{g}^*$ and, if $m(t)$ denotes the trajectory of the hamiltonian system passing by $m(0)=m$, $\dot{m}(t) = V_H|_{m(t)}$, the images $\gamma(t) = J(m(t))$ lies completely on the coadjoint orbit through $J(m)$ and the equation of motion there is

$$\dot{\gamma}(t) = -ad^*_{\mathcal{L}_\mathsf{h}(\gamma(t))}\gamma(t)$$

that can be regarded as a hamiltonian system on the coadjoint orbits on $\mathfrak{g}^*$, with hamiltonian function $H_{\mathfrak{g}^*}=\mathsf{h}$.

Proposition: *Let $\gamma(t)$ a curve in $\mathfrak{g}^*$, and $\mathcal{L}_\mathsf{h}(\gamma(t))\subset\mathfrak{g}$. Define the curve $g(t)$ in G such that*

$$\gamma(t) = Ad^*_{g^{-1}(t)}\gamma(0) \tag{11}$$

Then, this curve is a solution of the differential equation on G

$$\dot{g}(t)g^{-1}(t) = \mathcal{L}_{\mathfrak{h}}(\gamma(t)) \qquad (12)$$
$$g(0) = e$$

Hence, $m(t) = \varphi(g(t), m)$ is the solution to the original hamiltonian system. Moreover, if $\mathfrak{g}$ is supplied with an invariant non degenerate bilinear form $(\,,)$: $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ thus providing an isomorphism from $\mathfrak{g}^* \to \mathfrak{g}$, and denoting $\tilde{\gamma}(t) \subset \mathfrak{g}$ the image of $\gamma(t) \subset \mathfrak{g}^*$ through the bilinear form, the equation of motion turns into the Lax form

$$\frac{d\tilde{\gamma}(t)}{dt} = [\tilde{\gamma}(t), \mathcal{L}_{\mathfrak{h}}(\gamma(t))] \qquad (13)$$

showing this systems are integrable.

Coming back to our systems on T^*N, T^*N* and T^*H*, with dynamics given in *collective hamiltonian form*, ensures that the corresponding hamiltonian vector fields are tangent to the H$_C$ orbits. Further, a hamiltonian vector field at $\widehat{Ad}_{l^{-1}}^{\mathrm{H}*}(0,1) \in \mathcal{O}_c(0,1)$ is alike $\widehat{ad}_{\mathcal{L}_{\mathfrak{h}}}^{\mathrm{H}*}\widehat{Ad}_{l^{-1}}^{\mathrm{H}*}(0,1)$, and the solution curves are determined from the solution of the differential equation on H

$$\dot{l}(t)l^{-1}(t) = \mathcal{L}_{\mathfrak{h}}(\gamma(t)) \qquad (14)$$

Thus, the solutions for the collective hamiltonian vector fields on T^*N, $M_c^{(0,1)}$ and $T^{*\mathrm{N}*}$ are

$$\begin{cases} \hat{\mathsf{d}}\left(l(t), (g_o, \eta_o)\right) \\ [l(t)l_o] \\ \hat{\mathsf{b}}\left(l(t), \left(\tilde{h}_o, Z_o\right)\right) \end{cases}$$

respectively, with $l(t)$ given by (14) and for $(g_o, \eta_o) \in \mu^{-1}(\gamma(0))$, $[l_0] \in \hat{\Phi}^{-1}(\gamma(0))$, $\left(\tilde{h}_0, Z_0\right) \in \tilde{\mu}^{-1}(\gamma(0))$.

Also note that, in order to have a non trivial duality, restriction to the common sector in $\mathfrak{h}_c^*$ where all the momentum maps intersect is required. Hence, in this case we consider the coadjoint orbit $\mathcal{O}_c(0,1)$, focus our attention on the pre images $\mu^{-1}(\mathcal{O}_c(0,1))$ and $\tilde{\mu}^{-1}(\mathcal{O}_c(0,1))$. We shall refer to these pre-images as *dualizable* or *admissible* subspaces.

Let us work out an example of hamiltonian function. For the symplectic manifold $(\mathrm{H} \times \mathfrak{h}^*, \omega_c)$ and their reduced space $M_c^{(0,1)}$, we consider the quadratic Hamilton function

$$\mathcal{H}_c(l,\eta) = \frac{1}{2}(Ad_{l^{-1}}^*\eta, \mathbb{L}_3^* Ad_{l^{-1}}^*\eta)_{\mathfrak{h}^*} + (Ad_{l^{-1}}^*\eta, \mathbb{L}_2^* C(l))_{\mathfrak{h}^*} - \frac{1}{2}(C(l), \mathbb{L}_2^* C(l))_{\mathfrak{h}^*}$$

that when restricted to $M_c^{(0,1)}$, $\eta = Ad_l^* C(l)$, takes the collective form

$$\mathcal{H}_c(l,\eta)|_{M_c^{(0,1)}} = \frac{1}{2}(C(l), (\mathbb{L}_2^* + \mathbb{L}_3^*) C(l))_{\mathfrak{h}^*}$$

in terms of the momentum map $\hat{\Phi}_c(l \cdot \mathrm{H}_c) = (C(l), 1)$. The Hamilton equations of motion reduces to

$$\begin{array}{l} l^{-1}\dot{l} = \bar{\psi}\left(\left(\mathbb{L}_3^{l*} + \mathbb{L}_2^{l*}\right)\eta\right) \\ \dot{\eta} = ad^*_{\bar{\psi}\left(\left(\mathbb{L}_3^{l*}+\mathbb{L}_2^{l*}\right)\eta\right)}\eta - \hat{c}\left(\bar{\psi}\left(\left(\mathbb{L}_3^{l*} + \mathbb{L}_2^{l*}\right)\eta\right)\right) \end{array}$$

Analogously, let us now consider the hamiltonian space and $(\mathrm{N}\times\mathfrak{n}^*, \omega_{o,}, \mathrm{H}_C, \mu, \mathsf{h}\circ\mu)$. For h quadratic as above, the hamiltonian for the sigma model system must be

$$\mathcal{H}_\sigma = \frac{1}{2}(\mu, \mathcal{E}\mu)_{\mathfrak{h}}$$

where we named by $\mathcal{E}$ the symmetric part of $(\mathbb{L}_2 + \mathbb{L}_3)$. The Hamilton equation of motion yield

$$g^{-1}\dot{g} = \Pi_{\mathfrak{n}}\psi\left(\left(Ad_g^{\mathrm{H}} \circ \mathcal{E} \circ Ad_{g^{-1}}^{\mathrm{H}}\right)\left(Ad_g^{\mathrm{H}}\lambda + \bar{\psi}(C(g))\right)\right)$$

For more details on the Hamilton and Lagrange functions for these models, see [11].

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