

THE \mathfrak{n} -HOMOLOGY OF REPRESENTATIONS.

TIM BRATTEN

ABSTRACT. The \mathfrak{n} -homology groups of a \mathfrak{g} -module provide a natural and fruitful extension of the concept of highest weight to the representation theory of a noncompact reductive Lie group. In this article we give an introduction to the \mathfrak{n} -homology groups and a survey of some developments, with a particular emphasis on results pertaining to the problem of calculating \mathfrak{n} -homology groups.

1. INTRODUCTION

The concept of a highest weight and its use to classify irreducible representations of compact Lie groups can be traced back nearly a century, to seminal work by E. Cartan and H. Weyl. For a compact, connected Lie group, the highest weight theory gives a tight parametrization of irreducible representations in terms of specific invariants associated to the group. If one tries to extend this concept to the representation theory of a noncompact, real reductive group one immediately encounters two problems. On the one hand, in the noncompact case, it turns out there are several conjugacy classes of complex Borel subalgebras, and what might be called a highest weight depends on the choice of a conjugacy class. On the other hand, it is quite common that what should be called a highest weight turns out to be zero for every choice of Borel subalgebra. This means, in the traditional sense, the highest weight does not exist for a great majority of irreducible representations.

Although there is no way to avoid the first problem, representation theorists have confronted the second problem by considering the highest weight to be a functorial construction and studying the related derived functors. This has proved to be especially fruitful, producing a strong and useful family of invariants associated to a representation. In this article we give a brief introduction to the \mathfrak{n} -homology (and \mathfrak{n} -cohomology) groups, followed by a survey of some results, focusing on developments related to the problem of calculating the \mathfrak{n} -homology of representations.

The author would like to take this opportunity to thank the organizers of the 2007 meeting of the UMA for giving him an opportunity to present some results in the form of a conference and for asking him to submit this article. He would also like to acknowledge the help and encouragement he has received from Jorge Vargas. This article is dedicated to the memory of Mischa Cotlar, with a special recognition to Dr. Cotlar's role as advisor and mentor to the late José Pererya.

2. \mathfrak{n} -HOMOLOGY AND \mathfrak{n} -COHOMOLOGY

In this section we introduce the \mathfrak{n} -homology and \mathfrak{n} -cohomology of \mathfrak{g} -modules (for more details see [10]).

Let \mathfrak{g} be a complex reductive Lie algebra. By definition, a *Borel subalgebra* of \mathfrak{g} is a maximal solvable subalgebra and a *parabolic subalgebra* of \mathfrak{g} is a subalgebra that contains a Borel subalgebra. If $\mathfrak{p} \subseteq \mathfrak{g}$ is a parabolic subalgebra then the *nilradical* \mathfrak{n} of \mathfrak{p} is the largest solvable ideal in $[\mathfrak{p}, \mathfrak{p}]$. A *Levi factor* is a complementary subalgebra to \mathfrak{n} in \mathfrak{p} . One knows that Levi factors exist and that they are exactly the subalgebras which are maximal with respect to being reductive in \mathfrak{p} . When \mathfrak{l} is a Levi factor then

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$$

is called a *Levi decomposition*.

Fix a parabolic subalgebra \mathfrak{p} with nilradical \mathfrak{n} and Levi factor \mathfrak{l} . Let $U(\mathfrak{n})$ denote the enveloping algebra of \mathfrak{n} and let \mathbb{C} be the 1-dimensional trivial module. If M is a \mathfrak{g} -module then the *zero \mathfrak{n} -homology of M* is the \mathfrak{l} -module

$$H_0(\mathfrak{n}, M) = \mathbb{C} \otimes_{U(\mathfrak{n})} M.$$

This \mathfrak{l} -module is sometimes referred to as the *space of coinvariants*, although it clearly depends on the choice of parabolic subalgebra. The definition of the zero homology determines a right exact functor from the category of \mathfrak{g} -modules to the category of \mathfrak{l} -modules. The *\mathfrak{n} -homology groups of M* are the \mathfrak{l} -modules obtained as the corresponding derived functors. There is a standard complex for calculating these homology groups, defined as follows. The *right standard resolution of \mathbb{C}* is the complex of free right $U(\mathfrak{n})$ -modules given by

$$\cdots \rightarrow \Lambda^{p+1}\mathfrak{n} \otimes U(\mathfrak{n}) \rightarrow \Lambda^p\mathfrak{n} \otimes U(\mathfrak{n}) \rightarrow \cdots \rightarrow \mathfrak{n} \otimes U(\mathfrak{n}) \rightarrow U(\mathfrak{n}) \rightarrow 0.$$

Applying the functor

$$- \otimes_{U(\mathfrak{n})} M$$

to the standard resolution we obtain a complex

$$\cdots \rightarrow \Lambda^{p+1}\mathfrak{n} \otimes M \rightarrow \Lambda^p\mathfrak{n} \otimes M \rightarrow \cdots \rightarrow \mathfrak{n} \otimes M \rightarrow M \rightarrow 0$$

of left \mathfrak{l} -modules called the *standard \mathfrak{n} -homology complex*. Here \mathfrak{l} acts via the tensor product of the adjoint action on $\Lambda^p\mathfrak{n}$ with the given action on M . Since $U(\mathfrak{g})$ is a free $U(\mathfrak{n})$ -module, a routine homological argument identifies the p th homology of the standard complex with the p th \mathfrak{n} -homology group

$$H_p(\mathfrak{n}, M).$$

One can prove that the induced \mathfrak{l} -action on the homology groups of the standard complex is the correct one.

The *zero \mathfrak{n} -cohomology* of a \mathfrak{g} -module M is the \mathfrak{l} -module

$$H^0(\mathfrak{n}, M) = \text{Hom}_{U(\mathfrak{n})}(\mathbb{C}, M).$$

This \mathfrak{l} -module is sometimes referred to as the *space of invariants*, and also clearly depends on the choice of parabolic subalgebra. The definition of the zero cohomology determines a left exact functor from the category of \mathfrak{g} -modules to the category

of \mathfrak{l} -modules. By definition, the \mathfrak{n} -cohomology groups of M are the \mathfrak{l} -modules obtained as the corresponding derived functors. These \mathfrak{l} -modules can be calculated by applying the functor

$$\text{Hom}_{U(\mathfrak{n})}(-, M)$$

to the standard resolution of \mathbb{C} , this time by free left $U(\mathfrak{n})$ -modules. In a natural way, one obtains a complex of \mathfrak{l} -modules and the p th cohomology of this complex realizes the p th \mathfrak{n} -cohomology group

$$H^p(\mathfrak{n}, M).$$

It turns out that the structure of the \mathfrak{n} -cohomology is determined by the structure of the \mathfrak{n} -homology, in a simple way. Thus, it is often a matter of convenience whether one works with homology groups or cohomology groups. In this article, we will focus on results framed in terms of homology. The following proposition, whose proof is established by an analysis of standard complexes, can be used to translate results about \mathfrak{n} -homology into results about \mathfrak{n} -cohomology [9, Section 2].

Proposition 2.1. *Suppose M is a \mathfrak{g} -module. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra with nilradical \mathfrak{n} and Levi factor \mathfrak{l} . Let d denote the dimension of \mathfrak{n} . Then there are natural isomorphisms*

$$H_p(\mathfrak{n}, M) \cong H^{d-p}(\mathfrak{n}, M) \otimes \Lambda^d \mathfrak{n}.$$

3. REPRESENTATIONS OF LINEAR REDUCTIVE LIE GROUPS

In this section we review some classical results about the representation theory of reductive Lie groups (for details see [16]) and introduce the canonical globalizations.

For simplicity we work with a class of reductive Lie groups we call *linear*, although there is no problem working in the more general context of a reductive group of Harish-Chandra class. In particular, we assume the following setup. G will denote a connected, complex reductive group. This means G is a connected, complex Lie group with the property that the maximal compact subgroups are real forms of G . The group G_0 will denote a real form of G and is assumed to have finitely many connected components. We call G_0 a *linear reductive Lie group*. The Lie algebras of G and G_0 will be denoted \mathfrak{g} and \mathfrak{g}_0 , respectively. For the remainder of this article we fix a maximal compact subgroup K_0 of G_0 and let $K \subseteq G$ be the complexification of K_0 . In general, we write K, L etc. to indicate complex subgroups of G and denote the corresponding Lie algebras by $\mathfrak{k}, \mathfrak{l}$ etc. Subgroups of G_0 will be denoted by K_0, L_0 etc. with the corresponding real Lie algebras written as $\mathfrak{k}_0, \mathfrak{l}_0$ etc.

A *representation* of G_0 will mean a continuous linear action of G_0 in a complete, locally convex topological vector space. When we speak of irreducible or finite length representations, the corresponding definitions should be framed in terms of invariant closed subspaces. A vector v in a representation V is called *smooth* when

$$\lim_{t \rightarrow 0} \frac{\exp(t\xi)v - v}{t} \text{ exists for each } \xi \in \mathfrak{g}_0.$$

In order to define \mathfrak{n} -homology groups, we will be primarily interested in *smooth representations*. These are representations where every vector is smooth. In a natural way a smooth representation carries a compatible \mathfrak{g} -action. For a compact Lie group one can show that a finite length representation is finite-dimensional and therefore smooth.

We recall some basic results about the infinite-dimensional representations of reductive groups. In the 1950s, Harish-Chandra proved that an irreducible unitary representation V has the property that each irreducible K_0 -submodule has finite multiplicity in V . This led him to define and study *admissible representations*. By definition, this means each irreducible K_0 -submodule has finite multiplicity. Harish-Chandra then considered *the subspace of K_0 -finite vectors*. By definition, a vector v in a representation is called *K_0 -finite* if the span of the K_0 -orbit of v is finite-dimensional. Although the subspace of K_0 -finite vectors is not G_0 -invariant, Harish-Chandra proved that K_0 -finite vectors are smooth, and thus form a (\mathfrak{g}, K_0) -module called *the underlying Harish-Chandra module*.

On the other hand, it is possible to define abstractly the concept of a Harish-Chandra module. This is a \mathfrak{g} -module equipped with a compatible, locally finite K_0 -action. Harish-Chandra proved that an irreducible Harish-Chandra module appears as the underlying (\mathfrak{g}, K_0) -module of K_0 -finite vectors in an irreducible admissible Banach space representation for G_0 and W . Casselman proved that the same holds for any finite-length Harish-Chandra module. By now we know more. In particular, given a Harish-Chandra module M we define a globalization M_{glob} of M to be an admissible representation for G_0 whose underlying Harish-Chandra is M . We assume our Harish-Chandra modules have finite-length. Then we can assert that several canonical and functorial globalizations exist on the category of Harish-Chandra modules. These are: the smooth globalization of Casselman and Wallach [5], its dual (called: the distribution globalization), Schmid's minimal globalization [14] and its dual (the maximal globalization). All four globalizations are smooth. We will let M_{min} , M_{max} , M_{∞} and M_{dis} denote respectively, the minimal, the maximal, the smooth and the distribution globalizations of a Harish-Chandra module M . If M_{glob} denotes a Banach globalization of M , then there is a natural chain of inclusions

$$M \subseteq M_{\text{min}} \subseteq M_{\infty} \subseteq M_{\text{glob}} \subseteq M_{\text{dis}} \subseteq M_{\text{max}}.$$

In this chain the minimal globalization is known to coincide with the analytic vectors in M_{glob} while M_{∞} coincides with the smooth vectors in M_{glob} . In particular, one knows that a finite-length admissible Banach space representation for G_0 is smooth if and only if it is finite dimensional. Later in this article we will review various results, often called *comparison theorems*, relating the \mathfrak{n} -homologies of a Harish-Chandra module to the \mathfrak{n} -homologies of a canonical globalization.

4. SOME STRUCTURAL DETAILS

In this section we recall some structure theory and an important technical result about the decomposition of \mathfrak{n} -homology groups for certain \mathfrak{g} -modules, giving special emphasis on the case of a Borel subalgebra. Recall that a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$

is a maximal abelian subalgebra whose elements are semisimple under the adjoint representation of \mathfrak{h} in \mathfrak{g} . A nonzero eigenvalue $\alpha \in \mathfrak{h}^*$ for the adjoint representation is called a root. Σ will denote the set of roots. Thus

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha$$

where \mathfrak{g}^α is the eigenspace corresponding to root α . One knows that $\alpha \in \Sigma$ if and only if $-\alpha \in \Sigma$. If \mathfrak{b} is a Borel subalgebra of \mathfrak{g} containing \mathfrak{h} then the roots of \mathfrak{h} in \mathfrak{b} define a subset $\Sigma^+ \subseteq \Sigma$ called *the corresponding set of positive roots*. When the sum of two positive roots is a root, then that sum is positive. One also knows that Σ is a disjoint union:

$$\Sigma = \Sigma^+ \cup -\Sigma^+.$$

One can show there is a unique $H_\alpha \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. We use this element to define the value of *the dual root*. In particular, the dual root is given by

$$\check{\alpha}(\mu) = \mu(H_\alpha) \text{ for } \mu \in \mathfrak{h}^*.$$

The linear reflection $s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ corresponding to $\alpha \in \Sigma$ is defined as

$$s_\alpha(\mu) = \mu - \check{\alpha}(\mu)\alpha.$$

These reflections generate a finite subgroup of the general linear group of \mathfrak{h}^* , denoted W and called *the Weyl group of \mathfrak{h} in \mathfrak{g}* .

Let $Z(\mathfrak{g})$ denote the center of the enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . A \mathfrak{g} -infinitesimal character Θ is a homomorphism of algebras

$$\Theta : Z(\mathfrak{g}) \rightarrow \mathbb{C}.$$

Since $Z(\mathfrak{g})$ acts on an irreducible Harish-Chandra module (and also any corresponding smooth globalization) by a scalar, the infinitesimal character is an important invariant associated to an irreducible, admissible representation. We now recall Harish-Chandra's parametrization of infinitesimal characters. We choose a Borel subalgebra \mathfrak{b} containing \mathfrak{h} . Thus

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \text{ where } \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \text{ is the nilradical of } \mathfrak{b}.$$

Then one knows that $Z(\mathfrak{g}) \subseteq U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}$ and that the corresponding projection of $Z(\mathfrak{g})$ in $U(\mathfrak{h})$ defines an injective morphism of algebras called *the unnormalized Harish-Chandra map*. We can use this morphism to identify infinitesimal characters with Weyl group orbits in \mathfrak{h}^* in the following way. Let ρ denote one-half the sum of the positive roots and suppose $\lambda \in \mathfrak{h}^*$. Then, via the unnormalized Harish-Chandra map, the composition

$$\Theta : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h}) \xrightarrow{\lambda + \rho} \mathbb{C}$$

defines an infinitesimal character Θ . One knows that for $w \in W$, the element $w\lambda \in \mathfrak{h}^*$ defines the same infinitesimal character Θ . Abusing notation somewhat, we write $\Theta = W \cdot \lambda$. The infinitesimal character is called *regular* when the only element of W fixing an element in the orbit $W \cdot \lambda$, is the identity. This is equivalent to the condition that

$$\check{\alpha}(\lambda) \neq 0 \text{ for each } \alpha \in \Sigma.$$

For a \mathfrak{g} -module M with regular infinitesimal character one has the following result. The notes by D. Milicic [13] contain a proof.

Theorem 4.1. *Let M be a \mathfrak{g} -module with regular infinitesimal character Θ . Suppose \mathfrak{b} is a Borel subalgebra of \mathfrak{g} with Levi decomposition*

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}.$$

Let $\lambda \in \mathfrak{h}^$ such that $\Theta = W \cdot \lambda$ and let ρ be one half the sum of the positive roots. Then the Cartan subalgebra \mathfrak{h} acts semisimply on the \mathfrak{n} -homology groups $H_p(\mathfrak{n}, M)$ with eigenvalues of the form $w\lambda + \rho$ for $w \in W$. In particular*

$$H_p(\mathfrak{n}, M) = \bigoplus_{w \in W} H_p(\mathfrak{n}, M)_{w\lambda + \rho}$$

where

$$H_p(\mathfrak{n}, M)_{w\lambda + \rho} = \{v \in H_p(\mathfrak{n}, M) : \xi \cdot v = (w\lambda + \rho)(\xi)v \text{ for each } \xi \in \mathfrak{h}\}.$$

A generalization of this result works for any parabolic subalgebra \mathfrak{p} with Levi decomposition

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$$

although the parametrization is not quite as tight. In particular, if M is a \mathfrak{g} -module M with regular infinitesimal character Θ and if $Z(\mathfrak{l})$ denotes the center of the enveloping algebra of \mathfrak{l} then $H_p(\mathfrak{n}, M)$ is a semisimple $Z(\mathfrak{l})$ -module and decomposes into a direct sum of $Z(\mathfrak{l})$ -eigenspaces, where the associated \mathfrak{l} -infinitesimal characters that appear are related to Θ by an appropriately defined Harish-Chandra map.

5. KOSTANT'S THEOREM

When G_0 is a connected, compact Lie group, there is a result, called Kostant's theorem, that calculates the \mathfrak{n} -homology groups of an irreducible representation. In this section we review that result, with special emphasis on the case of a Borel subalgebra.

Assume G_0 is a compact real form of G . Fix a Borel subalgebra \mathfrak{b} . Then the normalizer of \mathfrak{b} in G_0 is a maximal torus H_0 and a real form for a Cartan subgroup H of G . We let \mathfrak{h} be the Lie algebra of H and \mathfrak{n} the nilradical of \mathfrak{b} . Σ is the set of roots. The roots of \mathfrak{h} in \mathfrak{b} determine a set of positive roots $\Sigma^+ \subseteq \Sigma$. Let ρ be one-half the sum of the positive roots. Suppose

$$\chi : H_0 \rightarrow \mathbb{C}$$

is a continuous character and let $\mu \in \mathfrak{h}^*$ denote the complexification of the derivative of χ . To be consistent with the notation in Section 7 we use *the shifted parameter*

$$\lambda = \mu - \rho.$$

One knows that

$$\check{\alpha}(\lambda) \text{ is an integer for each } \alpha \in \Sigma.$$

The character χ is called *antidominant and regular* if

$$\check{\alpha}(\lambda) \notin \{0, 1, 2, 3, \dots\} \text{ for each } \alpha \in \Sigma^+.$$

The Cartan-Weyl parametrization of irreducible representations is as follows.

Theorem 5.1. *Maintain the established notations.*

(a) *Suppose M is an irreducible G_0 -module. Then the space of coinvariants $H_0(\mathfrak{n}, M)$ is an irreducible H_0 -module and the associated character χ is antidominant and regular. This character is called the lowest weight.*

(b) *If two irreducible representations have the same lowest weight then they are isomorphic.*

(c) *To each antidominant and regular character there is an irreducible G_0 -module with the given character as its lowest weight.*

We need to define the length function on the Weyl group. One knows that the Weyl group permutes the roots of \mathfrak{h} in \mathfrak{g} . We can define *the length of $w \in W$* to be the number of roots in

$$-\Sigma^+ \cap w\Sigma^+.$$

Kostant's theorem is the following:

Theorem 5.2. *Suppose M is the irreducible representation for G_0 with lowest weight χ and let $\lambda \in \mathfrak{h}^*$ be the shifted parameter. Then $H_p(\mathfrak{n}, M)$ is a sum of irreducible H_0 -modules each having multiplicity one. The characters of H_0 that show up as eigenvalues in $H_p(\mathfrak{n}, M)$ are exactly those whose derivative have the form $w\lambda + \rho$ where the length of w is p .*

In the more general case of a parabolic subalgebra \mathfrak{p} , let L_0 be the normalizer of \mathfrak{p} in G_0 and let \mathfrak{l} be the complexified Lie algebra of L_0 . One knows that L_0 is connected and that \mathfrak{l} is a Levi factor of \mathfrak{p} . Indeed, if L is the connected subgroup of G with Lie algebra \mathfrak{l} then L_0 is the compact real form of L . Suppose M is an irreducible representation for G_0 and let \mathfrak{n} be the nilradical of \mathfrak{p} . Then Kostant's Theorem describes the structure of the p th homology group $H_p(\mathfrak{n}, M)$ as an L_0 -module. In particular, the theorem states that an irreducible representation V of L_0 has, at most, multiplicity one in $H_p(\mathfrak{n}, M)$ and gives a precise condition when V appears, in terms of the degree p , the lowest weight of M and the lowest weight of V . We refer the reader to [10, Chapter IV, Section 9] for more details.

6. FLAG MANIFOLDS AND COMPARISON THEOREMS

As we mentioned before, when G_0 is noncompact, there are several conjugacy classes of Borel subalgebras and the structure of the \mathfrak{n} -homology groups of a representation can depend on the choice of G_0 -conjugacy class. On the other hand, when M is a Harish-Chandra module, then the locally finite K_0 -action on M extends naturally to a locally holomorphic K -action, and it turns out that the \mathfrak{n} -homology groups of M depend on the K -conjugacy classes of Borel subalgebras. In order to compare the \mathfrak{n} -homology groups of M with the \mathfrak{n} -homology groups of a smooth globalization, we therefore need to know something about the relationship between G_0 -conjugacy classes and K -conjugacy classes. There is an elegant geometric result, referred to as *Matsuki duality*, that gives us the needed information. We now review that result.

One knows that the group G acts transitively on the set of Borel subalgebras of \mathfrak{g} . The corresponding G -homogeneous complex manifold X is called *the full flag space*. In general, if \mathfrak{p} is a parabolic subalgebra of \mathfrak{g} then the normalizer of \mathfrak{p} in G is the connected subgroup P with Lie algebra \mathfrak{p} and the corresponding quotient

$$Y = G/P$$

is called *a flag manifold*. The points in Y are naturally identified with the G -conjugates to \mathfrak{p} .

Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be the complexification of a Cartan involution of \mathfrak{g}_0 corresponding to the maximal compact subgroup K_0 . A Cartan subalgebra \mathfrak{h} of \mathfrak{g} is called *stable* if $\mathfrak{g}_0 \cap \mathfrak{h}$ is a real form and if $\theta(\mathfrak{h}) = \mathfrak{h}$. A Borel subalgebra is called *very special* if it contains a stable Cartan subalgebra. A stable Cartan subalgebra of a Borel subalgebra is unique. A point in the full flag space is called *very special* if the corresponding Borel subalgebra is.

Matsuki has established the following [11].

Theorem 6.1. *Let X be the full flag space. Then*

- (a) *The subset of very special points in a G_0 -orbit is a nonempty K_0 -orbit.*
- (b) *The subset of very special points in a K -orbit is a nonempty K_0 -orbit.*

It follows that the very special points give a one-to-one correspondence between the G_0 -orbits and the K -orbits on X , defined by the following duality. A G_0 -orbit S is said to be *dual* to a K -orbit Q when $S \cap Q$ contains a special point. In this duality, open G_0 -orbits correspond to closed K -orbits and the (unique) closed G_0 -orbit corresponds to the (unique) open K -orbit. We note that Matsuki has established a similar result for any flag manifold [12].

Example 6.2. *Suppose $G = SL(2, \mathbb{C})$, the group of 2×2 complex matrices with determinant 1 and let $G_0 = SL(2, \mathbb{R})$. Then the full flag space X is isomorphic to the Riemann sphere. G_0 has three orbits on X . The closed G_0 -orbit can be identified with an equatorial circle and the other two orbits are the corresponding open hemispheres. It turns out every point in the closed orbit is very special, independent of the choice of K_0 (this is true in general). Put $K_0 = SO(2, \mathbb{R})$. Thus $K = SO(2, \mathbb{C})$. Then the three K -orbits on X are a punctured plane, containing the closed G_0 -orbit, and two fixed points, which can be identified with the respective poles in each of the open hemispheres. These two poles are the other very special points.*

When M is a Harish-Chandra module and \mathfrak{n} is the nilradical of a Borel subalgebra then one knows that the homology groups $H_p(\mathfrak{n}, M)$ are finite-dimensional, so it may seem reasonable to ask when $H_p(\mathfrak{n}, M)$ coincides with the \mathfrak{n} -homology groups of a smooth globalization. It turns out this not only depends on the choice of Borel subalgebra, but also in the the choice of smooth globalization. When \mathfrak{n} is the nilradical of a very special Borel subalgebra, M is a Harish-Chandra module, and M_{\min} is the minimal globalization, then H. Hecht and J. Taylor have shown [8] that the natural map

$$M \rightarrow M_{\min} \text{ induces isomorphisms } H_p(\mathfrak{n}, M) \rightarrow H_p(\mathfrak{n}, M_{\min}).$$

On the other hand, for the maximal globalization, there are counterexamples to this result.

The result of Hecht and Taylor has been generalized in the following form. A Levi factor \mathfrak{l} of a parabolic subalgebra \mathfrak{p} is called *stable* if $\mathfrak{l} \cap \mathfrak{g}_0$ is a real form \mathfrak{l}_0 of \mathfrak{l} and if $\theta(\mathfrak{l}) = \mathfrak{l}$. The parabolic subalgebra \mathfrak{p} is called *very special* if it contains a stable Levi factor. Such a Levi factor is unique. Unlike the case of the full flag space, there may be parabolic subalgebras which are not G_0 -conjugate to a very special parabolic subalgebra, so we are not considering all orbits on every flag manifold. However, suppose \mathfrak{p} is very special and \mathfrak{l} is the stable Levi factor. Define L_0 to be the subgroup of G_0 that normalizes \mathfrak{p} and normalizes \mathfrak{l} . Then L_0 is a linear reductive Lie group with complexified Lie algebra \mathfrak{l} and maximal compact subgroup $L_0 \cap K_0$, called *the associated real Levi subgroup*. We have the following result [9, Proposition 2.24].

Proposition 6.3. *Suppose \mathfrak{p} is a very special parabolic subalgebra with L_0 and \mathfrak{l} defined as above. Let \mathfrak{n} be the nilradical of \mathfrak{p} and suppose M is a Harish-Chandra module for (\mathfrak{g}, K_0) . Then the \mathfrak{n} -homology groups are Harish-Chandra modules for $(\mathfrak{l}, K_0 \cap L_0)$.*

For the minimal globalization, we have the following [2].

Theorem 6.4. *Maintain the hypothesis of the previous proposition. Then the standard complex induces a Hausdorff topology on $H_p(\mathfrak{n}, M_{min})$ and the natural map $M \rightarrow M_{min}$ induces isomorphisms*

$$H_p(\mathfrak{n}, M)_{min} \cong H_p(\mathfrak{n}, M_{min}).$$

One might conjecture that above theorem works for the smooth globalization, and W. Casselman has informed the author that he has proven something along these lines, although details are unclear. Two partial comparison theorems about smooth globalizations have been published by other mathematicians. H. Hecht and J. Taylor have shown the result for minimal parabolic subgroups of G_0 [6], while U. Bunke and M. Olbrich have shown the result for any real parabolic subgroup [4].

D. Vogan has conjectured that all four canonical globalizations commute with the \mathfrak{n} -homology groups of a very special parabolic subalgebra when the corresponding G_0 -orbit on the flag manifold is open [15]. We remark that it has recently been shown that Vogan's conjecture is true for one globalization if and only it's true for the dual [3]. Thus the conjecture is proven for both the minimal and maximal globalization.

7. THE \mathfrak{n} -HOMOLOGY OF STANDARD MODULES

In the noncompact case, the problem of calculating \mathfrak{n} -homology groups can be quite complicated and there seems to be little hope of just writing down a formula that generalizes Kostant's theorem for all irreducible representations. However, there are certain representations, called *standard modules*, whose \mathfrak{n} -homology

groups are a bit more predictable. These standard modules are generically irreducible, coincide with irreducibles when G_0 is compact, and can be used to classify the irreducible representations. In this section we define the standard representations and consider their \mathfrak{n} -homology groups, focusing on the case of the full flag space.

In particular, we use the construction of minimal globalizations given in [7]. Let X be the full flag space and, since we need to keep track of points, introduce the following notation. For $x \in X$ we let \mathfrak{b}_x be the corresponding Borel subalgebra and let \mathfrak{n}_x denote the nilradical of \mathfrak{b}_x . When we are interested in calculating the \mathfrak{n}_x -homology of Harish-Chandra modules, we can assume \mathfrak{b}_x is a very special Borel subalgebra. In that case, \mathfrak{h}_x denotes the stable Cartan subalgebra of \mathfrak{b}_x and $H_0 \subseteq G_0$ is the corresponding real Cartan subgroup (thus H_0 is the associated Levi subgroup). By our linear assumptions on G_0 , it follows that H_0 is abelian, so that an irreducible, admissible representation of H_0 is a continuous character

$$\chi : H_0 \rightarrow \mathbb{C}.$$

Let $S \subseteq X$ be the G_0 -orbit of x . In a natural way, χ extends to a character of the normalizer of \mathfrak{b}_x in G_0 (we note that H_0 and the normalizer of \mathfrak{b}_x coincide exactly when S is open). Thus χ determines a G_0 -homogeneous analytic line bundle over S . One can then define the concept of a *polarized section* [7, Section 8]. When S is open, the polarized sections are holomorphic sections, and in general the polarized sections are locally isomorphic with the restricted holomorphic functions. Let $\mathcal{A}(x, \chi)$ denote the sheaf of polarized sections on S and, for $p = 0, 1, 2, 3, \dots$ let

$$H_c^p(S, \mathcal{A}(x, \chi))$$

denote the corresponding compactly supported sheaf cohomology group. Suppose $\mu \in \mathfrak{h}_x^*$ is the complexified differential of χ , let ρ be one-half the sum of the positive roots for \mathfrak{h}_x in \mathfrak{b}_x and let $\lambda \in \mathfrak{h}_x^*$ denote the corresponding shifted parameter. Thus

$$\mu = \lambda + \rho.$$

We have the following theorem [7].

Theorem 7.1. *Maintain the previously defined notations. Let q be the codimension of the K -orbit of x in X .*

(a) $H_c^p(S, \mathcal{A}(x, \chi))$ carries a natural topology and a continuous G_0 -action, so that the resulting representation is a minimal globalization.

(b) $H_c^p(S, \mathcal{A}(x, \chi))$ has infinitesimal character $\Theta = W \cdot \lambda$.

(c) When λ is antidominant then $H_c^p(S, \mathcal{A}(x, \chi)) = 0$ when $p \neq q$.

(d) When λ is antidominant and Θ is regular then $H_c^q(S, \mathcal{A}(x, \chi))$ contains a unique irreducible submodule. In particular, $H_c^q(S, \mathcal{A}(x, \chi)) \neq 0$.

When λ is antidominant and Θ is regular, we call $H_c^q(S, \mathcal{A}(x, \chi))$ a *regular standard module*. These modules can be used to parametrize irreducible representations with regular infinitesimal character. For the remainder of this article we will make some remarks about how to calculate the \mathfrak{n} -homology of regular standard modules.

But we first note that, in the case of a singular infinitesimal character, the definition of standard module is more subtle, and the calculation of \mathfrak{n} -homology is more elusive.

To state results we will need to differentiate points where we calculate \mathfrak{n} -homology and the corresponding parameters for eigenvalues of a Cartan subalgebra (see Theorem 4.1). In particular, we fix a very special point $x \in X$ as a base point. For $\lambda \in \mathfrak{h}_x^*$ we put $\lambda(x) = \lambda$. When \mathfrak{b}_y is a very special Borel subalgebra and $\mathfrak{h}_y \subseteq \mathfrak{b}_y$ is the stable Cartan subalgebra, then there exists $g \in G$ such that

$$g\mathfrak{b}_xg^{-1} = \mathfrak{b}_y \quad \text{and} \quad g\mathfrak{h}_xg^{-1} = \mathfrak{h}_y.$$

Thus

$$(\mathfrak{h}_x^*)^g = \mathfrak{h}_y$$

This isomorphism is independent of the choice of $g \in G$. For $\lambda \in \mathfrak{h}_x^*$ put

$$\lambda(y) = \lambda^g \in \mathfrak{h}_y^*.$$

We note that $\alpha \in \mathfrak{h}_x^*$ is a root of \mathfrak{h}_x in $\mathfrak{g} \Leftrightarrow \alpha(y)$ is a root of \mathfrak{h}_y and that α is positive at $x \Leftrightarrow \alpha(y)$ is positive at y . In particular, $\rho(y)$ is one-half the sum of the positive roots for \mathfrak{h}_y in \mathfrak{b}_y .

The circle of ideas utilized in [7] depend on an identification of the derived functor of \mathfrak{n} -homology, in a certain weight (Theorem 4.1), with the geometric fiber applied to a certain, corresponding localization functor. These ideas originate in the an elegant generalization of Casselman’s submodule theorem, given by A. Beilinson and J. Bernstein in [1]. This identification, together with some functorial rigmarole, immediately leads to the following result.

Proposition 7.2. *Suppose $V = H_c^q(S, \mathcal{A}(x, \chi))$ is a regular standard module. Maintain the previously introduced notations. Let \mathbb{C}_χ denote the 1-dimensional representation of H_0 corresponding to χ and let \mathfrak{n}_x be the nilradical of \mathfrak{b}_x . Then we have the following.*

(a)

$$H_p(\mathfrak{n}_x, V)_{\lambda+\rho} = 0 \quad \text{for } p \neq q \text{ and } H_q(\mathfrak{n}_x, V)_{\lambda+\rho} = \mathbb{C}_\chi.$$

(b) *If \mathfrak{b}_y is a very special Borel subalgebra with nilradical \mathfrak{n}_y and $y \notin S$ then*

$$H_p(\mathfrak{n}_y, V)_{\lambda(y)+\rho(y)} = 0 \quad \text{for each } p.$$

According to Theorem 4.1, the problem of calculating of the \mathfrak{n}_y -homology groups of V , at a special point $y \in X$, reduces to the problem of calculating the values in the weights $(w\lambda + \rho)(y)$ for $w \in W$. In geometric terms, this means calculating the geometric fibers of certain localizations of V or, equivalently, calculating the result of the so called *intertwining functor*. We briefly consider this problem.

In general, a positive root is called *simple* if it cannot be decomposed into a nontrivial sum of positive roots. Let Σ_y^+ be the positive roots associated to \mathfrak{h}_y in \mathfrak{b}_y . For a simple root $\alpha \in \Sigma_x^+$, the problem of calculating the values of the \mathfrak{n}_y -homology groups in the weight $(s_\alpha\lambda + \rho)(y)$ can be geometrically reduced to specific calculations for certain real subgroups of $SL(2, \mathbb{C})$. To a large extent, this

idea is already exploited and explained in [7] and some of the necessary calculations are dealt with there.

We finish with an example where, using these ideas, a general formula, like Kostant's, can be actually written down. Assume G_0 is a connected, complex reductive Lie group. Fix a very special Borel subalgebra \mathfrak{b}_x with stable Cartan subalgebra \mathfrak{h}_x and let Σ_x^+ denote the corresponding positive roots. For each $w \in W$, the set

$$w(\Sigma_x^+) = \Sigma_{w \cdot x}^+$$

defines a new set of positive roots and thus a corresponding Borel subalgebra $\mathfrak{b}_{w \cdot x}$ of \mathfrak{g} containing the stable Cartan subalgebra \mathfrak{h}_x . Thus the point $w \cdot x \in X$ is very special. Because G_0 is a complex reductive group, one knows that each Borel subalgebra of \mathfrak{g} is G_0 -conjugate to a Borel subalgebra of the form $\mathfrak{b}_{w \cdot x}$. Suppose H is the Cartan subgroup of G with Lie algebra \mathfrak{h}_x . Then each $\alpha \in \Sigma$ defines a holomorphic character of H and by restriction, a corresponding character of H_0 . We write χ_α for this character of H_0 . If we let λ be the shifted parameter and put $y = w \cdot x$ then $(w\lambda)(y) = \lambda$.

Using the above ideas, one can deduce the following.

Theorem 7.3. *Let G_0 be a connected, complex reductive group. Suppose $V = H_c^q(S, \mathcal{A}(x, \chi))$ is the previously defined regular standard module and assume the G_0 -orbit of x is open in X . We define a chain $\alpha_1, \dots, \alpha_k$ of simple roots to be a finite sequence of roots of \mathfrak{h}_x such that for each j , α_{j+1} is simple for the set of positive roots defined by*

$$s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_j}(\Sigma_x^+).$$

Suppose $\alpha_1, \dots, \alpha_k$ is a chain of simple roots and let $w \in W$ be the ordered product of reflections given by this chain. Let χ_w be the character of H_0 defined by

$$\chi_w = \chi_{\alpha_1}^{-1} \chi_{\alpha_2}^{-1} \cdots \chi_{\alpha_k}^{-1}$$

and let $\mathbb{C}_\chi \otimes \mathbb{C}_{\chi_w}$ be the 1-dimensional representation of H_0 corresponding to the character $\chi \cdot \chi_w$. Let q^w denote the codimension of the K -orbit of $y = w \cdot x$ in X . Then

$$H_p(\mathfrak{n}_y, V) = 0 \text{ for } p \neq q^w \text{ and } H_{q^w}(\mathfrak{n}_y, V) = \mathbb{C}_\chi \otimes \mathbb{C}_{\chi_w}.$$

We note that the hypothesis of the theorem implies that the representation V is irreducible, and also remark that any attempt to write down a similar result for other orbits (even in the case of a connected, complex reductive group), when the standard module is reducible, is considerably more complicated.

REFERENCES

- [1] Beilinson, A. and Bernstein, J.: *A generalization of Casselman's submodule theorem*. Birkhäuser, Progress in Math. 40, Boston, 1983, pp. 35-52. [7](#)
- [2] Bratten, T.: *A comparison theorem for Lie algebra homology groups*. Pacific J. Math. **182** (1998) 23-36. [6](#)
- [3] Bratten, T and Corti, S.: *A simple proof of the algebraic version of a conjecture by Vogan*. J. of Lie Theory **18** (2008) 83-91. [6](#)
- [4] Bunke, U. and Olbrich, M.: *Cohomological properties of the canonical globalizations of Harish-Chandra modules*. Ann. of Global Analysis and Geometry. 15 (1997) 401-418. [6](#)
- [5] Casselman, W.; *Canonical extensions of Harish-Chandra modules to representations of G* . Can. J. Math. **41** (1989) 385-438. [3](#)
- [6] Hecht, H and Taylor, J.: *A remark on Caselman's comparison theorem*. In the book "Geometry and Representation Theory of p-Adic Groups", editors: J. Tirao, D. Vogan and J. Wolf. Birkhäuser, 1998. [6](#)
- [7] Hecht, H and Taylor, J.: *Analytic localization of group representations*. Advances in Math. **79** (1990) 139-212. [7](#), [7](#), [7](#)
- [8] Hecht, H. and Taylor, J.: *A comparison theorem for \mathfrak{n} -homology*. Compositio Math. **86** (1993) 187-207. [6](#)
- [9] Hecht, H. and Schmid, W.: *Characters, asymptotics and \mathfrak{n} -homology of Harish-Chandra modules*. Acta Math. **151** (1983) 49-151. [2](#), [6](#)
- [10] Knapp, A and Vogan, D.: *Cohomological Induction and Unitary Representations*. Princeton University Press. Princeton, 1995. [2](#), [5](#)
- [11] Matsuki, T.: *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*. J. Math. Soc. Japan. **31** (1979) 331-357. [6](#)
- [12] Matsuki, T.: *Orbits on affine symmetric spaces under the action of parabolic subgroups*. Hiroshima Math. J. **12** (1982) 307-320. [6](#)
- [13] Milicic, D.: *Localization and Representation Theory of Reductive Lie Groups*. Notes dated 1994 and available on Milicic's web page. [4](#)
- [14] Schmid, W.: *Boundary value problems for group invariant differential equations*. The mathematical heritage of Elie Cartan, Astérisque (1985) 311-321. [3](#)
- [15] Vogan, D.: *Unitary representations and complex analysis*. Notes from the Cime summer school, Venice, Italy 2004. Available on Vogan's web page <http://www-math.mit.edu/~dav/paper.html>. [6](#)
- [16] Warner, G.: *Harmonic Analysis on Semi-Simple Lie Groups I*. Springer-Verlag, 1972.

[3](#)

Tim Bratten
 Facultad de Ciencias Exactas,
 UNICEN. Tandil, Argentina
bratten@exa.unicen.edu.ar

Recibido: 10 de abril de 2008
Aceptado: 6 de mayo de 2008