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MATRIX SPHERICAL FUNCTIONS AND ORTHOGONAL POLYNOMIALS: AN INSTRUCTIVE EXAMPLE

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Abstract. In the scalar case, it is well known that the zonal spherical functions of any compact Riemannian symmetric space of rank one can be expressed in terms of the Jacobi polynomials. The main purpose of this paper is to revisit the matrix valued spherical functions associated to the complex projective plane to exhibit the interplay among these functions, the matrix hypergeometric functions and the matrix orthogonal polynomials. We also obtain very explicit expressions for the entries of the spherical functions in the case of 2×2 matrices and exhibit a natural sequence of matrix orthogonal polynomials, beyond the group parameters.

1. Introduction

The well known Legendre polynomials are a special case of *spherical harmonics*: the homogeneous harmonic polynomials of $\mathbb{R}^3$, considered as functions on the unit sphere S^2. Let (r,θ,ϕ) be ordinary polar coordinates in $\mathbb{R}^3$: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. In terms of these coordinates the Riemannian structure of $\mathbb{R}^3$ is given by the symmetric differential form $ds^2 = dr^2 + r^2 d\theta^2 + r^2(\sin\theta)^2 d\phi^2$, and the Laplace operator is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2}{\partial\phi^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}.$$

If f is a homogeneous harmonic polynomial of degree n which does not depend on the variable ϕ, then

$$\frac{d^2 f}{d\,\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{df}{d\theta} + n(n+1)f = 0.$$

By making the change of variables $y = (1+\cos\theta)/2$ we get

$$y(1-y)\frac{d^2 f}{d\,y^2} + (1-2y)\frac{df}{dy} + n(n+1)f = 0.$$

The bounded solution at $y = 0$, up to a constant, is ${}_2F_1(-n, n+1, 1; y)$. Since the Legendre polynomial of degree n is given by

$$P_n(x) = {}_2F_1\left(\begin{matrix}-n\,,\ n+1\\ 1\end{matrix}; (1+x)/2\right),$$

we get that $f(\theta) = P_n(\cos\theta)f(0)$.
Let $o = (0,0,1)$ be the north pole of S^2, and let $d(o,p)$ be the geodesic distance

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from a point $p \in S^2$ to o. Let $\phi(p) = P_n(\cos(d(o,p)))$. Then we have proved that ϕ is the unique spherical harmonic of degree n, constant along parallels and such that $\phi(o) = 1$. Moreover the set of all complex linear combinations of translates $\phi_g(p) = \phi(g \cdot p)$, $g \in \mathrm{SO}(3)$, is the linear space of all spherical harmonics of degree n.

Legendre and Laplace found that the Legendre polynomials satisfy the following addition formula

$$\begin{aligned} &P_n(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi) \\ &\quad = P_n(\cos\alpha)P_n(\cos\beta) + 2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_n^k(\cos\alpha)P_n^k(\cos\beta)\cos k\phi, \end{aligned} \tag{1}$$

where the P_n^k's are the associated Legendre polynomials.

By integrating (1) we get

$$P_n(\cos\alpha)P_n(\cos\beta) = \frac{1}{2\pi}\int_0^{2\pi} P_n(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi)d\phi. \tag{2}$$

Moreover the Legendre polynomials can be determined as solutions to (2). This integral equation can now be expressed in terms of the function ϕ on $\mathrm{SO}(3)$ defined by $\phi(g) = \phi(g \cdot o) = P_n(\cos(d(o, g \cdot o)))$. In fact (2) is equivalent to

$$\phi(g)\phi(h) = \int_K \phi(gkh)\, dk, \tag{3}$$

where K denotes the compact subgroup of $\mathrm{SO}(3)$ of all elements which fix the north pole o, and dk denotes the normalized Haar measure of K.

In fact, let A denote the subgroup of all elements of $\mathrm{SO}(3)$ which fix the point $(0,1,0)$. Then $\mathrm{SO}(3) = KAK$. Thus to prove (3) it is enough to consider rotations g and h around the y-axis through the angles α and β, respectively. Then if k denotes the rotation of angle ϕ around the z-axis we have

$$gkh \cdot o = (-\cos\alpha\cos\phi\sin\beta + \sin\alpha\cos\beta, -\sin\phi\sin\beta, \sin\alpha\cos\phi\sin\beta + \cos\alpha\cos\beta).$$

Thus $\cos(d(o, g \cdot o)) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi$ and

$$\phi(gkh) = P_n(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi).$$

The functional equation (3) has been generalized to many different settings. One is the following. Let G be a locally compact unimodular group and let K be a compact subgroup. A nontrivial complex valued continuous function ϕ on G is a *zonal spherical function* if (3) holds for all $g, h \in G$. Note that then $\phi(k_1 g k_2) = \phi(g)$ for all $k_1, k_2 \in K$ and all $g \in G$, and that $\phi(e) = 1$ where e is the identity element of G.

The example above arises when $G = \mathrm{SO}(3)$ $K = \mathrm{SO}(2)$ and $S^2 = G/K$. The other compact connected rank one symmetric spaces have zonal spherical functions which are orthogonal polynomials in an appropriate variable. These polynomials are special cases of Jacobi polynomials and they can be given explicitly as hypergeometric functions.

The complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$ is another rank one symmetric space. In this case the zonal spherical functions are $P_n^{0,1}(\cos\phi)$.

A very fruitful generalization of the functional equation (3) is the following (see [T1] and [GV]. Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ and $d(\delta)$ denote, respectively, the character and the dimension of any representation in the class δ, and set $\chi_\delta = d(\delta)\xi_\delta$. We shall denote by V a finite dimensional complex vector space and by $\mathrm{End}(V)$ the space of all linear transformations of V into V.

A *spherical function* Φ on G of type $\delta \in \hat{K}$ is a continuous function $\Phi : G \longrightarrow \mathrm{End}(V)$ such that $\Phi(e) = I$, (I= identity transformation) and

$$\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\,dk, \text{ for all } x, y \in G.$$

When δ is the class of the trivial representation of K and $V = \mathbb{C}$, the corresponding spherical functions are precisely the zonal spherical functions. From the definition it follows that $\pi(k) = \Phi(k)$ is a representation of K, equivalent to the direct sum of n representations in the class δ, and that $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$ for all $k_1, k_2 \in K$ and all $g \in G$. The number n is the height of Φ. The height and the type are uniquely determined by the spherical function.

2. Matrix valued spherical functions associated to $P_2(\mathbb{C})$

In [GPT1] the authors consider the problem of determining all irreducible spherical functions associated to the complex projective plane $P_2(\mathbb{C})$. This space can be realized as the homogeneous space G/K, $G = \mathrm{SU}(3)$ and $K = \mathrm{S}((\mathrm{U}(2) \times \mathrm{U}(1)) \simeq \mathrm{U}(2)$. In this case all irreducible spherical functions are of height one. Let (V_π, π) be any irreducible representation of K in the class δ. Then an irreducible spherical function can be characterized as a function $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ such that

i) Φ is analytic,
ii) $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$,
iii) $[\Delta\Phi](g) = \lambda(\Delta)\Phi(g)$, for all $g \in G$ and $\Delta \in D(G)^G$.

Here $D(G)^G$ denotes the algebra of all left and right invariant differential operators on G. In our case it is known that the algebra $D(G)^G$ is a polynomial algebra in two algebraically independent generators Δ_2 and Δ_3, explicitly given in [GPT1].

The set $\hat{K}$ can be identified with $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ in the following way: If $k \in K$ then

$$\pi(k) = \pi_{n,\ell}(k) = (\det k)^n k^\ell,$$

where k^ℓ denotes the ℓ-symmetric power of the matrix k.

For any $g \in \mathrm{SU}(3)$ we denote by $A(g)$ the left upper 2×2 block of g, and we consider the open dense subset $\mathcal{A} = \{g \in G : \det(A(g)) \neq 0\}$. Then $\mathcal{A}$ is left and right invariant under K. For any $\pi = \pi_{n,\ell}$ we introduce the following function defined on $\mathcal{A}$:

$$\Phi_\pi(g) = \pi(A(g)),$$

where π above denotes the unique holomorphic representation of $\mathrm{GL}(2,\mathbb{C})$ which extends the given representation of $\mathrm{U}(2)$.

To determine all irreducible spherical functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ of type $\pi = \pi_{n,\ell}$, we use the function Φ_π in the following way: in the open set $\mathcal{A} \subset G$ we define the function H by

$$H(g) = \Phi(g)\,\Phi_\pi(g)^{-1}, \tag{4}$$

where Φ is supposed to be a spherical function of type δ. Then H satisfies

i) $H(e) = I$,
ii) $H(gk) = H(g)$, for all $g \in \mathcal{A}, k \in K$,
iii) $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in \mathcal{A}, k \in K$.

The canonical projection $p : G \longrightarrow P_2(\mathbb{C})$ defined by $p(g) = g \cdot o$ where $o = (0,0,1)$ maps the open dense subset $\mathcal{A}$ onto the affine space $\mathbb{C}^2$ of those points in $P_2(\mathbb{C})$ whose last homogeneous coordinate is not zero. Then property ii) says that H may be considered as a function on $\mathbb{C}^2$, and moreover from iii) it follows that H is determined by its restriction $H = H(r)$ to the cross section $\{(r,0) \in \mathbb{C}^2 : r \geq 0\}$ of the K-orbits in $\mathbb{C}^2$, which are the spheres of radius $r \geq 0$ centered at the origin. That is H is determined by the function $r \mapsto H(r) = H(r,0)$ on the interval $[0,+\infty)$. Let M be the closed subgroup of K of all diagonal matrices of the form $\mathrm{diag}(e^{i\theta}, e^{-2i\theta}, e^{i\theta})$, $\theta \in \mathbb{R}$. Then M fixes all points $(r,0) \in \mathbb{C}^2$. Therefore iii) also implies that $H(r) = \pi(m)H(r)\pi(m^{-1})$ for all $m \in M$. Since any V_π as an M-module is multiplicity free, it follows that there exists a basis of V_π such that $H(r)$ is simultaneously represented by a diagonal matrix for all $r \geq 0$. Thus, if $\pi = \pi_{n,\ell}$, we can identify $H(r) \in \mathrm{End}(V_\pi)$ with a vector

$$H(r) = (h_0(r), \dots, h_\ell(r))^t \in \mathbb{C}^{\ell+1}.$$

The fact that Φ is an eigenfunction of Δ_2 and Δ_3 makes $H = H(r)$ into an eigenfunction of certain differential operators $\tilde{D}$ and $\tilde{E}$ on $(0,\infty)$.

Making the change of variables $t = 1/(1+r^2) \in (0,1)$ these operators become

$$\tilde{D}H = t(1-t)H'' + (A_0 - tA_1)H' + \frac{1}{1-t}(B_0 - tB_1)H, \tag{5}$$

$$\tilde{E}H = t(1-t)MH'' + (C_0 - tC_1)H' + \frac{1}{1-t}(D_0 + tD_1)H. \tag{6}$$

If we denote by E_{ij} the $(\ell+1)\times(\ell+1)$ matrix with entry (i,j) equal to 1 and 0 elsewhere, then the coefficient matrices are

$$A_0 = \textstyle\sum_{i=0}^{\ell}(n+\ell-i+1)E_{i,i} \qquad A_1 = \sum_{i=0}^{\ell}(n+\ell-i+3)E_{i,i},$$
$$B_0 = \textstyle\sum_{i=0}^{\ell}(i+1)(\ell-i)(E_{i,i+1} - E_{i,i}), \quad B_1 = \sum_{i=0}^{\ell} i(\ell-i+1)(E_{i,i} - E_{i,i-1}),$$
$$M = \textstyle\sum_{i=0}^{\ell}(n-\ell+3i)E_{i,i},$$
$$C_0 = \textstyle\sum_{i=0}^{\ell}(n-\ell+3i)(n+\ell-i+1)E_{i,i} - 3\sum_{i=0}^{\ell}(i+1)(\ell-i)E_{i,i+1},$$
$$C_1 = \textstyle\sum_{i=0}^{\ell}(n-\ell+3i)(n+\ell-i+3)E_{i,i} - 3\sum_{i=0}^{\ell} i(\ell-i+1)E_{i,i-1},$$

$$
\begin{aligned}
D_0 &= \textstyle\sum_{i=0}^{\ell}(n+2\ell-3i)(i+1)(\ell-i)(E_{i,i+1}-E_{i,i}) \\
&\quad - 3\textstyle\sum_{i=0}^{\ell}(n+\ell-i+1)i(\ell-i+1)(E_{i,i}-E_{i,i-1}), \\
D_1 &= (2n+\ell+3)B_1.
\end{aligned}
$$

The following result, which characterizes the spherical functions associated to the complex projective plane is taken from Theorem 3.8 of [RT], see also [GPT1].

Theorem 2.1. *The irreducible spherical functions Φ of $\mathrm{SU}(3)$ of type (n,ℓ), correspond precisely to the simultaneous $\mathbb{C}^{\ell+1}$-valued polynomial eigenfunctions H of the differential operators $\tilde{D}$ and $\tilde{E}$, introduced in (5) and (6), such that $h_i(t) = t^{i-n-\ell}g_i(t)$ for all $n+\ell+1 \le i \le \ell$ with g_i polynomial and $H(1) = (1,\dots,1)^t$.*

We also obtain, from [GPT1] or [PT1], that there is a bijective correspondence between the equivalence classes of all irreducible spherical functions Φ of type (n,ℓ) and the set of pairs of integers

$$\big\{(w,k) \in \mathbb{Z}\times\mathbb{Z} \,:\; 0 \le w,\; 0 \le k \le \ell\,, 0 \le w+n+k\big\}. \tag{7}$$

Under this correspondence the function H associated to the spherical function Φ satisfies $\tilde{D}H = \lambda H$ and $\tilde{E}H = \mu H$ where

$$
\begin{aligned}
\lambda &= \lambda_k(w) = -w(w+n+\ell+k+2) - k(n+k+1), \\
\mu &= \mu_k(w) = \lambda(n-\ell+3k) - 3k(\ell-k+1)(n+k+1),
\end{aligned} \tag{8}
$$

2.1. **Hypergeometric operators.** A key result to characterize the spherical functions of $\mathrm{SU}(3)$ of any type (n,ℓ) is the fact that the differential operator $\tilde{D}$ is conjugated, by a matrix polynomial function $\psi(t)$, to a hypergeometric operator D. From [RT], (or [PT2], for a more general situation) we obtain that the function $\psi(t) = XT(t)$, where

$$X = \sum_{0\le j\le i\le \ell} \tbinom{i}{j} E_{ij} \qquad T = \sum_{0\le i\le \ell} (1-t)^i E_{ii},$$

satisfies that the differential operator $D = \psi^{-1}\tilde{D}\psi$ takes the hypergeometric form

$$D = t(1-t)\frac{d^2}{du^2} + (C - tU)\frac{d}{du} - V, \tag{9}$$

where the coefficient matrices are

$$
\begin{aligned}
C &= \textstyle\sum_{i=0}^{\ell}(n+\ell+i+1)E_{i,i} + \sum_{i=0}^{\ell} iE_{i,i-1}, \qquad U = \sum_{i=0}^{\ell}(n+\ell+i+3)E_{i,i}, \\
V &= \textstyle\sum_{i=0}^{\ell} i(n+i+1)E_{i,i} - \sum_{i=0}^{\ell}(\ell-i)(i+1)E_{i,i+1}
\end{aligned}
$$

This fact allows us to describe the eigenfunctions of the differential operator $\tilde{D}$ in term of the matrix valued hypergeometric functions, introduced in [T2]: Let W be a d-dimensional complex vector space, and let A, B and $C \in \mathrm{End}(W)$. The hypergeometric equation is

$$z(1-z)F'' + (C - z(I+A+B))F' - ABF = 0, \tag{10}$$

where F stands for a function of z with values in W.

More generally we can consider the equation

$$z(1-z)F''+(C-zU)F'-VF=0. \tag{11}$$

In the scalar case the differential operator (11) is always of the form (10). Nevertheless in a noncommutative setting the equations $U=1+A+B$ and $V=AB$ may have no solutions A, B.

If the eigenvalues of C are not in $-\mathbb{N}_0$ we define the function

$${}_2H_1\left({U\,;\,V \atop C}\,;z\right)=\sum_{m=0}^{\infty}\frac{z^m}{m!}[C;U;V]_m, \tag{12}$$

where the symbol $[C,U,V]_m$ is defined inductively by $[C;U;V]_0=I$ and

$$[C;U;V]_{m+1}=(C+m)^{-1}(m^2+m(U-1)+V)[C;U;V]_m,$$

for all $m\geq 0$. The function ${}_2H_1\left({U\,;\,V \atop C}\,;z\right)$ is analytic on $|z|<1$, with values in $\mathrm{End}(W)$. Moreover if $F_0\in W$ then ${}_2H_1\left({U\,;\,V \atop C}\,;z\right)F_0$ is a solution of the hypergeometric equation (11) such that $F(0)=F_0$. Conversely any solution F of (11), analytic at $z=0$ is of this form.

2.2. **Spherical functions as matrix hypergeometric functions.** The irreducible spherical functions of SU(3) of type (n,ℓ) are in a one to one correspondence with certain simultaneous $\mathbb{C}^{\ell+1}$-polynomial eigenfunctions H of the differential operators $\tilde{D}$ and $\tilde{E}$ (see Theorem 2.1).

A delicate fact establish in [RT] is that the functions $F(t)=\psi(t)^{-1}H(t)$ are also polynomials functions which are eigenfunctions of the differential operators $D=\psi^{-1}\tilde{D}\psi$ and $E=\psi^{-1}\tilde{E}\psi$.
In the variable $u=1-t$, these operators have the form

$$D=\psi^{-1}\tilde{D}\psi=u(1-u)\frac{d^2}{du^2}+(C-uU)\frac{d}{du}-V, \tag{13}$$

$$E=\psi^{-1}\tilde{E}\psi=u(Q_0+uQ_1)\frac{d^2}{du^2}+(P_0+uP_1)\frac{d}{du}-(n+2\ell+3)V, \tag{14}$$

where the coefficient matrices are

$$\begin{aligned} C&=\textstyle\sum_{i=0}^{\ell}2(i+1)E_{i,i}+\sum_{i=0}^{\ell}iE_{i,i-1},\\ U&=\textstyle\sum_{i=0}^{\ell}(n+\ell+i+3)E_{i,i},\\ V&=\textstyle\sum_{i=0}^{\ell}i(n+i+1)E_{i,i}-\sum_{i=0}^{\ell}(\ell-i)(i+1)E_{i,i+1}\end{aligned} \tag{15}$$

$$Q_0=\textstyle\sum_{i=0}^{\ell}3iE_{i,i-1}, \qquad Q_1=\sum_{i=0}^{\ell}(n-\ell+3i)E_{i,i},$$
$$P_0=\textstyle\sum_{i=0}^{\ell}\big(2(i+1)(n+2\ell)-3(\ell-i)-3i^2\big)E_{i,i}-\sum_{i=0}^{\ell}i(3i+3+\ell+2n)E_{i,i-1},$$
$$P_1=\textstyle\sum_{i=0}^{\ell}-(n-\ell+3i)(n+\ell+i+3)E_{i,i}+\sum_{i=0}^{\ell}3(i+1)(\ell-i)E_{i,i+1},$$

To describe all simultaneous $\mathbb{C}^{\ell+1}$-polynomial eigenfunctions of the differential operators D and E we start by considering the eigenfunctions of D of eigenvalues

$\lambda = -w(w+n+\ell+k+2) - k(n+k+1)$, with $w, k \in \mathbb{N}_0, 0 \le k \le \ell$ (see (8)). We let

$$V_\lambda = \{F = F(u) : DF = \lambda F,\ F \text{ polynomial}\}.$$

Remark. It is not difficult to prove that that $V_\lambda \neq 0$ if and only if $\lambda = -w(w+n+\ell+k+2) - k(n+k+1)$, for some $0 \le k \le \ell$.

Therefore if $F \in V_\lambda$ then it is of the form

$$F(u) = {}_2H_1\left({U\,;\,V+\lambda \atop C} ; u \right) F_0,$$

for some $F_0 \in \mathbb{C}^\ell$. The simultaneous eigenfunctions of D and E will correspond to particular choices of F_0.

Since the initial value $F(0) = F_0$ determines $F \in V_\lambda$, we have that the linear map $\nu : V_\lambda :\to \mathbb{C}^{\ell+1}$ defined by $\nu(F) = F(0)$ is a surjective isomorphism. Since Δ_2 and Δ_3 commute, the differential operators D and E also commute. Moreover, since E has polynomial coefficients whose degrees are less or equal to the corresponding orders of differentiation, E restricts to a linear operator of W_λ. Thus we have the following commutative diagram

$$\begin{array}{ccc} V_\lambda & \xrightarrow{E} & V_\lambda \\ {\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\ \mathbb{C}^{\ell+1} & \xrightarrow{M(\lambda)} & \mathbb{C}^{\ell+1} \end{array} \tag{16}$$

where $M(\lambda)$ is the $(\ell+1)\times(\ell+1)$ matrix given by

$$M(\lambda) = Q_0(C+1)^{-1}(U+V+\lambda)C^{-1}(V+\lambda) + P_0C^{-1}(V+\lambda) - (n+2\ell+3)V. \tag{17}$$

The eigenvalues of M are given by (see Theorem 10.3 in [GPT1])

$$\mu_k(\lambda) = \lambda(n-\ell+3k) - 3k(\ell-k+1)(n+k+1), \qquad k = 0, 1, \dots, \ell.$$

Moreover, all eigenvalues $\mu_k(\lambda)$ of $M(\lambda)$ have geometric multiplicity one. In other words all eigenspaces are one dimensional. Moreover if $v = (v_0, ..., v_\ell)^t$ is a nonzero μ-eigenvector of $M(\lambda)$, then $v_0 \neq 0$.

The irreducible spherical functions of SU(3) of type (n,ℓ) are parameterized by two nonnegative integers w, k with $0 \le k \le \ell$ and $0 \le w+n+k$ (see (7)). Under this correspondence the function H associated to the spherical function satisfies $\tilde{D}H = \lambda H$ and $\tilde{E}H = \mu H$ where

$$\begin{aligned} \lambda &= \lambda_k(w) = -w(w+n+\ell+k+2) - k(n+k+1), \\ \mu &= \mu_k(w) = \lambda(n-\ell+3k) - 3k(\ell-k+1)(n+k+1), \end{aligned} \tag{18}$$

Then the characterization of the irreducible spherical functions is summarize in the following theorem, taking from [RT].

Theorem 2.2. *The function H associated to a spherical function of type (n,ℓ) and parameters w, k is of the form $H(u) = XT(u)F(u)$, where*

$$X = \sum_{0 \le j \le i \le \ell} \binom{i}{j} E_{ij}, \qquad T(u) = \sum_{0 \le i \le \ell} u^i E_{ii}$$

$$F(u) = {}_2H_1\left({U\,;\,V+\lambda \atop C};u\right)F_0, \tag{19}$$

and F_0 is the unique μ-eigenvector of $M(\lambda)$ normalized by $F_0 = (1, x_1, \dots, x_\ell)^t$. The expressions of the matrices C, U, V are given in (15) *and the eigenvalues λ and μ are given in* (18).

2.3. **Orthogonality.** Let $(C(G) \otimes \mathrm{End}(V_\pi))^{K\times K}$ be the space of all continuous functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ such that $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$. Let us equip V_π with an inner product such that $\pi(k)$ becomes unitary for all $k \in K$. We have the following inner product in $(C(G) \otimes \mathrm{End}(V_\pi))^{K\times K}$:

$$\langle \Phi, \Psi \rangle = \int_G \mathrm{Tr}(\Phi(g)\Psi(g)^*)\, dg\,, \tag{20}$$

where $\Psi(g)^*$ denotes the adjoint of $\Psi(g)$ with respect to the inner product in V_π. Then we have the following inner product on the corresponding functions H's associated to the spherical functions

$$\langle H, K\rangle = \sum_{i=0}^{\ell} \int_0^1 (1-t)t^{n+\ell-i}\, h_i(t)\overline{k_i(t)}\, dt = \int_0^1 K(t)^* \tilde W(t) H(t)\, dt, \tag{21}$$

where

$$\tilde W(t) = \sum_{0\le i\le \ell} (1-t)t^{n+\ell-i} E_{ii}.$$

Since the Casimir operator is symmetric with respect to the L^2-inner product for matrix valued functions on G given in (20), it follows that the differential operators $\tilde D$ and $\tilde E$ are symmetric with respect to the weight function $\tilde W$, that is they satisfy

$$\langle DH, K\rangle = \langle H, DK\rangle.$$

Now it is easy to verify that the differential operators $D = \psi^{-1}\tilde D\psi$ and $E = \psi^{-1}\tilde E\psi$ are symmetric with respect to the weight function $W = \psi^* \tilde W \psi$

$$W(u) = \sum_{i,j=0}^{\ell} \left(\sum_{r=0}^{\ell} \tbinom{r}{i}\tbinom{r}{j}(1-u)^{n+\ell-r} u^{i+j+1} \right) E_{ij}. \tag{22}$$

3. The explicit expressions

To illustrate the above result we will display the cases $\ell = 0$ (the scalar case) and $\ell = 1$, where the size of our matrices will be 2×2.

3.1. **The case $\ell = 0$.** In this case the functions $H(t) = h(t)$ are scalar functions. If the parameter n is 0 then we have the zonal spherical functions.
The operator $\tilde E$ is proportional to $\tilde D$, ($\tilde E = n\tilde D$) and

$$\tilde D h = t(1-t)h'' + (n+1-t(n+3))h'$$

To find the eigenfunctions of $\tilde D$ we put $\lambda = -w(w+n+2)$. Then h should be a solution of the hypergeometric equation with

$$a = -w, \quad b = w+n+2 \quad \text{and} \quad c = n+1.$$

For generic values of the parameters the functions

$$ {}_2F_1\left({-w, w+n+2 \atop n+1}; t\right) \quad \text{and} \quad t^{-n}{}_2F_1\left({-w-n, w+2 \atop 1-n}; t\right) $$

are linearly independent solutions. By Theorem 2.1 we have that h should be a polynomial function such that $h(1) = 1$. Moreover if $n < 0$ the function h have to satisfies $h(t) = t^{-n}g(t)$ with g a polynomial function. Therefore we get: For $n \geq 0$ and $w = 0, 1, 2, \dots$

$$ h_w(t) = \tfrac{(n+1)_w}{(-w-1)_w}\,{}_2F_1\left({-w, w+n+2 \atop n+1}; t\right) $$

For $n < 0$ and $w = -n, -n+1, \dots$

$$ h_w(t) = \tfrac{(1-n)_{w+n}}{(-w-n-1)_{w+n}}\,t^{-n}{}_2F_1\left({-w-n, w+2 \atop 1-n}; t\right) $$

where $(a)_m = a(a+1)\dots(a+m-1)$, for $m \in \mathbb{N}$ and $(a)_0 = 1$.
By using the Pfaff's identity we get

$$ \tfrac{(n+1)_w}{(-w-1)_w}\,{}_2F_1\left({-w, w+n+2 \atop n+1}; t\right) = {}_2F_1\left({-w, w+n+2 \atop 2}; 1-t\right) $$

and

$$ \tfrac{(1-n)_{w+n}}{(-w-n-1)_{w+n}}\,t^{-n}{}_2F_1\left({-w-n, w+2 \atop 1-n}; t\right) = t^{-n}{}_2F_1\left({-w-n, w+2 \atop 2}; 1-t\right) $$

Now by using the Euler transformation we get

$$ t^{-n}{}_2F_1\left({-w-n, w+2 \atop 2}; 1-t\right) = {}_2F_1\left({-w, w+n+2 \atop 2}; 1-t\right) $$

Therefore we obtain that

Proposition 3.1. *The spherical functions associated to the complex projective plane of type* $(n, 0)$ *are*

$$ h_w(t) = {}_2F_1\left({-w, w+n+2 \atop 2}; 1-t\right) $$

under the conditions $w \in \mathbb{Z}$, $w \geq 0$ *and* $w+n \geq 0$. *Moreover these functions satisfy*

$$ \tilde{D}h_w = -w(w+n+2)h_w \qquad \tilde{E}h_w = -nw(w+n+2)h_w. $$

3.2. **The case $\ell = 1$.** In this case the operators $\tilde{D}$ and $\tilde{E}$ are

$$ \begin{aligned} \tilde{D} =& t(1-t)\frac{d^2}{dt^2} + \begin{pmatrix} n+2-t(n+4) & 0 \\ 0 & n+1-t(n+3) \end{pmatrix}\frac{d}{dt} + \frac{1}{1-t}\begin{pmatrix} -1 & 1 \\ t & -t \end{pmatrix} \\ \tilde{E} =& t(1-t)\begin{pmatrix} (n-1) & 0 \\ 0 & (n+2) \end{pmatrix}\frac{d^2}{dt^2} \\ &+ \begin{pmatrix} (n-1)\big(n+2-t(n+4)\big) & -3 \\ 3t & (n+2)\big(n+1-t(n+3)\big) \end{pmatrix}\frac{d}{dt} \\ &+ \frac{1}{1-t}\begin{pmatrix} -n+2 & n+2 \\ 3n+3-2(n+2)t & -3n-3+2(n+2)t \end{pmatrix} \end{aligned} \tag{23} $$

In [GPT1], Section 11.1 we exhibit the complete list of spherical function of type $(n, 1)$. We have two families of such functions, corresponding with the choice of the parameter $k = 0$ or $k = 1$. For $n \geq 0$ the parameter w is in the range $w = 0, 1, 2, \dots$, and if $n < 0$ $w = -n, -n+1, \dots$.

First family. For $k = 0$ we have $\lambda = -w(w+n+3)$, $\mu = \lambda(n-1)$. The (vector valued) function H is given by, up to the normalizing constant such that $H(1) = (1,1)$.

$$H = \begin{cases} \left((1 - \frac{\lambda}{n+1})\, {}_3F_2\left({-w,\ w+n+3,\ \lambda-n \atop n+2,\ \lambda-n-1} ; t \right) ,\ {}_2F_1\left({-w,\ w+n+3 \atop n+1} ; t \right) \right) & \text{if } n \geq 0 \\ \left(n t^{-n-1}\, {}_3F_2\left({w+2,\ -w-n-1,\ a+1 \atop -n,\ a} ; t \right) ,\ t^{-n}\, {}_2F_1\left({w+3,\ -w-n \atop 1-n} ; t \right) \right) & \text{if } n < 0 \end{cases}$$

with $a = -w(w+n+3) - 2n - 2$.

Second family. For $k = 0$ we have $\lambda = -w(w+n+4) - n - 2$ and $\mu = (\lambda - 3)(n+2)$. The functions H is

$$H = \begin{cases} \left({}_2F_1\left({-w,\ w+n+4 \atop n+2} ; t \right) ,\ -(n+1)\, {}_3F_2\left({-w-1,\ w+n+3,\ \lambda \atop n+1,\ \lambda-1} ; t \right) \right) & \text{if } n \geq 0 \\ \left(t^{-n-1}\, {}_2F_1\left({w+3,\ -w-n-1 \atop -n} ; t \right) ,\ \frac{b}{n} t^{-n}\, {}_3F_2\left({w+3,\ -w-n-1,\ b+1 \atop 1-n,\ b} ; t \right) \right) & \text{if } n < 0 \end{cases}$$

with $b = -w(w+n+4) - 2n - 3$.

By taking the Taylor expansion at $t = 1$ these functions takes the following unified expression. We recall that $t = 1$ corresponds to the identity of the group G.

Theorem 3.2. *The complete list of spherical functions associated to* $\mathrm{SU}(3)$ *of type* $(n,1)$ *are given by*

(1) *For* $k = 0$ *we have* $\lambda = -w(w+n+3)$, $\mu = \lambda(n-1)$ *and*

$$H(t) = \begin{pmatrix} {}_3F_2\left({-w,\ w+n+3,\ 2 \atop 3,\ 1} ; 1-t \right) \\ {}_2F_1\left({-w,\ w+n+3 \atop 3} ; 1-t \right) \end{pmatrix}$$

The parameter w *is an integer that satisfies* $w \geq 0$ *and* $w + n \geq 0$.

(2) *For* $k = 1$, *we have* $\lambda = -w(w+n+4) - n - 2$, $\mu = (\lambda - 3)(n+2)$ *and*

$$H(t) = \begin{pmatrix} {}_2F_1\left({-w,\ w+n+4 \atop 3} ; 1-t \right) \\ {}_3F_2\left({-w-1,\ w+n+3,\ c+1 \atop 3,\ c} ; 1-t \right) \end{pmatrix}$$

where $c = \frac{(w+1)(w+n+3)}{(w+1)(w+n+3)+n}$.
The parameter w *is an integer that satisfies* $w \geq 0$ *and* $w + n + 1 \geq 0$.

In this case the function $\psi(u) = XT(u)$, where $X = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $T(u) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right)$ is

$$\psi(u) = \begin{pmatrix} 1 & 0 \\ 1 & u \end{pmatrix}. \tag{24}$$

In the variable $u = 1-t$, the conjugated operators $D = \psi^{-1}\tilde{D}\psi$ and $E = \psi^{-1}\tilde{E}\psi$ are

$$\begin{aligned} D &= u(1-u)\frac{d^2}{du^2} + \begin{pmatrix} 2-(n+4)u & 0 \\ 1 & 4-(n+5)u \end{pmatrix}\frac{d}{du} - \begin{pmatrix} 0 & -1 \\ 0 & n+2 \end{pmatrix}, \\ E &= (1-u)\begin{pmatrix} (n-1)u & 0 \\ 3 & (n+2)u \end{pmatrix}\frac{d^2}{du^2} \\ &\quad + \begin{pmatrix} 2n+1-(n-1)(n+4)u & 3u \\ -(2n+7) & 4n+5-(n+2)(n+5)u \end{pmatrix}\frac{d}{du} \\ &\quad - (n+5)\begin{pmatrix} 0 & -1 \\ 0 & n+2 \end{pmatrix} \end{aligned} \tag{25}$$

The matrix $M(\lambda)$ (see (17)), is

$$M(\lambda) = \begin{pmatrix} \lambda(n+\frac{1}{2}) & \frac{9}{2} \\ \lambda(\frac{\lambda}{2}-n-2) & \lambda(n+\frac{1}{2})-3(n+2) \end{pmatrix}.$$

The eigenvalues of $M(\lambda)$ are $\mu_0 = \lambda(n-1)$ and $\mu_1 = (n+2)(\lambda-3)$ and the respective normalized eigenvectors are

$$F_{0,0} = \begin{pmatrix} 1 \\ -\frac{\lambda}{3} \end{pmatrix}, \quad F_{0,1} = \begin{pmatrix} 1 \\ \frac{\lambda-2(n+2)}{3} \end{pmatrix}.$$

Therefore the functions F associated to the spherical functions are, for $k = 0, 1$,

$$F(u) = {}_2H_1\left({U\,;\,V+\lambda_k \atop C} ; u \right) F_{0,k},$$

where $\lambda_k = -w(w+n+3+k) - k(n+k+1)$,

$$C = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}, \qquad U = \begin{pmatrix} n+4 & 0 \\ 0 & n+5 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 1 \\ 0 & -n-2 \end{pmatrix}.$$

The explicit expression of the entries of these functions F's are given in the following theorem.

Theorem 3.3. *The functions $F = \psi^{-1}H$ associated to the spherical functions of the pair $(\mathrm{SU}(3), \mathrm{U}(2))$ of type $(n, 1)$ are given by*

(1) *For $k = 0$ we have $\lambda = -w(w+n+3)$, $\mu = \lambda(n-1)$ and*

$$F(u) = \begin{pmatrix} {}_3F_2\left({-w,\ w+n+3,\ 2 \atop 3,\ 1} ; u \right) \\ \frac{w(w+n+3)}{3}\, {}_2F_1\left({-w+1,\ w+n+4 \atop 4} ; u \right) \end{pmatrix}$$

The parameter w is an integer that satisfies $w \geq 0$ and $w + n \geq 0$.

(2) *For $k = 1$, we have $\lambda = -w(w+n+4) - n - 2$, $\mu = (\lambda - 3)(n+2)$ and*

$$F(u) = \begin{pmatrix} {}_2F_1\left({-w,\ w+n+4 \atop 3} ; u \right) \\ -\frac{s_w}{3}\, {}_3F_2\left({-w,\ w+n+4,\ s_w+1 \atop 4,\ s_w} ; u \right) \end{pmatrix}$$

where $s_w = w(w+n+4) + 3(n+2)$.
The parameter w is an integer that satisfies $w \geq 0$ and $w + n + 1 \geq 0$.

Proof. If $H = (h_1, h_2)$ is an eigenfunction of $\tilde{D}$ then $F(u) = \psi^{-1}(u)H(u)$ is an eigenfunction of D with the same eigenvalue. Explicitly the function F is

$$F(u) = \Big(h_1(u), \tfrac{h_2(u)-h_1(u)}{u}\Big) \tag{26}$$

From (26) we only have to prove the expression for the second entry of the function F. For the first family, from Theorem 3.2 we get

$$\begin{aligned} f_2(u) = \frac{h_2(u)-h_1(u)}{u} &= \frac{1}{u}\Big({}_2F_1\left(\begin{smallmatrix} -w,\ w+n+3 \\ 3 \end{smallmatrix}; u\right) - {}_3F_2\left(\begin{smallmatrix} -w,\ w+n+3,\ 2 \\ 3,\ 1 \end{smallmatrix}; u\right)\Big) \\ &= \frac{1}{u}\Big(\sum_{j\geq 0} \tfrac{(-w)_j (w+n+3)_j (1-(j+1))}{j!\,(3)_j} u^j\Big) = \tfrac{w(w+n+3)}{3}\, {}_2F_1\left(\begin{smallmatrix} -w+1,\ w+n+4 \\ 4 \end{smallmatrix}; u\right) \end{aligned}$$

For the second family we obtain, with $c = \frac{(w+1)(w+n+3)}{(w+1)(w+n+3)+n}$

$$\begin{aligned} f_2(u) = \frac{h_2(u)-h_1(u)}{u} &= \frac{1}{u}\Big({}_3F_2\left(\begin{smallmatrix} -w-1,\ w+n+3,\ c+1 \\ 3,\ c \end{smallmatrix}; u\right) - {}_2F_1\left(\begin{smallmatrix} -w,\ w+n+4 \\ 3 \end{smallmatrix}; u\right)\Big) \\ &= \frac{1}{u}\Big(\sum_{j\geq 0} \tfrac{(-w)_{j-1}(w+n+4)_{j-1}}{(3)_j\, j!}\big(-(w+1)(w+n+3)(1+j/c) \\ &\qquad\qquad - (-w+j-1)(w+n+3+j)\big)u^j\Big) \\ &= \sum_{j\geq 0} \tfrac{(-w)_{j-1}(w+n+4)_{j-1}}{(3)_j\, j!}(-j)\big(w(w+n+4)+3(n+2)+j-1\big)u^{j-1} \\ &= -\sum_{j\geq 0} \tfrac{(-w)_j (w+n+4)_j}{3(4)_j\, j!}\big(w(w+n+4)+3(n+2)+j\big)u^j \\ &= -\sum_{j\geq 0} \tfrac{s_w}{3}\tfrac{(-w)_j (w+n+4)_j (s_w+j)}{j!\,(4)_j\, s_w} u^j = -\tfrac{s_w}{3}\, {}_3F_2\left(\begin{smallmatrix} -w,\ w+n+4,\ s_w+1 \\ 4,\ s_w \end{smallmatrix}; u\right) \end{aligned}$$

This concludes the proof of the theorem. □

3.3. **Matrix valued orthogonal polynomials coming from spherical functions.** In the scalar case, it is well known that the zonal spherical functions of the sphere $S^d = \mathrm{SO}(d+1)/\mathrm{SO}(d)$ are given, in spherical coordinates, in terms of Gegenbauer polynomials. Therefore, it is not surprising that in the matrix valued setting the same phenomenon occurs: the matrix spherical functions are closely related to matrix orthogonal polynomials.

For a given nonnegative integers n and w we define the matrix polynomial $P_w(u)$ as the 2×2 matrix function whose k-row is the polynomial $F_{w,k}(u)$, associated to the spherical functions of type $(n, 1)$, given in the previous section. In other words

$$P_w(u) = \begin{pmatrix} {}_3F_2\left(\begin{smallmatrix} -w,\ w+n+3,\ 2 \\ 3,\ 1 \end{smallmatrix}; u\right) & \tfrac{w(w+n+3)}{3}\, {}_2F_1\left(\begin{smallmatrix} -w+1,\ w+n+4 \\ 4 \end{smallmatrix}; u\right) \\ {}_2F_1\left(\begin{smallmatrix} -w,\ w+n+4 \\ 3 \end{smallmatrix}; u\right) & -\tfrac{s_w}{3}\, {}_3F_2\left(\begin{smallmatrix} -w,\ w+n+4,\ s_w+1 \\ 4,\ s_w \end{smallmatrix}; u\right) \end{pmatrix}.$$

where $s_w = w(w+n+4)+3(n+1)$.

Since different spherical functions are orthogonal with respect to the natural inner product among these functions, we obtain that the matrices P_w are orthogonal with respect to the weight function $W = W(u)$:

$$W(u) = u(1-u)^n \begin{pmatrix} 2-u & u \\ u & u^2 \end{pmatrix},$$

explicitly we have

$$(P_w, P_{w'}) = \int_0^1 P_w(u)W(u)P_{w'}(u)^*\,du = 0, \quad \text{for all } w \neq w'.$$

A look at the definition shows that the leading coefficient of P_w is a triangular nonsingular matrix. Therefore $(P_w)_w$ is a sequence of matrix valued orthogonal polynomials with respect to the weight matrix W.

The columns of P_w^* are eigenfunctions of the differential operators D and E given in (25), thus we have that $P_w(u)$ satisfies

$$DP_w(u)^* = P_w(u)^* \begin{pmatrix} \lambda_0(w) & 0 \\ 0 & \lambda_1(w) \end{pmatrix}, \qquad EP_w(u)^* = P_w(u)^* \begin{pmatrix} \mu_0(w) & 0 \\ 0 & \mu_1(w) \end{pmatrix},$$

where the eigenvalues $\lambda_k(w)$ and $\mu_k(w)$ are

$$\begin{aligned} \lambda_0(w) &= -w(w+n+3), & \lambda_1(w) &= -w(w+n+4) - n - 2 \\ \mu_0(w) &= \lambda_0(w)(n-1), & \mu_1(w) &= (\lambda_1(w) - 3)(n+2) \end{aligned}$$

3.4. **Extension of the group parameters.** These results have a direct and fruitful generalization by replacing the complex projective plane by the d-dimensional complex projective space $P_d(\mathbb{C})$, which can be realized as the homogeneous space G/K, where $G = \mathrm{SU}(d+1)$ and $K = \mathrm{S}(\mathrm{U(d)} \times \mathrm{U}(1)) \simeq \mathrm{U}(d)$.

In this case, the finite dimensional irreducible representations of K, are parameterized by the d-tuples of integers $\pi = (m_1, m_2, \ldots, m_d) \in \hat{K}$ such that $m_1 \geq m_2 \geq \cdots \geq m_d$. By considering the irreducible spherical functions of type $\pi = (n+1, n, \ldots, n)$, and proceeding as we explained for the complex projective plane, one obtains a situation that generalizes the one of $d = 2$. Then by extending the parameters $\alpha = n$, $\beta = d-1$ we have the following results.

Theorem 3.4. *Let* $s_w = \frac{w(w+\alpha+\beta+3)+(\beta+2)(\alpha+\beta+1)}{\beta}$ *and let us define*

$$P_w(t) = \begin{pmatrix} {}_3F_2\left({-w, w+\alpha+\beta+2, 2 \atop \beta+2, 1}; 1-t\right) & \frac{w(w+\alpha+\beta+2)}{\beta+2}\, {}_2F_1\left({-w+1, w+\alpha+\beta+3 \atop \beta+3}; 1-t\right) \\ {}_2F_1\left({-w, w+\alpha+\beta+3 \atop \beta+2}; 1-t\right) & -\frac{s_w}{\beta+2}\, {}_3F_2\left({-w, w+\alpha+\beta+3, s_w+1 \atop \beta+3, s_w}; 1-t\right) \end{pmatrix}$$

Then $\{P_w\}_{w\geq 0}$ *is a sequence of orthogonal polynomials with respect the weight matrix*

$$W(t) = t^\alpha (1-t)^\beta \begin{pmatrix} \beta + t & \beta(1-t) \\ \beta(1-t) & \beta(1-t)^2 \end{pmatrix}, \qquad (\alpha, \beta > -1).$$

Let D be the following second order differential operator

$$D = t(1-t)\frac{d^2}{dt^2} + \begin{pmatrix} \alpha+2-t(\alpha+\beta+3) & 0 \\ -1 & \alpha+1-t(\alpha+\beta+4) \end{pmatrix} \frac{d}{dt} + \begin{pmatrix} 0 & \beta \\ 0 & -(\alpha+\beta+1) \end{pmatrix} I$$

The polynomials P_w satisfies

$$DP_w^* = P_w^* \Lambda_w$$

where

$$\Lambda_w = \begin{pmatrix} -w(w+\alpha+\beta+2) & 0 \\ 0 & -w(w+\alpha+\beta+3)-(\alpha+\beta+1) \end{pmatrix}$$

In [PT1] for SU(3) or in general in [P08], we obtain a multiplication formula for spherical functions by tensoring certain irreducible representations of SU($n+1$) and decomposing them into irreducible representations. From this formula we derive a three term recursion relation for the "packages" of spherical functions. Restricting this to the variable t (the variable that parameterizes a section of the K-orbits in $P_2(\mathbb{C})$), we obtain a three term recursion relation for the packages of functions F associated to the spherical functions. In this case we obtain the following

Theorem 3.5. *The sequence $\{P_w(t)\}_{w\geq 0}$ satisfies the following three term recursion relation*

$$A_w P_{w-1}(t) + B_w P_w(t) + C_w P_{w+1}(t) = tP_w(t).$$

with

$$A_w = \begin{pmatrix} \frac{w(w+\alpha)(w+\alpha+\beta+1)}{(w+\alpha+\beta)(2w+\alpha+\beta+1)(2w+\alpha+\beta+2)} & \frac{w\beta}{(w+1)(w+\alpha+\beta)(2w+\alpha+\beta+2)} \\ 0 & \frac{w(w+2)(w+\alpha+1)}{(w+1)(2w+\alpha+\beta+2)(2w+\alpha+\beta+3)} \end{pmatrix}$$

$$C_w = \begin{pmatrix} \frac{(w+1)(w+\beta+2)(w+\alpha+\beta+2)}{(w+2)(2w+\alpha+\beta+2)(2w+\alpha+\beta+3)} & 0 \\ \frac{(w+\beta+2)}{(w+2)(w+\alpha+\beta+2)(2w+\alpha+\beta+3)} & \frac{(w+\beta+2)(w+\alpha+\beta+1)(w+\alpha+\beta+3)}{(w+\alpha+\beta+2)(2w+\alpha+\beta+3)(2w+\alpha+\beta+4)} \end{pmatrix}$$

$$B_w = \begin{pmatrix} B_w^{11} & \frac{\beta(w+\alpha+\beta+2)}{(w+2)(w+\alpha+\beta+1)(2w+\alpha+\beta+2)} \\ \frac{w+\alpha+1}{(w+1)(w+\alpha+\beta+1)(2w+\alpha+\beta+3)} & B_w^{22} \end{pmatrix}$$

where

$$\begin{aligned} B_w^{11} &= \frac{(w+1)^2(w+\beta+2)}{(w+2)(2w+\alpha+\beta+2)(2w+\alpha+\beta+3)} + \frac{\beta}{(w+1)(w+2)(w+\alpha+\beta)(w+\alpha+\beta+1)} \\ &\quad + \frac{(w+\alpha)(w+\alpha+\beta+1)^2}{(w+\alpha+\beta)(2w+\alpha+\beta+1)(2w+\alpha+\beta+2)} \\ B_w^{22} &= \frac{(w+1)(w+3)(w+\beta+2)}{(w+2)(2w+\alpha+\beta+3)(2w+\alpha+\beta+4)} + \frac{(w+\alpha+1)(w+\alpha+\beta)(w+\alpha+\beta+2)}{(w+\alpha+\beta+1)(2w+\alpha+\beta+2)(2w+\alpha+\beta+3)} \end{aligned}$$

Remark 3.6. The three term recursion relation can be seen as a difference operator in the variable w, given by a semiinfinite matrix L. The vector matrix $P = (P_0, P_1, \dots, P_w, \dots)$ is an eigenfunction of L because it satisfies $LP = tP$. We observe that the semiinfinte matrix L have the interesting property that the sum of all the matrix elements in any row is equal to one. Moreover all the entries of L are nonnegative real numbers. This has important applications in the modeling of some stochastic phenomena.

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