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ON THE NOTION OF BANDLIMITEDNESS AND ITS GENERALIZATIONS

AHMED I. ZAYED

Abstract. In this survey article we introduce the Paley-Wiener space of bandlimited functions PW_ω, and review some of its generalizations. Some of these generalizations are new and will be presented without proof because the proofs will be published somewhere else.

Guided by the role that the differentiation operator plays in some of the characterizations of the Paley-Wiener space, we construct a subspace of vectors $PW_\omega(D)$ in a Hilbert space H using a self-adjoint operator D. We then show that the space $PW_\omega(D)$ has similar properties to those of the space PW_ω.

The paper is concluded with an application to show how to apply the abstract results to integral transforms associated with singular Sturm-Liouville problems.

1. Introduction

The term bandlimited functions came from electrical engineering where it means that the frequency content of a signal $f(t)$ is limited by certain bounds from below and above. More precisely, if $f(t)$ is a function of time, its Fourier transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

is called the amplitude spectrum of f. It represents the frequency content of the signal. The energy of the signal is measured by

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \|f\|^2 .$$

A signal is said to be bandlimited to $[-\sigma, \sigma]$ if $\hat{f}$ vanishes outside $[-\sigma, \sigma]$. σ is called the bandwidth. Hence, the space of all finite energy, bandlimited signals is a subspace of $L^2(I\!R)$ consisting of all functions with Fourier transforms supported on finite intervals symmetric around the origin. This space, which is known in harmonic analysis as the Paley-Wiener space, will be denoted by PW_σ or B_σ^2. P for Paley, W for Wiener, and B for Bernstein.

In this survey article we shall give an overview of some of the generalizations of this space, of which some are new and will be presented without proof since the

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proofs will be published somewhere else. For some related work, see [1, 2, 3, 4, 8, 9, 18, 19]

We begin with the following fundamental result by Paley and Wiener on band-limited functions, which gives a nice characterization of the space PW_σ.

Theorem 1 (Paley-Wiener,[13]). *A function f is band-limited to $[-\sigma,\sigma]$ if and only if*

$$f(t)=\int_{-\sigma}^{\sigma} e^{-i\omega t} g(\omega)\,d\omega \qquad (t\in \mathbb{R}),$$

for some function $g\in L^2(-\sigma,\sigma)$ and if and only if f is an entire function of exponential type that is square integrable on the real line, i.e., f is an entire function such that

$$|f(z)|\le \sup_{x\in\mathbb{R}} |f(x)|\exp(\sigma\,|y|), \quad z=x+iy,$$

and

$$\int_{\mathbb{R}} |f(x)|^2\,dx<\infty.$$

Another important property of the space PW_σ is given by the Whittaker-Shannon-Koteln'nikov (WSK) sampling theorem, which can be stated as follows [22]:

Theorem 2. *If $f\in PW_\sigma$, then f can be reconstructed from its samples, $f(t_k)$, where $t_k=k\pi/\sigma$ via the formula*

$$f(t)=\sum_{k=-\infty}^{\infty} f(t_k)\,\frac{\sin\sigma(t-t_k)}{\sigma(t-t_k)} \qquad (t\in\mathbb{R}), \tag{1.1}$$

with the series being absolutely and uniformly convergent on $\mathbb{R}$.

One of the earliest generalizations of the Paley-Wiener space is the Bernstein space. Let $\sigma>0$ and $1\le p\le\infty$. The Bernstein space B_σ^p is a Banach space consisting of all entire functions f of exponential type with type at most σ that belong to $L^p(\mathbb{R})$ when restricted to the real line. It is known [5, p. 98] that $f\in B_\sigma^p$ if and only if f is an entire function satisfying

$$\|f(x+iy)\|_p \le \|f\|_p \exp(\sigma|y|), \quad z=x+iy,$$

where the norm on the left is taken with respect to x for any fixed y and

$$\|f\|_p=\left(\int_{-\infty}^{\infty}|f(x)|^p\,dx\right)^{1/p}<\infty, \quad \text{if } 1\le p<\infty$$

and

$$\|f\|_\infty = \text{ess.sup}_{x\in\mathbb{R}}|f(x)|<\infty, \text{ if } p=\infty.$$

Unlike the spaces $L^p(\mathbb{R})$, the spaces B_σ^p are closed under differentiation and the differentiation operator plays a vital role in their characterization. The Bernstein spaces have been characterized in a number of different ways and one can prove that the following are equivalent:

A) A function $f \in L^p(\mathbb{R})$ belongs to B^p_σ if and only if its distributional Fourier transform has support $[-\sigma, \sigma]$ in the sense of distributions.

B) Let $f \in C^\infty(\mathbb{R})$ be such that $f^{(n)} \in L^p(\mathbb{R})$ for all $n = 0, 1, \cdots$, and some $1 \le p \le \infty$, then $f \in B^p_\sigma$ if and only if f satisfies the Bernstein's inequality [12, p. 116]

$$\left\| f^{(n)} \right\|_p \le \sigma^n \left\| f \right\|_p, \quad n = 0, 1, 2, \cdots; 1 \le p \le \infty. \tag{1.2}$$

C) Let $f \in C^\infty(\mathbb{R})$ be such that $f^{(n)} \in L^p(\mathbb{R})$ for all $n = 0, 1, \cdots$, and some $1 \le p \le \infty$. Then

$$\lim_{n\to\infty} \left\| f^{(n)} \right\|_p^{1/n} \le \infty, \quad \text{exists}$$

and $f \in B^p_\sigma$ if and only if $\lim_{n\to\infty} \left\| f^{(n)} \right\|_p^{1/n} = \sigma < \infty$.

D) Let $f \in C^\infty(\mathbb{R})$ be such that $f \in L^p(\mathbb{R})$ for some $1 \le p \le \infty$. Then $f \in B^p_\sigma$ if and only if it satisfies the Riesz interpolation formula

$$f^{(1)}(x) = \frac{\sigma}{\pi^2} \sum_{k\in\mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} f\left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \tag{1.3}$$

where the series converges in $L^p(\mathbb{R})$. Because this characterization is not well known, we will prove it. We have

$$\left\| f^{(1)} \right\|_p = \left\| \frac{\sigma}{\pi^2} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{(k-1/2)^2} f\left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p \tag{1.4}$$

$$\le \frac{\sigma}{\pi^2} \sum_{k\in\mathbb{Z}} \frac{1}{(k-1/2)^2} \left\| f\left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p. \tag{1.5}$$

But

$$\left\| f\left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p = \left\| f(x) \right\|_p,$$

and $\sum_k \frac{1}{(k-1/2)^2} = \pi^2$; hence

$$\left\| f^{(1)} \right\|_p \le \sigma \left\| f \right\|_p,$$

which shows that $f^{(1)} \in L^p(\mathbb{R})$. Now by differentiating the Riesz interpolation formula once more, we obtain formally

$$f^{(2)}(x) = \frac{\sigma}{\pi^2} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{(k-1/2)^2} f^{(1)}\left(x + \frac{\pi}{\sigma}(k - 1/2)\right),$$

but the series on the right-hand side converges because

$$\left\| \frac{\sigma}{\pi^2} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{(k-1/2)^2} f^{(1)}\left(x + \frac{\pi}{\sigma}(k-1/2)\right)\right\|_p \tag{1.6}$$

$$\leq \frac{\sigma}{\pi^2} \sum_{k\in\mathbb{Z}} \frac{1}{(k-1/2)^2} \left\| f^{(1)}\left(x + \frac{\pi}{\sigma}(k-1/2)\right)\right\|_p \tag{1.7}$$

$$\leq \sigma \left\| f^{(1)} \right\|_p . \tag{1.8}$$

Therefore, it follows that

$$\left\| f^{(2)} \right\|_p \leq \sigma \left\| f^{(1)} \right\|_p ,$$

which shows that $f^{(2)} \in L^p(\mathbb{R})$ and in addition

$$\left\| f^{(2)} \right\|_p \leq \sigma^2 \left\| f \right\|_p .$$

Now an induction argument shows that

$$\left\| f^{(n)} \right\|_p \leq \sigma^n \left\| f \right\|_p , \qquad \text{for all } n = 1, 2, \cdots ,$$

that is f satisfies the Bernstein inequality; hence, $f \in B^p_\sigma$. The converse is shown in [12].

The space B^2_σ is the Paley-Wiener space PW_σ. Hence, a function f in $L^2(\mathbb{R})$ belongs to the Paley-Wiener space $PW_\sigma(\mathbb{R})$ if and only if

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \hat{f}(\omega) e^{it\omega}\, d\omega.$$

In other words, $f \in L^2(\mathbb{R})$ belongs to $PW_\sigma(\mathbb{R})$ if it has an extension to the complex plane as an entire function of exponential type not exceeding σ. We also have a generalization of the WSK sampling theorem.

Theorem 3. *Let $f \in B^p_\sigma$, $1 \leq p < \infty$ and $0 < \sigma$. Then*

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)} \qquad (t \in \mathbb{R}).$$

The result is not true for $p = \infty$. For, $f(t) = \sin(\sigma t)$ vanishes at all t_k but it is not identically zero. However, the theorem is true for $f \in B^\infty_{\sigma-\delta}$, $0 < \delta < \sigma$.

Now we introduce the Zakai Space of Bandlimited Functions [21].

Definition 4. *A function f is said to be bandlimited with bandwidth σ in the sense of Zakai if it is entire of exponential type satisfying $|f(z)| \leq Be^{A|z|}$ and*

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} dx < \infty, \tag{1.9}$$

for some $0 < A, B$, where σ is the infimum of all W such that the Fourier transform of $(f(z) - f(0))/z$ vanishes outside $(-W, W)$.

It should be noted that if f is σ-bandlimited in the sense of Zakai, then $g(z) = (f(z) - f(0))/z \in PW_\sigma$. Let us denote the Zakai space by H_σ. Clearly, $B_\sigma^\infty \subset H_\sigma$ since if f is bounded on the real line, the integral in Eq. (1.9) is finite. Examples of functions in H_σ are $\sin(\sigma z)$ and $Si(t) = \int_0^t \frac{\sin x}{x} dx$, which can be written as a Fourier transform of a function with compact support, namely, $F(\omega) = \left(\sqrt{2\pi}/2i\omega\right) \chi_{(-1,1)}$, since

$$Si(t) = \frac{1}{2i} \int_{-1}^{1} \frac{1}{\omega} e^{i\omega t} d\omega.$$

The function $F(\omega)$ is not in L^p for any $1 \le p$, and the Fourier transform is taken in the sense of distributions.

Another generalization of the class of bandlimited functions is the class H_σ^k which is defined as follows. Let H_σ^k be the class of all entire functions of exponential type satisfying

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)^k} dt$$

and $|f(z)| \le C(1+|z|)^k \exp(\sigma|\Im z|)$. Then $f \in H_\sigma^k$ is equivalent to either of the following

(1)

$$f(t) = \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} t^j + \frac{t^k}{k!} g(t), \quad g \in B_\sigma^2$$

(2) f is a temperate distribution whose Fourier transform has support in $[-\sigma, \sigma]$.

The class H_σ^0 is the same as B_σ^2 and H_σ^1 is the same as the Zakai class H_σ. The class

$$H_\sigma^\infty = \cup_{k=0}^\infty H_\sigma^k$$

consists of all functions that are temperate distributions having Fourier transform with support in $[-\sigma, \sigma]$. Moreover, $f \in H_\sigma^\infty$ is such that

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)^k} dt < \infty$$

if and only if the order of its distributional Fourier transform is less than or equal to k.

Moreover, the following sampling theorem holds [10]:

Theorem 5. *Let* $f \in H_\sigma^k$,

$$0 < \tau < \pi/\sigma, \quad \textit{and } 0 < \beta < \frac{\pi}{k}(\frac{1}{\tau} - \frac{\sigma}{\pi}).$$

Then

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\tau) \frac{\sin[(\pi/\tau)(t-n\tau)] \sin^k[\beta(t-n\tau)]}{[(\pi/\tau)(t-n\tau)][\beta(t-n\tau)]^k}$$

2. Bandlimited Vectors in a Hilbert Space

In this section we introduce a space of Paley-Wiener vectors in a Hilbert space H. As can be seen from (1.2) and (1.3) the differentiation operator plays a vital role in the characterization of classical Paley-Wiener space. In our abstract setting, the differentiation operator will be replaced by a self-adjoint operator D in a Hilbert space H. Furthermore, from the abstract setting we will be able to derive a new characterization of the classical Paley-Wiener space that connects Paley-Wiener functions to analytic solutions of a Cauchy problem involving Schrödinger equation.

According to the spectral theory [6], there exist a direct integral of Hilbert spaces $A = \int A(\lambda)dm(\lambda)$ and a unitary operator $\mathcal{F}_D$ from H onto A, which transforms the domain $\mathcal{D}_k$ of the operator D^k onto $A_k = \{a \in A | \lambda^k a \in A\}$ with norm

$$\|a(\lambda)\|_{A_k} = \left(\int_{-\infty}^{\infty} \lambda^{2k} \|a(\lambda)\|^2_{A(\lambda)} dm(\lambda) \right)^{1/2}$$

and $\mathcal{F}_D(Df) = \lambda(\mathcal{F}_D f), f \in \mathcal{D}_1$.

Definition 6. *The unitary operator $\mathcal{F}_D$ will be called the Spectral Fourier transform and $a = \mathcal{F}_D f$ will be called the Spectral Fourier transform of $f \in H$.*

Definition 7. *We will say that a vector f in H belongs to the space $PW_\omega(D)$ if its Spectral Fourier transform $\mathcal{F}_D f = a$ has support in $[-\omega, \omega]$.*

The next proposition is evident.

Proposition 8. *The following properties hold true:*
a) The linear set $\bigcup_{\omega>0} PW_\omega(D)$ is dense in H.
b) The set $PW_\omega(D)$ is a linear closed subspace in H.

In the following theorems we describe some basic properties of Paley-Wiener vectors and show that they share similar properties to those of the classical Paley-Wiener functions. The next theorem, whose proof can be found in [15], shows that the space $PW_\omega(D)$ has properties (A) and (B). See also [14, 16]

Theorem 9. *The following conditions are equivalent:*
1)$f \in PW_\omega(D)$;
2) f belongs to the set

$$\mathcal{D}_\infty = \bigcap_{k=1}^{\infty} \mathcal{D}_k,$$

and for all $k \in \mathbb{N}$, the following Bernstein inequality holds

$$\|D^k f\| \leq \omega^k \|f\|; \tag{2.1}$$

3) for every $g \in H$ the scalar-valued function $< e^{itD} f, g >$ of the real variable $t \in \mathbb{R}^1$ is bounded on the real line and has an extension to the complex plane as an entire function of exponential type ω;

4) the abstract-valued function $e^{itD} f$ is bounded on the real line and has an extension to the complex plane as an entire function of exponential type ω.

To show that the space $PW_\omega(D)$ has property (C), we will need the following Lemma.

Lemma 10. *Let D be a self-adjoint operator in a Hilbert space H and $f \in \mathcal{D}_\infty$. If for some $\omega > 0$ the upper bound*

$$\sup_{k \in N} (\omega^{-k} \|D^k f\|) = B(f, \omega), \tag{2.2}$$

is finite, then $f \in PW_\omega$ and $B(f,\omega) \leq \|f\|$.

Definition 11. *Let $f \in PW_\omega(D)$ for some positive number ω. We denote by ω_f the smallest positive number such that the interval $[-\omega_f, \omega_f]$ contains the support of the Spectral Fourier transform $\mathcal{F}_D f$.*

It is easy to see that $f \in PW_{\omega_f}(D)$ and that $PW_{\omega_f}(D)$ is the smallest space to which f belongs among all the spaces $PW_\omega(D)$. For,

$$\begin{aligned} \|D^k f\| &= \left(\int_{-\infty}^{\infty} \lambda^{2k} \|a(\lambda)\|^2_{A(\lambda)} dm(\lambda) \right)^{1/2} \\ &= \left(\int_{-\omega_f}^{\omega_f} \lambda^{2k} \|a(\lambda)\|^2_{A(\lambda)} dm(\lambda) \right)^{1/2} \leq \omega_f^k \left(\|a\| \right)_A . \end{aligned}$$

Hence, by Theorem 9, $f \in PW_{\omega_f}(D)$. Moreover, if $f \in PW_\omega(D)$ for some $\omega < \omega_f$, then from Definition 7 the spectral Fourier transform of f has support in $[-\omega, \omega]$ which contradicts the definition of $[-\omega_f, \omega_f]$. The next theorem shows that the space $PW_\omega(D)$ has property (C).

Theorem 12. *Let $f \in H$ belong to the space $PW_\omega(D)$, for some $0 < \omega < \infty$. Then*

$$d_f = \lim_{k \to \infty} \|D^k f\|^{1/k} \tag{2.3}$$

exists and is finite. Moreover, $d_f = \omega_f$. Conversely, if $f \in \mathcal{D}_\infty$ and $d_f = \lim_{k\to\infty} \|D^k f\|^{1/k}$, exists and is finite, then $f \in PW_{\omega_f}$ and $d_f = \omega_f$.

Finally, we have another characterization of the space $PW_\omega(D)$. Consider the Cauchy problem for the abstract Schrödinger equation

$$\frac{\partial u(t)}{\partial t} = iDu(t), u(0) = f, i = \sqrt{-1}, \tag{2.4}$$

where $u : I\!R \to H$ is an abstract function with values in H.

The next theorem gives another characterization of the space $PW_\omega(D)$, from which we obtain a new characterization of the space PW_ω.

Theorem 13. *A vector $f \in H$, belongs to $PW_\omega(D)$ if and only if the solution $u(t)$ of the corresponding Cauchy problem (2.4) has the following properties:*

1) as a function of t, it has an analytic extension $u(z), z \in \mathbb{C}$ to the complex plane $\mathbb{C}$ as an entire function;

2) it has exponential type ω in the variable z, that is

$$\|u(z)\|_H \leq e^{\omega|z|} \|f\|_H .$$

and it is bounded on the real line.

3. Applications To Sturm-Liouville Operators

In this section we apply the general results obtained in previous sections to specific examples involving differential operators. We specify our characterization of Paley-Wiener functions that are defined by integral transforms other than the Fourier transform. For related material, see [20, 23].

3.1. **Integral Transforms Associated with Sturm-Liouville Operators on a Half-line.** Consider the singular Sturm-Liouville problem on the half line

$$L_x y = Ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda y, \quad 0 \le x < \infty, \tag{3.1}$$

with

$$y(0)\cos\alpha + y'(0)\sin\alpha \quad = \quad 0, \qquad \text{for some } 0 \le \alpha < 2\pi, \tag{3.2}$$

and q is assumed to be real-valued.

Let $\phi(x,\lambda)$ be a solution of equation (3.1) satisfying the initial conditions $\phi(0,\lambda) = \sin\alpha, \quad \phi'(0,\lambda) = -\cos\alpha$. Clearly, $\phi(x,\lambda)$ is a solution of (3.1) and (3.2). It is easy to see that $\phi(x,\lambda)$ and $\phi'(x,\lambda)$ are bounded as functions of x for $\lambda > 0$ [17]. It is known [17, 11] that if $f \in L^2(\mathbb{R}^+)$, then

$$F(\lambda) = \hat{f}(\lambda) = \int_0^\infty f(x)\phi(x,\lambda)\,dx \tag{3.3}$$

is well-defined (in the mean) and belongs to $L^2(\mathbb{R}, d\rho)$, and

$$f(x) = \int_{-\infty}^\infty \hat{f}(\lambda)\phi(x,\lambda)\,d\rho(\lambda), \tag{3.4}$$

with

$$\|f\|_{L_2(\mathbb{R}^+)} = \|\hat{f}\|_{L_2(\mathbb{R},d\rho)}. \tag{3.5}$$

The measure $\rho(\lambda)$ is called the spectral function of the problem. In many cases of interest the support of $d\rho$ is $\mathbb{R}^+$. In this case the transform (3.4) takes the form

$$f(x) = \int_0^\infty \hat{f}(\lambda)\phi(x,\lambda)\,d\rho(\lambda), \tag{3.6}$$

and the Parseval equality (3.5) becomes $\|f\|_{L^2(\mathbb{R}^+)} = \|\hat{f}\|_{L^2(\mathbb{R}^+,d\rho)}$. Hereafter, we assume that q is real-valued, bounded and $C^\infty(\mathbb{R}^+)$. Because we are interested in the case where the spectrum of the problem is continuous, we shall focus on the case in which the differential equation (3.1) is in the limit-point case at infinity. Restrictions on q to guarantee continuous spectra can be found in [11, 17]. The condition $q \in L^1(R^+)$ will suffice. In such a case the problem (3.1) and (3.2) is self-adjoint [7, p. 158,], i.e., $\langle Lf, g\rangle = \langle f, Lg\rangle$ for all $f, g \in \mathcal{D}_L$, where $\mathcal{D}_L$ consists of all functions u satisfying

(1) u is differentiable and u' is absolutely continuous on $0 \le x \le b$ for all $b < \infty$,
(2) u and Lu are in $L^2(\mathbb{R}^+)$,

(3) $u(0)\cos\alpha + u'(0)\sin\alpha = 0.$

Now consider the initial-boundary-value problem involving the Schrodinger equation

$$i\frac{\partial u(x,t)}{\partial t} = -L_x u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - q(x)u(x,t),\ 0 \le x < \infty,\ t \ge 0, \tag{3.7}$$

with

$$u(x,0) = f(x) \tag{3.8}$$

and

$$u(0,t)\cos\alpha + \frac{\partial u(0,t)}{\partial x}\sin\alpha = 0 \qquad \text{for all } 0 < t, \tag{3.9}$$

where $f(x) \in L^2(\mathbb{R}^+)$,

Set

$$u(x,t) = \int_{-\infty}^{\infty} e^{i\lambda t}\hat{f}(\lambda)\phi(x,\lambda)d\rho(\lambda)\,, \tag{3.10}$$

Formally, if f and Lf are in $L^2(\mathbb{R}^+)$, then

$$i\frac{\partial u(x,t)}{\partial t} = \int_{-\infty}^{\infty}(-\lambda)e^{i\lambda t}\hat{f}(\lambda)\phi(x,\lambda)d\rho(\lambda) = -L_x u(x,t),$$

$$u(x,0) = \int_{-\infty}^{\infty}\hat{f}(\rho)\varphi(x,\lambda)d\rho = f(x),$$

and

$$u(0,t)\cos\alpha + \frac{\partial u(0,t)}{\partial x}\sin\alpha = 0. \tag{3.11}$$

Therefore, $u(x,t)$ is a solution of the initial-boundary-value problem (3.7) -(3.9), in the sense of $L^2(\mathbb{R}^+)$.

Definition 14. *We say that $f(x) \in L^2(\mathbb{R}^+)$ is bandlimited with bandwidth ω or $f \in PW_\omega(L)$ if its spectral Fourier transform $\hat{f}(\lambda)$ according to Definition 6, has support $[-\omega,\omega]$, where L is given by (3.1) and (3.2).*

It follows from the definition that if f is bandlimited to $[-\omega,\omega]$, then

$$f(x) = \int_{-\infty}^{\infty}\hat{f}\,(\lambda)\,\varphi(x,\lambda)d\rho = \int_{-\omega}^{\omega}\hat{f}(\lambda)\varphi(x,\lambda)d\rho.$$

In order to apply Theorem 13, we have to define the domain $\mathcal{D}^\infty$ on which all iterations of L are self-adjoint. It is easy to see that $\mathcal{D}^\infty$ consists of all functions u satisfying the following conditions:

i) u is infinitely differentiable on $\mathbb{R}^+$,
ii) $L^k u$ is in $L^2(\mathbb{R}^+)$, for all $k = 0, 1, 2, \cdots$,
iii) $\left(L^k u\right)(0)\cos\alpha + \left(\frac{d}{dx}L^k u\right)(0)\sin\alpha = 0.$

Hence, if f is bandlimited according to Definition 14, $L^n f(x) = \int_{-\omega}^{\omega} \hat{f}(\lambda)\,(\lambda)^n\,\varphi(x,\lambda)d\rho$, which exists for all $n = 0, 1, 2, \cdots$. Thus, by Parseval's equality

$$\|L^n f\|^2_{L^2(\mathbb{R}^+)} = \int_{-\omega}^{\omega} \left|\hat{f}(\lambda)\right|^2 \lambda^{2n} d\rho \le \omega^{2n} \int_{-\infty}^{\infty} \left|\hat{f}\,(\lambda)\right|^2 d\rho$$
$$= \omega^{2n} \|\hat{f}\|^2_{L^2(\mathbb{R},d\rho)} = \omega^{2n}\,\|f\|^2_{L^2(\mathbb{R}^+)}\,. \tag{3.12}$$

That is,

$$\|L^n f\| \le \omega^n\,\|f\|\,, n = 0, 1, 2, \cdots , \tag{3.13}$$

which is a generalization of Bernstein inequality (1.2).

Theorem 15. *A function $f \in L^2(\mathbb{R}^+)$ is bandlimited in the sense of Definition 14 with bandwidth ω if and only if the solution $u(x,t)$ of the initial-boundary-value problem (3.7) - (3.9) with $f \in \mathcal{D}^\infty$ has the following properties:*

(1) *As a function of t it has analytic extension $u(x,z)$ to the complex plane as entire function of exponential type ω,*
(2) *It satisfies the estimate*

$$\|u(.,z)\|_{L^2(\mathbb{R}^+)} \le e^{\omega|\Im z|}\,\|f\|_{L^2(\mathbb{R}^+)} \le e^{\omega|z|}\,\|f\|_{L^2(\mathbb{R}^+)}\,.$$

In particular, $u(x,z)$ is bounded on the real t-line.

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Ahmed I. Zayed
Department of Mathematical Sciences,
DePaul University,
Chicago, IL 60614, USA
azayed@math.depaul.edu

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