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SATURATED NEIGHBOURHOOD MODELS OF MONOTONIC MODAL LOGICS

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Abstract. In this paper we shall introduce the notions of point-closed, point-compact, and m-saturated monotonic neighbourhood models. We will give some characterizations, and we will prove that the ultrafilter extension and the valuation extension of a model are m-saturated.

1. Introduction

Monotonic neighbourhood semantics (cf. [1] and [6]) is a generalization of Kripke semantics. It is also the standard tool for reasoning about monotonic modal logics in which some (Kripke valid) principles such as $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$, do not hold. A *monotonic neighbourhood model*, or monotonic model, is a structure $\mathcal{M} = \langle X, R, V \rangle$ where $R \subseteq X \times \mathcal{P}(X)$, $R(x)$ is closed under supersets for each $x \in X$, and V is a valuation defined on X.

The main objective of this paper will be the identification and study of some properties of saturation of monotonic models. It is also intented to prove that the ultrafilter and valuation extension of a monotonic model is m-saturated. We will define the image-compact, point-compact, point-closed, and modally saturated (or m-saturated) models. These notions are defined in topological terms. For each monotonic model $\mathcal{M} = \langle X, R, V \rangle$ we will define a topology $\mathcal{T}_{D_V}$ in the set

$$\mathcal{K}_R = \{Y \subseteq X : \exists x \in X \ (Y \in R(x))\},$$

using the Boolean algebra $D_V = \{V(\varphi) : \varphi \in Fm\}$, and taking as sub-basis the collection of all sets of the form

$$L_{V(\varphi)} = \{Y \in \mathcal{K}_R : Y \cap V(\varphi) \neq \emptyset\}.$$

The topological space $\mathcal{K}_R = \langle \mathcal{K}_R, \mathcal{T}_{D_V} \rangle$ is called the *hyperspace* of $\langle X, D_V \rangle$ relative to $\mathcal{K}_R$. The notions of point-compact, point-closed, and m-saturated monotonic models are defined relative to this space. For instance, a model $\mathcal{M}$ is m-saturated if $R(x)$ is a compact subset of $\mathcal{K}_R$ for each $x \in X$, and for each $Y \in R(x)$, there exists a compact set of Z of $\langle X, D_V \rangle$ such that $Z \subseteq Y$ and $(x, Z) \in R$. With this notion of m-saturation we will be able to prove that the ultrafilter extension of a monotonic model is m-saturated. This question has already been addressed in [6], but with a different notion of m-saturation.

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In Section 2 we will recall the principal results on the relational and algebraic semantics for monotonic modal logic. In Section 3 we will introduce the notions of compact, image-compact, point-compact, point-closed, and m-saturated monotonic models. We will give a characterizations of point-compact models. We will show that some of the notions introduced are invariant by surjective bounded morphisms. Section 4 is the core of this paper. We shall prove that for a monotonic model $\mathcal{M}$ there exists at least two saturated extensions: the *valuation extension* and the *ultrafilter extension.* The valuation extension of a model is a bounded image of the ultrafilter extension, and as the property of m-saturation is invariant under surjective bounded morphism, we will deduce that the valuation extension is also m-saturated. Saturated extensions of monotonic models may be seen as a completation of the underlying frame structure. For Kripke models, the saturated extensions are modally saturated structures, which implies that modally states are bisimilar (see [7] for Kripke models and [6] for monotonic models).

2. Preliminaries

A *monotonic algebra* is a pair $A = \langle A, \Diamond \rangle$, where A is a Boolean algebra, and $\Diamond : A \to A$ is a monotonic function, i.e. if $a \leq b$ then $\Diamond(a) \leq \Diamond(b)$, for all $a, b \in A$. The dual operator $\Box$ is defined as $\Box a = \neg \Diamond \neg a$. The filter (ideal) generated by a set $H \subseteq A$ will be denoted by $[H)$ $((H])$. The set of all prime filters or *ultrafilters* of A is denoted by $\mathrm{Ul}(A)$.

Given a set X, we denote by $\mathcal{P}(X)$ the powerset of X, and for a subset Y of X, we write Y^c for the complement $X \backslash Y$ of Y in X. We will call a *space* a pair $\mathcal{X} = \langle X, D \rangle$, where X is a set, and D a subalgebra of the Boolean algebra of $\mathcal{P}(X)$. We note that D is a basis of a topology $\mathcal{T}_D$ on X whose open sets are the unions of subsets of D. All the elements of D are clopen (closed and open) subsets of X, because D is a Boolean algebra, but an arbitrary clopen set does not to be need an element of D. Given a space $\langle X, D \rangle$ and $Y \subseteq X$, we will use the notation $cl(Y)$ to express the closure of Y. The set of all closed subsets (compact subsets) of $\mathcal{X}$ will be denoted by $\mathcal{C}(X)$ $(\mathcal{K}(X))$. We note that $\mathcal{C}(X)$ and $\mathcal{K}(X)$ are posets under the inclusion relation. Some topological properties of $\mathcal{X}$ can be characterized in terms of the map $\varepsilon_D : X \to \mathrm{Ul}\,(D)$ given by $\varepsilon_D\,(x) = \{U \in D : x \in U\}$. The map ε_D is called the *insertion map* in [2]. A space $\mathcal{X}$ is called a *Boolean space* if it is *compact and totally disconnected.* If $\mathcal{X}$ is a Boolean space, then the family D of clopen subsets is a basis for $\mathcal{X}$.

To each Boolean algebra A we can associate a Boolean space whose points are the elements of $\mathrm{Ul}\,(A)$ with the topology determined by the basis $\beta_A\,(A) = \{\beta_A\,(a) : a \in A\}$, where $\beta_A\,(a) = \{x \in \mathrm{Ul}\,(A) : a \in x\}$. It is known that $\beta_A\,(A)$ is a Boolean subalgebra of $\mathcal{P}(\mathrm{Ul}\,(A))$. By the explanation above we have that, if $\mathcal{X}$ is a Boolean space, then $\mathcal{X} \cong \mathrm{Ul}\,(D)$, by means of the map ε_D, and if A is a Boolean algebra, then $A \cong \beta_A\,(A)$, by means of the map β_A. Moreover, it is known that the map $F \to \hat{F} = \{x \in \mathrm{Ul}\,(A) : F \subseteq x\}$ establishes a bijective correspondence between the lattice of all filters of A and the lattice $\mathcal{C}(\mathrm{Ul}\,(A))$ of all closed subsets of $\mathrm{Ul}\,(A)$.

Definition 1. Let $\mathcal{X} = \langle X, D \rangle\rangle$ be a space. The *lower topology* $\mathcal{T}_D$ on a subset $\mathcal{K}$ of $\mathcal{P}(X)$ has as sub-base the collection of all sets of the form $L_U = \{Y \in \mathcal{K} : Y \cap U \neq \emptyset\}$, for $U \in D$. The pair $\mathcal{K} = \langle \mathcal{K}, \mathcal{T}_D \rangle$ is called the *hyperspace* of $\mathcal{X}$ relative to $\mathcal{K}$.

Let X be a set. A *neighbourhood relation,* or *multirelation,* defined on X is a relation $R \subseteq X \times \mathcal{P}(X)$.

Definition 2. *A monotonic neighbourhood frame,* or monotonic frame*,* is a structure $\mathcal{F} = \langle X, R \rangle$ where R is a multirelation on X such that $R(x) = \{Z \in \mathcal{P}(X) : (x, Z) \in R\}$ is an increasing subset of $\mathcal{P}(X)$, for all $x \in X$.

Every monotonic frame $\mathcal{F}$ gives rise to a monotonic algebra of sets. Consider the monotonic map $\Diamond_R : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by:

$$\Diamond_R(U) = \{x \in X : \exists Y \subseteq X \ (Y \subseteq U \text{ and } (x, Y) \in R)\},$$

for each $U \in \mathcal{P}(X)$. It is clear that the pair $\langle \mathcal{P}(X), \Diamond_R \rangle$ is a monotonic algebra. Using the notation introduced in Definition 1 the map $\Diamond_R$ can be defined also as $\Diamond_R(U) = \{x \in X : R(x) \cap (L_{U^c})^c \neq \emptyset\}$, for each $U \in \mathcal{P}(X)$. The dual map $\Box_R$ is defined by

$$\Box_R(U) = \{x \in X : R(x) \subseteq L_U\},$$

for each $U \in \mathcal{P}(X)$.

Next we show that any monotonic algebra with monotone gives rise to a monotonic frame by invoking the basic Stone representation. In other words, we represent the elements of a monotonic algebra as subsets of some universal set, namely the set of all ultrafilters, and then define a multirelation over this universe.

Let $\langle A, \Diamond \rangle$ be a monotonic algebra. Let us define a multirelation $R_\Diamond \subseteq \mathrm{Ul}(A) \times \mathcal{P}(\mathrm{Ul}(A)$ by:

$$(x, Y) \in R_\Diamond \quad \Leftrightarrow \quad \exists F \in Fi(A) \ (\hat{F} \subseteq Y \text{ and } F \subseteq \Diamond^{-1}(x). \tag{2.1}$$

where $\hat{F} = \{y \in \mathrm{Ul}(A) : F \subseteq x\}$. We note that for any filter $F \in Fi(A)$, and for all $x \in \mathrm{Ul}(A)$,

$$(x, \hat{F}) \in R_\Diamond \text{ iff } F \subseteq \Diamond^{-1}(x).$$

Theorem 3. *Let* $\langle A, \Diamond \rangle$ *be a monotonic algebra. Then*

(1) $\langle \mathrm{Ul}(A), R_\Diamond \rangle$ *is a monotonic frame.*
(2) $\Diamond_{R_\Diamond}(\beta_A(a)) = \beta_A(\Diamond(a))$*, for all* $a \in A$.

Proof. (1) Clearly $R_\Diamond(x)$ is an increasing subset of $\mathcal{P}(\mathrm{Ul}(A))$, for each $x \in \mathrm{Ul}(A)$.

(2) We prove that for all $a \in A$, and for all $x \in \mathrm{Ul}(A)$,

$$\Diamond(a) \in x \text{ iff } \exists F \in F_i(A) : (x, \hat{F}) \in R_\Diamond \text{ and } a \in F.$$

If $\Diamond(a) \in x$, then $F(a) \subseteq \Diamond^{-1}(x)$. So, $(x, F(a)) \in R_\Diamond$. If there exists $F \in F_i(A)$ such that $(x, \hat{F}) \in R_\Diamond$ and $a \in F$, then $a \in F \subseteq \Diamond^{-1}(x)$. Thus, $\Diamond(a) \in x$. As consequence of this we have that $\Diamond_{R_\Diamond}(\beta_A(a)) = \beta_A(\Diamond a)$, for all $a \in A$. Now it is easy to prove $\Diamond_{R_\Diamond}(\beta_A(a)) = \Diamond_{R_\Diamond}(\beta_A(a))$. ∎

3. m-saturated models

Let us consider a propositional language $\mathcal{L}_\Diamond$ defined by using a denumerable set of propositional variables Var, the connectives $\vee$ and $\wedge$, the negation $\neg$, the modal connective $\Diamond$, and the propositional constant $\top$. We shall denote by $\Box$ the operator defined as $\Box p = \neg\Diamond\neg p$, for $p \in Var$. The set of all well formed formulas will be denoted by Fm.

Definition 4. A *monotonic model* in the language $\mathcal{L}$ is a structure $\mathcal{M} = \langle X, R, V \rangle$, where $\langle X, R \rangle$ is a monotonic frame, and $V : Var \to \mathcal{P}(X)$ is a valuation.

Every valuation can be extended to Fm by means of the following clauses:

(1) $V(\top) = X$,
(2) $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
(3) $V(\neg\varphi) = X - V(\varphi) = V(\varphi)^c$,
(4) $V(\Diamond\varphi) = \Diamond_R(V(\varphi))$.

The notions of truth at a point, validity in a model and validity in a frame for formulas are defined as is usual. A formula φ is *valid at point* x in a model $\mathcal{M}$, in symbols $\mathcal{M} \vDash_x \varphi$ if $x \in V(\varphi)$. The formula φ is *valid in a model* $\mathcal{M}$, in symbols $\mathcal{M} \vDash \varphi$, if $V(\varphi) = X$.

Let $\mathcal{M}$ be a monotonic model. As $D_V = \{V(\varphi) : \varphi \in Fm\}$ is a Boolean algebra of sets, and $V(\Diamond\varphi) = \Diamond_R(V(\varphi)) \in D_V$, for every $\varphi \in Fm$, $\langle D_V, \Diamond_R \rangle$ is a monotonic subalgebra of the algebra $\langle \mathcal{P}(X), \Diamond_R \rangle$. We shall denote by $\mathcal{X}_V = \langle X, D_V \rangle$ the space generated taking D_V as a basis for a topology defined in X. Let

$$\mathcal{K}_R = \{Y \subseteq X : \exists x \in X \ ((x, Y) \in R)\}$$

the range of the multirelation R. Let $\mathcal{K}_R = \langle \mathcal{K}_R, \mathcal{T}_{D_V} \rangle$ the hyperspace of $\mathcal{X}_V = \langle X, D_V \rangle$ relative to $\mathcal{K}_R$. Recall that $\mathcal{K}(\mathcal{X}_V)$ denotes the set of all compact subsets of $\mathcal{X}_V$.

Definition 5. Let $\mathcal{M} = \langle X, R, V \rangle$ be a model. We shall say that:

(1) $\mathcal{M}$ is *compact* if the space $\mathcal{X}_V$ is compact,
(2) $\mathcal{M}$ is *image-compact* if for all $x \in X$ and for all $Y \in R(x)$, there exists $Z \in \mathcal{K}(\mathcal{X}_V)$ such that $Z \subseteq Y$ and $(x, Z) \in R$.
(3) $\mathcal{M}$ is *point-compact* if $R(x)$ is compact subset in the topological space $\mathcal{K}_R$, for each $x \in X$,
(4) $\mathcal{M}$ is *point-closed*, if $R(x)$ is a closed subset of the space $\mathcal{K}_R$, for each $x \in X$.
(5) $\mathcal{M}$ is *modally saturated (or m-saturated)* if it is image-compact and point-compact.

Remark 6. In [6] Hansen defines the notion of m-saturated model as follows:
A model $\mathcal{M} = \langle X, R, V \rangle$ is m-saturated if the following conditions hold:

(m1) For any $\Gamma \subseteq Fm$, $x \in X$ and $Y \subseteq X$ such that $(x, Y) \in R$, if $\bigcap\{V(\varphi) : \varphi \in \Gamma_0\} \cap Y \neq \emptyset$, for all finite subset Γ_0 of Γ, then $\bigcap\{V(\varphi) : \varphi \in \Gamma\} \cap Y \neq \emptyset$.

(m2) For any $\Gamma \subseteq Fm$, and $x \in X$, if for every finite subset Γ_0 of Γ there exists an $Y \in R(x)$ such that $Y \subseteq \bigcap \{V(\varphi) : \varphi \in \Gamma_0\}$, then there exists an $Z \in R(x)$ such that $Z \subseteq \bigcap \{V(\varphi) : \varphi \in \Gamma\}$.

The condition (m2) is equivalent to say that the model $\mathcal{M}$ is point-compact, as we will see. But the condition (1) is not equivalent to the notion of image-compact. Hansen's definition has the disadvantage that the ultrafilter extension of a model (see definition 15) fails to meet the condition (m1). Now we will see that the condition (m2) is equivalent to require that $\mathcal{M}$ should be point-compact.

Proposition 7. Let $\mathcal{M}$ be a model. Then $\mathcal{M}$ is *point-compact* iff it satisfies the condition (m2).

Proof. $\Rightarrow$) Let $\Gamma \subseteq Fm$ and $x \in X$, such that for every finite subset Γ_0 of Γ there exists an $Y \in R(x)$ such that $Y \subseteq \bigcap \{V(\varphi) : \varphi \in \Gamma_0\}$. Suppose that $Y \nsubseteq \bigcap \{V(\varphi) : \varphi \in \Gamma\}$, for any $Y \in R(x)$. Then for each $Y \in R(x)$ there exists $\varphi_y \in \Gamma$ such that $Y \cap V(\neg\varphi_y) \neq \emptyset$. So,

$$R(x) \subseteq \bigcup \left\{ L_{V(\neg\varphi_y)} : \varphi_y \in \Gamma \right\}.$$

As $R(x)$ is a compact subset of $\langle \mathcal{K}_R, \mathcal{T}_{D_V} \rangle$, there exists a finite subset $\{\varphi_{y_1}, ..., \varphi_{y_n}\} \subseteq \Gamma$ such that

$$R(x) \subseteq L_{V(\neg\varphi_{y_1})} \cup \ldots \cup L_{V(\neg\varphi_{y_n})}. \tag{3.1}$$

By hypothesis, there exists $Z \in R(x)$ such that $Z \subseteq V(\neg\varphi_{y_1}) \cap \ldots \cap V(\neg\varphi_{y_n})$, which is a contradiction to (3.1). Thus, there exists an $Z \in R(x)$ such that $Z \subseteq \bigcap \{V(\varphi) : \varphi \in \Gamma\}$.

$\Leftarrow$) Let $x \in X$. We prove that $R(x)$ is compact subset in the topological space $\langle \mathcal{K}_R, \mathcal{T}_{D_V} \rangle$. Let $\Gamma \subseteq Fm$ such that $R(x) \subseteq \bigcup \left\{ L_{V(\varphi)} : \varphi \in \Gamma \right\}$. Suppose that

$$R(x) \nsubseteq \bigcup \left\{ L_{V(\varphi)} : \varphi \in \Gamma_i \right\},$$

for any finite subset Γ_i of Γ. So for each finite subset Γ_i of Γ there exists $Y_i \in R(x)$ such that $Y_i \subseteq \bigcap \{V(\neg\varphi) : \varphi \in \Gamma_i\}$. From condition (m1), there exists $Z \in R(x)$ such that $Z \subseteq \bigcap \{V(\neg\varphi) : \varphi \in \Gamma\}$. Thus, $Z \notin \bigcup \left\{ L_{V(\varphi)} : \varphi \in \Gamma \right\}$, which is a contradiction. □

Let $\mathcal{M}$ be a model. Recall that the insertion map ε_{D_V} of $\langle X, D_V \rangle$ is defined by $\varepsilon_{D_V}(x) = \{V(\varphi) : x \in V(\varphi)\}$, for each $x \in X$. We write ε_V by ε_{D_V}.

Lemma 8. *Let $\mathcal{M}$ be a model. If $\mathcal{M}$ is point-closed, then*

$$F_{\varepsilon_V(Y)} = \bigcap \{\varepsilon_V(y) : y \in Y\} \subseteq \Diamond_R^{-1}(\varepsilon_V(x)) \textit{ implies that } (x, Y) \in R,$$

for all $x \in X$ and for all $Y \subseteq X$.

Proof. Let $x \in X$ and $Y \subseteq X$ such that $F_{\varepsilon_V(Y)} \subseteq \Diamond_R^{-1}(\varepsilon_V(x))$. Suppose that $Y \notin R(x)$. Since R is point-closed, there exists $V(\varphi) \in D_V$ such that $Y \subseteq V(\varphi)$ and $x \notin \Diamond_R(V(\varphi))$. Since $F_{\varepsilon_V(Y)} \subseteq \Diamond_R^{-1}(\varepsilon_V(x))$, there exists $y \in Y$ such that $y \notin V(\varphi)$, which is a contradiction, because $Y \subseteq V(\varphi)$. Therefore $(x, Y) \in R$. □

Let $\mathcal{M}$ be a model. Let us consider the following property:

(P): For all $x \in X$ and for all $Y \in \mathcal{P}(X)$, if $F_{\varepsilon_V(Y)} \subseteq \Diamond_R^{-1}(\varepsilon_V(x))$ then there exists $Z \subseteq X$ such that $(x, Z) \in R$ and $Z \subseteq cl(Y)$,

where $cl(Y)$ is the topological closure in the space $\mathcal{X}_V$.

Proposition 9. Let $\mathcal{M}$ be a model. If $\mathcal{M}$ is point-compact, then $\mathcal{M}$ satisfies the property **(P)**.

Proof. Let $x \in X$ and let $Y \in \mathcal{P}(X)$. Assume that

$$F_{\varepsilon_V(Y)} \subseteq \Diamond_R^{-1}(\varepsilon_V(x)). \tag{3.2}$$

Suppose that for all $Z_i \in R(x)$, $Z_i \nsubseteq cl(Y) = \bigcap \{V(\varphi) : Y \subseteq V(\varphi)\}$. So for each $Z_i \in R(x)$ there exists $V(\varphi_i) \in D_V$ such that $Y \subseteq V(\varphi_i)$ and $Z_i \nsubseteq V(\varphi_i)$, i.e. $Z_i \cap V(\neg\varphi_i) \neq \emptyset$. Thus,

$$R(x) \subseteq \bigcup \{L_{V(\neg\varphi_i)} : Y \subseteq V(\varphi_i)\}.$$

Since $R(x)$ is a compact subset of $\langle \mathcal{K}_R, \mathcal{T}_{D_V} \rangle$, there exists a some finite set of formulas $\{\varphi_1, ..., \varphi_n\}$ such that

$$R(x) \subseteq L_{V(\neg\varphi_1)} \cup \cdots \cup L_{V(\neg\varphi_n)}.$$

Then,

$$x \notin \Diamond_R (V(\varphi_1) \cap \ldots \cap V(\varphi_n)) = V(\Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n)),$$

and $Y \subseteq V(\Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n))$. By (3.2), $V(\varphi_1 \wedge \ldots \wedge \varphi_n) \in \Diamond_R^{-1}(\varepsilon_V(x))$, i.e., $x \in V(\Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n))$, which is a contradiction. Therefore, there exists $Z \in R(x)$ such that $Z \subseteq cl(Y)$. □

Proposition 10. Let $\mathcal{M}$ be a compact model. Then $\mathcal{M}$ satisfies the property (**P**) iff $\mathcal{M}$ is point-compact.

Proof. Suppose that $\mathcal{M}$ satisfies the property (**P**). Let $W \subseteq D_V$. Suppose that for every finite subset W_0 of W

$$R(x) \nsubseteq \bigcup \{L_{V(\varphi)} : V(\varphi) \in W_0\}. \tag{3.3}$$

We prove that $\bigcap \{V(\varphi)^c : V(\varphi) \in W\} \neq \emptyset$. Suppose the contrary. Then $X = \bigcup \{V(\varphi) : V(\varphi) \in W\}$. As $\mathcal{M}$ is compact, $X = V(\varphi_1) \cup \ldots \cup V(\varphi_n)$, for some $\{\varphi_1, ..., \varphi_n\}$. From (3.3), there exists $Y \in R(x)$ such that $Y \cap V(\varphi_1) = \emptyset, ..., Y \cap V(\varphi_n) = \emptyset$, i.e., $Y \cap (V(\varphi_1) \cup \ldots \cup V(\varphi_n)) = Y \cap X = \emptyset$, which is impossible. Thus, $\bigcap \{V(\varphi)^c : V(\varphi) \in W\} \neq \emptyset$. Let

$$Z = \bigcap \{V(\varphi)^c : V(\varphi) \in W\}.$$

It is clear that Z is a closed subset of $\mathcal{X}$, and by compacity, Z is compact. We prove that

$$\bigcap \{\varepsilon_V(z) : z \in Z\} \subseteq \Diamond_R^{-1}(\varepsilon_V(x)).$$

If $V(\psi) \in \bigcap \{\varepsilon_V(z) : z \in Z\}$, then $Z = \bigcap \{V(\varphi)^c : V(\varphi) \in W\} \subseteq V(\psi)$. By compacity, there exists a finite set $\{\varphi_1, ..., \varphi_n\}$ such that

$$V(\varphi_1)^c \cap \ldots \cap V(\varphi_n)^c \subseteq V(\psi).$$

Then $\Diamond_R(V(\varphi_1)^c \cap \ldots \cap V(\varphi_n)^c) \subseteq \Diamond_R(V(\psi))$. From (3.3), there exists $T \in R(x)$ such that $T \cap (V(\varphi_1) \cup \ldots \cup V(\varphi_n)) = \emptyset$, i.e., $T \subseteq V(\varphi_1)^c \cap \ldots \cap V(\varphi_n)^c \subseteq V(\psi)$. Thus, $x \in \Diamond_R(V(\psi))$. So,

$$\bigcap \{\varepsilon_V(z) : z \in Z\} \subseteq \Diamond_R^{-1}(\varepsilon_V(x)),$$

and by hypothesis there exists $Y \in R(x)$ such that $Y \subseteq cl(Z) = Z$. Then $Y \cap V(\varphi) = \emptyset$, for every $V(\varphi) \in W$. Thus,

$$R(x) \nsubseteq \bigcup \{L_{V(\varphi)} : V(\varphi) \in W\}.$$

The other direction is followed by Proposition 9. □

The maps between monotonic frames and monotonic models which preserve the modal structure will be referred to as bounded morphisms. These have previously been studied in [6] (see also [5]).

Definition 11. *([6]) A* *bounded morphism* between two monotonic models $\mathcal{M}_1$ and $\mathcal{M}_2$ is a function $f : X_1 \to X_2$ such that

(1) $f^{-1}(V_2(p) = V_1(p)$, for each propositional variable p,
(2) If $(x, Y) \in R_1$, then $(f(x), f(Y)) \in R_2$, and
(3) If $(f(x), Z) \in R_2$, then there exists $Y \subseteq X$ such that $(x, Y) \in R_1$ and $f(Y) \subseteq Z$.

It follows that truth of modal formulas is invariant under bounded morphisms ([6]).

Proposition 12. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be monotonic models. If $f : X_1 \to X_2$ is a bounded morphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ then for each formula φ, $f^{-1}(V_2(\varphi)) = V_1(\varphi)$.

The following technical lemma is needed in the next results.

Lemma 13. *Let $\mathcal{M}$ be a model. Then $\mathcal{X}_V$ is compact iff ε_V is surjective.*

Proof. $\Rightarrow$) Let $P \in \mathrm{Ul}(D_V)$. We prove that $\bigcap\{V(\varphi) : V(\varphi) \in P\} \neq \emptyset$. Let us suppose the opposite. Then $X = \bigcup \{V(\neg\varphi) : V(\varphi) \in P\}$. Since $\langle X, D_V\rangle$ is compact, $X = V(\neg\varphi_1) \cup \ldots \cup V(\neg\varphi_n)$, for some formulas $\varphi_1, \ldots, \varphi_n$. So, $V(\varphi_1 \wedge \ldots \wedge \varphi_n) = \emptyset \in P$, which is impossible. It follows that there exists $x \in \bigcap\{V(\varphi) : V(\varphi) \in P\}$. Now, it is easy to see that $\varepsilon_V(x) = P$.

$\Leftarrow$) Let $X = \bigcup\{V(\varphi) : \varphi \in \Gamma \subseteq Fm.\}$. Suppose that for any finite subset Γ_0 of Γ, $X \neq \bigcup\{V(\varphi) : \varphi \in \Gamma_0\}$. Let us consider the filter F of D_V generated by the set $\{V(\neg\varphi) : \varphi \in \Gamma\}$. It is not difficult to prove that F is proper. It follows that there exists an ultrafilter P of D_V such that $F \subseteq P$. Since the map ε_V is onto, there exists $x \in X$ such that $\varepsilon_V(x) = P$. Then, $x \in V(\neg\varphi)$ for all $\varphi \in \Gamma$. So, $x \in \bigcap_{\varphi\in\Gamma} V(\neg\varphi)$, which is a contradiction. Thus, there is a finite subset Γ_0 of Γ, such that $X = \bigcup\{V(\varphi) : \varphi \in \Gamma_0 \subseteq Fm.\}$. □

The concepts of compact and point-compact models are preserved by surjective bounded morphisms.

Proposition 14. Let $f : X_1 \to X_2$ be a bounded morphism between the monotonic models $\mathcal{M}_1$ and $\mathcal{M}_2$.

(1) If $\mathcal{M}_1$ is compact, then $\mathcal{M}_2$ is compact.
(2) If f is surjective and $\mathcal{M}_2$ is compact, then $\mathcal{M}_1$ is compact.
(3) If $\mathcal{M}_2$ is point-compact, then $\mathcal{M}_1$ is point compact.
(4) If f is surjective and $\mathcal{M}_1$ is point-compact, then $\mathcal{M}_2$ is point compact.

Proof. (1) By Lemma 13 we need to prove that the map $\varepsilon_{V_2} : X_2 \to \mathrm{Ul}(D_{V_2})$ is surjective. Let $Q \in \mathrm{Ul}(D_{V_2})$. Consider the set $Q' = \{V_1(\varphi) : V_2(\varphi) \in Q\}$. It is easy to prove that $Q' \in \mathrm{Ul}(D_{V_1})$. Since ε_{V_1} is surjective, there exists $x \in X_1$ such that $\varepsilon_{V_1}(x) = Q'$. Let $y = f(x)$. Then, it is easy to see that $\varepsilon_{V_2}(y) = Q$.

(2) By Lemma 13 we need to prove that the map $\varepsilon_{V_1} : X_1 \to \mathrm{Ul}(D_{V_1})$ is surjective. Let $P \in \mathrm{Ul}(D_{V_1})$. Consider $P' = \{V_2(\varphi) : V_1(\varphi) \in P\}$. It is easy to see that $P' \in \mathrm{Ul}(D_{V_2})$. As ε_{V_2} is surjective, there exists $y \in X_2$ such that $\varepsilon_{V_2}(y) = P'$. Since f is surjective, there exists $x \in X_1$ such that $f(x) = y$. We prove that $\varepsilon_{V_1}(x) = P$. For all $\varphi \in Fm$, $x \in V_1(\varphi) = f^{-1}(V_2(\varphi))$ iff $f(x) = y \in V_2(\varphi)$ iff $V_2(\varphi) \in \varepsilon_{V_2}(y) = P'$ iff $V_1(\varphi) \in P$. Thus, $\varepsilon_{V_1}(x) = P$.

(3) Let $x \in X_1$ and let $\Gamma \subseteq D_{V_1}$. Suppose that

$$R_1(x) \subseteq \bigcup \left\{L_{V_1(\varphi)} : V_1(\varphi) \in \Gamma\right\}.$$

Taking into account that $f^{-1}(V_2(\varphi)) = V_1(\varphi)$, for all $\varphi \in Fm$, it is easy to see that

$$R_2(f(x)) = R_2(y) \subseteq \bigcup \left\{L_{V_2(\varphi)} : V_1(\varphi) \in \Gamma\right\}.$$

As $\mathcal{M}_2$ is point-compact, there exists a finite set $\{V_2(\varphi_1), ..., V_2(\varphi_n)\}$ such that $R_2(y) \subseteq L_{V_2(\varphi_1)} \cup \ldots \cup L_{V_2(\varphi_1)}$. Let $Z \in R_1(x)$. Then $f(Z) \in R_2(y)$. So there exists $V_2(\varphi_i) \in \{V_2(\varphi_1), ..., V_2(\varphi_n)\}$ such that $Z \cap f^{-1}(V_2(\varphi_i)) = Z \cap V_1(\varphi_i) \neq \emptyset$. Thus $R_1(x) \subseteq L_{V_1(\varphi_1)} \cup \ldots \cup L_{V_1(\varphi_n)}$.

(4) Let $y \in X_2$ and let $\Gamma \subseteq D_{V_2}$. Suppose that

$$R_2(y) \subseteq \bigcup \left\{L_{V_2(\varphi)} : V_2(\varphi) \in \Gamma\right\}.$$

As f is surjective there exists $x \in X_1$ such that $f(x) = y$. We prove that

$$R_1(x) \subseteq \bigcup \left\{L_{f^{-1}(V_2(\varphi))} : V_2(\varphi) \in \Gamma\right\}. \tag{3.4}$$

Let $Z \in R_1(x)$. Since f is a bounded morphism, $f(Z) \in R_2(f(x)) = R_2(y)$. So, there exists $V_2(\varphi) \in \Gamma$ such that $f(Z) \cap V_2(\varphi) \neq \emptyset$, i.e. $Z \cap f^{-1}(V_2(\varphi)) \neq \emptyset$. Thus (3.4) is valid. As $\mathcal{M}_1$ is point-compact, there exists $\{V_2(\varphi_1), ..., V_2(\varphi_n)\} \subseteq \Gamma$ such that $R_1(x) \subseteq L_{f^{-1}(V_2(\varphi_1))} \cup \ldots \cup L_{f^{-1}(V_2(\varphi_n))}$. We prove that

$$R_2(y) \subseteq L_{V_2(\varphi_1)} \cup \ldots \cup L_{V_2(\varphi_n)}.$$

Let $Y \in R_2(y) = R_2(f(x))$. Since f is a bounded morphism, there exists $Z \subseteq X_1$ such that $Z \in R_1(x)$ and $f(Z) \subseteq Y$. As $R_1(x) \subseteq L_{V_2(\varphi_1)} \cup \ldots \cup L_{V_2(\varphi_1)}$, there exists $V_2(\varphi_i) \in \{V_2(\varphi_1), ..., V_2(\varphi_n)\}$ such that $Z \cap f^{-1}(V_2(\varphi_i)) \neq \emptyset$ .i.e. $f(Z) \cap V_2(\varphi_i) \neq \emptyset$.

It follows $Y \cap V_2(\varphi_i) \neq \emptyset$. Thus, $R_2(y) \subseteq L_{V_2(\varphi_1)} \cup \ldots \cup L_{V_2(\varphi_n)}$. So, $\mathcal{M}_2$ is point-compact. □

4. Ultrafilter and model extension

Let $\mathcal{M} = \langle X, R, V \rangle$ be a model. Let $\mathrm{Ul}(\mathcal{P}(X)) = \mathrm{Ul}(X)$. Let $\langle \mathrm{Ul}(X), \beta_{\mathcal{P}(X)}(\mathcal{P}(X)) \rangle$ the Boolean space of the algebra $\mathcal{P}(X)$. Let us consider the ultrafilter frame $\langle \mathrm{Ul}(X), R_{\mathcal{P}(X)} \rangle$ of the monotonic algebra $\langle \mathcal{P}(X), \Diamond_R \rangle$. Recall that $R_{\mathcal{P}(X)} \subseteq \mathrm{Ul}(X) \times \mathcal{P}(\mathrm{Ul}(X))$ is defined as:

$$(P, Y) \in R_{\mathcal{P}(X)} \text{ iff } \exists C \in \mathcal{C}(\mathrm{Ul}(X)) \ (C \subseteq Y \text{ and } F_C \subseteq \Diamond_R^{-1}(P)),$$

where $F_C = \bigcap \{Q : Q \in Y\}$, and $Y \subseteq \mathrm{Ul}(X)$. We write $R_{\mathcal{P}(X)} = R_U$. We note that if Y is a closed subset of $\mathrm{Ul}(X)$, then

$$(P, Y) \in R_U \text{ iff } F_Y = \bigcap \{Q : Q \in Y\} \subseteq \Diamond_R^{-1}(P).$$

Definition 15. The *ultrafilter extension* of a monotonic model $\mathcal{M} = \langle X, R, V \rangle$ is the structure

$$\boldsymbol{Ue}(\mathcal{M}) = \langle \mathrm{Ul}(X), R_U, V_U \rangle,$$

where $V_U : Var \to \mathcal{P}(\mathrm{Ul}(X))$ is a map defined by:

$$V_U(p) = \{P \in \mathrm{Ul}(X) : V(p) \in P\},$$

for every $p \in P$.

It is easy to see that $V_U(p) = \beta_{\mathcal{P}(X)}(V(p))$, for each $p \in Var$ (see [6]).

Given a model $\mathcal{M}$ we can define another extension taking the set $\mathrm{Ul}(D_V)$ as the base set of a model. Let $\langle \mathrm{Ul}(D_V), \beta_{D_V}(D_V) \rangle$ be the Boolean space of D_V. Let us consider the ultrafilter frame $\langle \mathrm{Ul}(D_V), R_{D_V} \rangle$ of the monotonic algebra $\langle D_V, \Diamond_R \rangle$, where $R_{D_V} \subseteq \mathrm{Ul}(D_V) \times \mathcal{P}(\mathrm{Ul}(D_V))$ is defined by

$$(P, Y) \in R_{D_V} \text{ iff } \exists C \in \mathcal{C}(\mathrm{Ul}(D_V)) \ (C \subseteq Y \text{ and } F_C \subseteq \Diamond_R^{-1}(P)),$$

where $F_C = \bigcap \{Q : Q \in Y\}$, and $Y \subseteq \mathrm{Ul}(D_V)$.

Definition 16. The *valuation extension, or valuation model,* of a model $\mathcal{M}$ is the structure

$$\boldsymbol{Ve}(\mathcal{M}) = \langle \mathrm{Ul}(D_V), R_{D_V}, V_{D_V} \rangle,$$

where the function $V_{D_V} : Var \to \mathcal{P}(\mathrm{Ul}(D_V))$ is defined by

$$V_{D_V}(p) = \{P \in \mathrm{Ul}(D_V) : V(p) \in P\},$$

for every $p \in P$.

We note that $V_{D_V}(p) = \beta_{D_V}(V(p))$, for each $p \in Var$.

Theorem 17. *Let $\mathcal{M}$ be a monotonic model. Then for any $\varphi \in Fm$,*

1. $V_U(\varphi) = \beta_{\mathcal{P}(X)}(V(\varphi))$, *and* $V_{D_V}(\varphi) = \beta_{D_V}(V(\varphi))$,
2. $\mathcal{M} \models \varphi$ *iff* $\boldsymbol{Ue}(\mathcal{M}) \models \varphi$, *and* $\mathcal{M} \models \varphi$ *iff* $\boldsymbol{Ve}(\mathcal{M}) \models \varphi$.

Proof. We prove $V_{D_V}(\varphi) = \beta_{D_V}(V(\varphi))$, for any $\varphi \in Fm$. The other proof is very similar. The proof is by induction on the complexity of φ. We consider the case $\Diamond\varphi$. Let $P \in \mathrm{Ul}(D_V)$. Let $V(\Diamond\varphi) = \Diamond_R(V(\varphi)) \in P$. Let us consider the filter $F = F(V(\varphi))$ in the Boolean algebra D_V. Then it is easy to see that $\left(P, \hat{F}\right) \in R_{D_V}$. For the other direction, suppose that $P \in V_{D_V}(\Diamond\varphi)$. Then there exists a filter F of D_V such that

$$(P, \hat{F}) \in R_{D_V} \text{ and } \hat{F} \subseteq V_{D_V}(\varphi).$$

By inductive hypothesis we have $V_{D_V}(\varphi) = \beta_{D_V}(V(\varphi))$. Thus, $V(\varphi) \in F \subseteq \Diamond_R^{-1}(P)$, i.e., $V(\Diamond\varphi) = \Diamond_R(V(\varphi)) \in P$.

(2). We prove $\mathcal{M} \vDash \varphi$ iff $\boldsymbol{Ve}(\mathcal{M}) \vDash \varphi$. The other proof is similar. Assume that $\mathcal{M} \vDash \varphi$. Then, $V(\varphi) = X$. But X belong to every ultrafilter of D_V. Then, $V_{D_V}(\varphi) = \mathrm{Ul}(D_V)$, i.e., $\boldsymbol{Ve}(\mathcal{M}) \vDash \varphi$. Now, if $\mathcal{M} \nvDash \varphi$, then there exists $x \in X$ such that $x \notin V(\varphi)$. Then, $V(\varphi) \notin \{V(\alpha) : x \in V(\alpha)\} \in \mathrm{Ul}(D_V)$. Thus, $\boldsymbol{Ve}(\mathcal{M}) \nvDash \varphi$. □

Theorem 18. *Let $\mathcal{M}$ be a model. Then the model $\boldsymbol{Ue}(\mathcal{M})$ is m-saturated.*

Proof. Let $\mathcal{M}$ be a monotonic model. We prove that $\boldsymbol{Ue}(\mathcal{M})$ is compact i.e., $\langle \mathrm{Ul}(X), D_{V_U}\rangle$ is a compact space. From Lemma 13 it is enough to prove that the map $\varepsilon_{V_U} : \mathrm{Ul}(X) \to \mathrm{Ul}(D_{V_U})$ is surjective. Let $P \in \mathrm{Ul}(D_{V_U})$. Consider the filter F in $\mathcal{P}(X)$ generated by $\{V(\varphi) : V_U(\varphi) \in P\}$. It is clear that F is proper. So there exists $Q \in \mathrm{Ul}(X)$ such that $F \subseteq Q$. By construction $\varepsilon_{V_U}(Q) \subseteq P$. So, $\varepsilon_{V_U}(Q) \subseteq P$, and consequently ε_{V_U} is surjective.

We prove that R_U is point-compact in the hyperspace $\left\langle \mathcal{K}_{R_U}, \mathcal{T}_{D_{V_U}} \right\rangle$, where $\mathcal{X}_{V_U} = \langle \mathrm{Ul}(X), D_{V_U}\rangle$. As $\langle \mathrm{Ul}(X), D_{V_U}\rangle$ is a compact space we will apply Proposition 10. Let $P \in \mathrm{Ul}(X)$ and let Y be a subset of $\mathrm{Ul}(X)$. Suppose that

$$\bigcap \{\varepsilon_U(Q) : Q \in Y\} \subseteq \Diamond_{R_U}^{-1}(\varepsilon_U(P)).$$

We need to prove that there exists a subset Z of $\mathrm{Ul}(X)$ such that $Z \in R_U(P)$ and $Z \subseteq cl(Y)$, where $cl(Y)$ is the topological closure of Y in the topological space $\langle \mathrm{Ul}(X), D_{V_U}\rangle$.

Consider the filter F generated by the set $\{V(\varphi) : Y \subseteq V_U(\varphi)\}$. We prove that

$$(P, \hat{F}) \in R_U.$$

Let $W \in F$. Then there exists $\varphi_1, ..., \varphi_n \in Fm$ such that

$$Y \subseteq V_U(\varphi) \cap \ldots \cap V_U(\varphi_n) \text{ and } V(\varphi) \cap \ldots \cap V(\varphi_n) \subseteq W.$$

So, $V_U(\varphi) \cap \ldots \cap V_U(\varphi_n) \in \bigcap\{\varepsilon_U(Q) : Q \in Y\} \subseteq \Diamond_{R_U}^{-1}(\varepsilon_U(P))$. Then

$$\Diamond_{R_U}(V_U(\varphi \wedge \ldots \varphi_n) = V_U(\Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n)) \in \varepsilon_U(P),$$

i.e. $V(\Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n)) = \Diamond_R(V(\varphi_1 \wedge \ldots \wedge \varphi_n)) \in P$. Then $\Diamond_R(W) \in P$. Thus $(P, \hat{F}) \in R_U$. Taking into account that the topological closure of Y in the space $\langle \mathrm{Ul}(X), D_{V_U}\rangle$ is $cl(Y) = \bigcap\{V_U(\varphi) : Y \subseteq V_U(\varphi)\}$, it is easy to prove that $\hat{F} \subseteq cl(Y)$. Therefore $\boldsymbol{U}_e(\mathcal{M})$ is m-saturated. □

Theorem 19. *Let $\mathcal{M}$ be a model. Then the model $\boldsymbol{V}_e(\mathcal{M})$ is m-saturated.*

Proof. Since $\langle D_V, \Diamond_R \rangle$ is a subalgebra of the algebra $\langle \mathcal{P}(X), \Diamond_R \rangle$, we can define a map $f : \mathrm{Ul}(X) \to \mathrm{Ul}(D_V)$ by $f(P) = P \cap D_V$. By well-established results in the duality theory of Boolean algebras the map f is the dual map of the inclusion homomorphism between D_V and $\mathcal{P}(X)$. From the results given by H. Hansen [6] we get that $f : \mathrm{Ul}(X) \to \mathrm{Ul}(D_V)$ is a surjective bounded morphism between the ultrafilter extension $\boldsymbol{U}_e(\mathcal{M})$ and the valuation extension $\boldsymbol{V}_e(\mathcal{M})$. Since $\boldsymbol{U}_e(\mathcal{M})$ is m-saturated, by Proposition 14 we get that $\boldsymbol{V}_e(\mathcal{M})$ is also m-saturated. □

References

[1] B. F. Chellas, **Modal Logic: an introduction,** Cambridge Univ. Press,1980. 1

[2] R. Goldblatt, Maps and Monads for Modal Frames, Studia Logica Volume 83, Numbers 1-3 (2006), pp.309-331. 2

[3] R. Goldblatt, **Mathematics of Modality**, CSLI Lectures Notes 43, 1993.

[4] R. Goldblatt, **Logics of Time and Computation,** CSLI Lectures Notes 7, Second Edition, 1992.

[5] H. H. Hansen and C. Kupke. A Coalgebraic Perspective on Monotone Modal Logic, Electronic Notes in Theoretical Computer Science, 106, pages 121-143, Elsevier, 2004. 3

[6] H. H. Hansen. Monotonic modal logic (Master's thesis). Preprint 2003-24, ILLC, University of Amsterdam, 2003. 1, 6, 3, 11, 3, 4, 4

[7] M. J. Hollenberg, Hennessey-Milner Classes and Process Algebra. In M. de Rijke A. Ponse and Y. Venema, editors, Modal Logic and Process Algebra: a Bisimulation Perspective, volume 53 of CSLI Lecture Notes, pages 187–216. CSLI Publications, 1995. 1

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