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RESTRICTION OF THE FOURIER TRANSFORM

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ABSTRACT. This paper contains a brief survey about the state of progress on the restriction of the Fourier transform and its connection with other conjectures. It contains also a description of recent related results that we have obtained.

1. Introduction

If $f \in L^1(R^n)$, the integral defining

$$\widehat{f}(\xi) = \int e^{-ix.\xi} f(x)\, dx$$

is absolutely convergent for every $\xi \in R^n$ and defines a continuos function on R^n.

For more general functions f the extension of the definition of $\widehat{f}$ requires density arguments. In particular if $f \in L^1(R^n) \cap L^2(R^n)$, the identity of Plancherel

$$\left\|\widehat{f}\right\|_2 = \|f\|_2\,,$$

allows us to extend the definition of $\widehat{f}$ to $L^2(R^n)$.

Moreover, since obviously

$$\left\|\widehat{f}\right\|_\infty \leq \|f\|_1\,,$$

from the Riez-Thorin theorem we obtain

$$\left\|\widehat{f}\right\|_{p'} \leq \|f\|_p\,,$$

for $f \in L^1(R^n) \cap L^p(R^n)$, $1 \leq p \leq 2$, and p' the Hölder conjugate of p. So we can extend the notion of $\widehat{f}$ to these $L^p(R^n)$.

Suppose that Σ is a given smooth submanifold of R^n and that μ is its induced Lebesgue measure .

If $1 \leq p \leq \infty$, we say that the L^p *restriction property* is valid for Σ if there exists $q = q(p)$, $1 \leq q \leq \infty$, so that the inequality

$$\left(\int_{\Sigma_0} \left|\widehat{f}(\xi)\right|^q d\mu\right)^{1/q} \leq A_{p,q}(\Sigma_0)\, \|f\|_{L^p(R^n)}$$

holds for each $f \in S(R^n)$ whenever Σ_0 is an open subset of Σ with compact closure in Σ. Because $S(R^n)$ is dense $L^p(R^n)$ we can, in this case, define $\widehat{f}$ on Σ (a.e. with respect to μ), for each $f \in L^p(R^n)$.

The determination of optimal ranges for the exponents p and q are difficult problems which have not yet been completely solved.

In paragraph 2 we describe some known results about certain submanifolds with this property, and we also describe the connection with the Kakeya and the Bochner Riesz conjectures.

In Paragraph 3 we state the results that we have obtained for hypersurfaces Σ given as the graph of certain homogeneous polynomial functions.

2. Some known results

From now on we will suppose that Σ is a compact submanifold of R^n and we wil study the restrition operator $\digamma : f \to \widehat{f}|_\Sigma$, where

$$\digamma f(\xi) = \widehat{f}|_\Sigma(\xi) = \int e^{-ix.\xi} f(x)\, dx \; \forall \xi \in \Sigma.$$

Remark: Since $|\digamma f| \leq \|f\|_{L^1(R^n)}$, the L^1 restriction property is obvious, taking $q = \infty$. Moreover, we can take any $1 \leq q \leq \infty$. Indeed

$$\|\digamma f\|_{L^q(\Sigma)} = \left[\int_\Sigma \left| \int_{R^n} e^{-ix\xi} f(x) dx \right|^q d\mu(\xi) \right]^{1/q}$$

$$\leq \left[\int_\Sigma \left(\int_{R^n} |f(x)|\, dx \right)^q d\mu(\xi) \right]^{1/q} = \|f\|_{L^1(R^n)} \mu(\Sigma)^{1/q}.$$

As usual, for $1 \leq p \leq \infty$, we define p' by $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem (P. A. Tomas, E. Stein, 1975) *Let S^{n-1} be the unit sphere of R^n, let $1 \leq p \leq \frac{2n+2}{2n+3}$ and $q = \left(\frac{n-1}{n+1}\right) p'$. There exists $A(p,q)$ such that, for $f \in S(R^n)$,*

$$\|\digamma f\|_{L^q(S^{n-1})} \leq A_{p,q} \|f\|_{L^p(R^n)} .$$

Remark. The statement of the above theorem still holds if $1 \leq p \leq \frac{2n+2}{n+3}$, $q \leq \left(\frac{n-1}{n+1}\right) p'$. Indeed,

$$\|\digamma f\|_{L^q(S^{n-1})} \leq \|\digamma f\|_{L^{\left(\frac{n-1}{n+1}\right)p'}(S^{n-1})}$$

The proof of the theorem extends naturally to submanifold Σ of R^n of dimension $n-1$, with never vanishing Gaussian curvature.

In general, it can be proved that the condition $q \leq \left(\frac{n-1}{n+1}\right) p'$ is necessary. It is not known if the condition about p is also necessary.

We have the following result. *If Σ is a compact submanifold of R^n and for some $1 \leq p, q \leq \infty$, $\digamma : L^p(R^n) \to L^q(\Sigma)$ is a bounded operator, then $\widehat{\mu} \in L^{p'}(R^n)$.*

In the case of the sphere, studying $\widehat{\mu}$, it can be checked that if $\widehat{\mu} \in L^{p'}(R^n)$, then $\frac{1}{p} > \frac{n+1}{2n}$, in other words, $\frac{1}{p} > \frac{n+1}{2n}$ is a necessary condition for $\digamma$ to have an

L^p restriction property. This result can also be proved for submanifols with never vanishing Gaussian curvature.

For these submanifols then, everything is done, except in the sector $\frac{n+1}{2n} < \frac{1}{p} < \frac{n+3}{2n+2}$. The *Stein conjecture* says that for submanifolds of codimension one in R^n, $n > 2$, with never vanishing Gaussian curvature we should be able to obain the statement of the theorem in that sector. For $n = 2$ this result has already been proved.

Theorem. (Fefferman 1970) *Let γ be a curve in R^2 with never vanishing curvature and let γ_0 be a subarc of γ. If $\frac{3}{4} < \frac{1}{p} \leq 1$ and $\frac{1}{3q} + \frac{1}{p} \geq 1$ then there exists $A_{p,q}(\gamma_0)$ such that, for $f \in S\left(R^2\right)$,*

$$\|F f\|_{L^q(\gamma_0)} \leq A_{p,q}(\gamma_0) \|f\|_{L^p(R^2)}.$$

In this case, $\frac{n+1}{2n} = \frac{3}{4}$, and we already know that these conditions about p and q are also necessary.

Back to the Stein conjecture, in the paper [4] there is a very interesting survey about the recent improvements that different authors have obtained , for the cases of the sphere and the paraboloid.

The restriction conjecture is related with the *Kakeya conjecture,* that is stated as follows **The Hausdorff dimension of a Kakeya set in R^n is n.** Up to these days, it is only known that this last conjecture is true for $n = 2$, but it is still an open problem for greater dimensions.

Definition. A Kakeya set, or a Besicovitch set is a compact set $E \subset R^n$, which contains a unitary segment in each direction, i.e

$$\forall e \in S^{n-1} \exists x \in R^n : x + te \in E,$$
$$\forall t \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

An old (from about 1920) and well known result due to Besicovitch asserts that for $n \geq 2$, there exist Kakeya sets in R^n with measure zero.

We define now the concept of *Hausdorff dimension.* For $\alpha > 0$ and $E \subset R^n$ we set $H_\alpha^\varepsilon(E) = \inf \sum_{j=1}^{\infty} (r_j)^\alpha$, where the infimum is taken over the countable coverings of E by discs $D(x_j, r_j)$ with $r_j < \varepsilon$.

We define $H_\alpha(E) = \lim_{\varepsilon \to 0} H_\alpha^\varepsilon(E)$. It is easy to check that there exists α_0, called the *Hausdorff dimension* of E such that $H_\alpha(E) = 0$ for $\alpha > \alpha_0$ and $H_\alpha(E) = \infty$ for $\alpha < \alpha_0$.

Fefferman y Bourgain proved that if the restriction conjecture holds for the sphere S^{n-1}, with $n > 2$ then the Kakeya conjecture also holds. A very nice approach to these subjects can be found in [5].

Another problem related with the restriction conjecture is the following. Fix $n \geq 2$, $1 \leq p \leq \infty$ and $\alpha > 0$, following [3] we use $BR(p, \alpha)$ to denote the statement that $S^{\delta(p)+\alpha}$ is bounded on L^p, where $\delta(p) = \max\left(n\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2},0\right)$ and S^{δ} is the Bochner Riesz multiplier

$$\widehat{S^{\delta}f}(\xi) = \left(1-|\xi|^2\right)_+^{\delta}\widehat{f}(\xi).$$

The *Bochner-Riesz conjecture* says that $BR(p,\varepsilon)$ holds for every $1 \le p \le \infty$ and for every $\varepsilon > 0$. In [3] the author proves that the Bochner Riesz conjecture implies the restriction conjecture.

3. Our results

We (jointly with Elida Ferreyra and Tomás Godoy) study hypersufaces Σ de R^3 given as a compact subset of the graph of a homogeneous polynomial function φ of degree $m \ge 2$,

$$\Sigma = \left\{\left(x_1, x_2, \varphi(x_1,x_2) : x_1^2 + x_2^2 \le 1\right)\right\}.$$

We denote by $Q = [0,1]\times[0,1]$. We try to obtain information about the *type set*

$$E = \left\{\left(\frac{1}{p},\frac{1}{q}\right) \in Q : \|\digamma f\|_{L^q(\Sigma)} \le c\,\|f\|_{L^p(R^3)}\right\}$$

for some $c > 0$ and for every $f \in S(R^3)$.

3.1. Necessary conditions.

A simple homogeneity argument shows that if $\left(\frac{1}{p},\frac{1}{q}\right) \in E$ then

$$\frac{1}{q} \ge -\left(\frac{m}{2}+1\right)\frac{1}{p} + \left(\frac{m}{2}+1\right).$$

The set of pairs $\left(\frac{1}{p},\frac{1}{q}\right)$ for which the equality holds is called the *homogeneity line.*

If $\det\varphi''$ does not vanish identically we know that the inequalities

$$\frac{1}{q} \ge -\frac{2}{p}+2 \qquad \textit{and} \qquad \frac{1}{p} > \frac{2}{3}$$

are neccesary conditions for a pair $\left(\frac{1}{p},\frac{1}{q}\right) \in E$. The first inequality is the same than the corresponding to homogeneity degree 2. Trying to obtain as much information as we could about E, (a sharp result would be to obtain that E is the set given by $\frac{1}{q} \ge -\left(\frac{m}{2}+1\right)\frac{1}{p} + \left(\frac{m}{2}+1\right)$ and $\frac{1}{p} > \frac{2}{3}$) we found some difficulties that suggested the existence of another line with greater slope than the slope of the homogeneity line, providing a better necessary condition. Indeed, if $\det\varphi''$ does not vanish identically on $R^2\setminus\{0\}$, but if it vanishes in some point $x_0 \ne 0$, it vanishes on a finite union of lines through the origin. If L_j, $1 \le j \le k$, is one of such lines, the vanishing order α_j of $\det\varphi''(x)$ in any point of L_j (α_j is independent of the point on L_j) plays a fundamental role. We define $\widetilde{m} = \max\{m,\ \alpha_1, ..., \alpha_k\}$ and we obtain that if $\left(\frac{1}{p},\frac{1}{q}\right) \in E$ then

$$\frac{1}{q} \ge -\left(\frac{\widetilde{m}}{2}+1\right)\frac{1}{p} + \left(\frac{\widetilde{m}}{2}+1\right).$$

We remark that in some cases, $\alpha_j > m$. For example, if $\varphi(x_1, x_2) = x_2^7 (x_1 + x_2)$, $\det \varphi''(x_1, x_2) = -49 x_2^{12}$, and its vanishing order on the x_1 axis is 12. In this case the line corresponding to $\widetilde{m}$ has bigger slope than the slope of the homogeneity line, and so we obtain a better necessary condition.

3.2. **Sufficient Conditions.** If $\det \varphi'' \equiv 0$, possibly after a linear change of coordinates that leaves E invariant, we have $\varphi(x_1, x_2) = x_2^m$, and it is easy to see that in this case the set E is the type set corresponding to the curve (t, t^m) in R^2. We obtained then the following result

Let $\varphi : R^2 \to R$ be a homogeneous polynomial function of degree $m \geq 2$ such that $\det \varphi''(x) \equiv 0$. *Then for $m \geq 3$*

$$E^\circ = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in Q : \frac{1}{q} > -\frac{m+1}{p} + m + 1 \right\}$$

and for $m = 2$

$$E^\circ = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in (\frac{3}{4}, 1] \times [0, 1] : \frac{1}{q} > -\frac{3}{p} + 3 \right\}.$$

If $\det \varphi''(x_1, x_2) \neq 0$ for $(x_1, x_2) \in R^2 \setminus \{0\}$, we obtain

(i) If $m \geq 6$, then

$$E^\circ = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in Q : \frac{1}{q} > -\left(\frac{m}{2} + 1 \right) \frac{1}{p} + \frac{m}{2} + 1 \right\},$$

(ii) if $m < 6$

$$E^\circ \cap \left(\left(\frac{3}{4}, 1 \right] \times [0, 1] \right) = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in Q : \frac{1}{q} > -\left(\frac{m}{2} + 1 \right) \frac{1}{p} + \frac{m}{2} + 1 \right\} \cap \left(\left(\frac{3}{4}, 1 \right] \times [0, 1] \right)$$

and also $\left(\frac{3}{4}, \frac{1}{q} \right) \in E$ for $\frac{\widetilde{m}+2}{8} < \frac{1}{q} \leq 1$.

In the region given by $\frac{1}{q} \geq -\left(\frac{m}{2} + 1 \right) \frac{1}{p} + \left(\frac{m}{2} + 1 \right)$, $\frac{2}{3} < \frac{1}{p} < \frac{3}{4}$, we can not give neither a positive nor a negative answer to the question if $\left(\frac{1}{p}, \frac{1}{q} \right)$ belongs to E. Also, we don't know wether $\left(\frac{3}{4}, \frac{m+2}{8} \right)$ belongs to E or not.

We did not expect to obtain positive results for $\frac{1}{p} < \frac{3}{4}$ since our proof basically consists in applying the Stein-Tomas theorem to the restriction of the Fourier transform to the shells

$$\Sigma_j = \left\{ \left(x_1, x_2, \varphi(x_1, x_2) : 2^{-j-1} \leq x_1^2 + x_2^2 \leq 2^{-j} \right) \right\},$$

that have non vanishing curvature, and then scaling.

If $\det \varphi''$ does not vanish identically on $R^2 \setminus \{0\}$, but if it vanishes in some point $x_0 \neq 0$, we obtain the same results than before, with m replaced by $\widetilde{m}$.

Finally, in every case we obtain a sharp $L^p(R^3) \to L^2(\Sigma)$ estimate.

The techniques that we use were:
- Asymptotic developments and Van der Corput lemmas for oscillatory integrals.
- Real and complex interpolation.
- Littlewood Paley theory.

These results are in the paper [2].

Lately, with E. Ferreyra, we studied the cases of anisotropically homogeneous surfaces. For $\beta_1, ..., \beta_n > 1$, and B the unit ball of R^n, we consider $\varphi : R^n \rightarrow R$ of the form $\varphi(x_1, ..., x_n) = \Sigma_{i=1}^n |x_i|^{\beta_i}$ and we studied the restriction of the Fourier transform to the surface S given by $S = \{(x, \varphi(x)) : x \in B\}$. We obtained a poligonal region contained in the type set E. In some cases this result is sharp (see [1]).

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