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A MODEL FOR THE THERMOELASTIC BEHAVIOR OF A JOINT-LEG-BEAM SYSTEM FOR SPACE APPLICATIONS

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Abstract. Rigidizable-Inflatable (RI) materials offer the possibility of deployable large space structures (C.H.M. Jenkins (ed.), Gossamer Spacecraft: Membrane and Inflatable Structures Technology for Space Applications, Progress in Aeronautics and Astronautics, 191, AIAA Pubs., 2001) and so are of interest in applications where large optical or RF apertures are needed. In particular, in recent years there has been renewed interest in inflatable-rigidizable truss-structures because of the efficiency they offer in packaging during boost-to-orbit. However, much research is still needed to better understand dynamic response characteristics, including inherent damping, of truss structures fabricated with these advanced material systems. One of the most important characteristics of such space systems is their response to changing thermal loads, as they move in and out of the Earth's shadow. We study a model for the thermoelastic behavior of a basic truss componentconsisting of two RI beams connected through a joint subject to solar heating. Axial and transverse motions as well as thermal response of the beams with thermoelastic damping are taking into account. The model results in a couple PDE-ODE system. Well-posedness and stability results are shown and analyzed.

1. Introduction

In recent years there has been renewed interest in Rigidizable-Inflatable (RI) space structures because of the efficiency they offer in packaging during boost-to-orbit. RI materials offer the possibility of deploying large space structures ([7]) and so are of interest in applications where large optical or RF apertures are needed. Several proposed space antenna systems will require ultra-light trusses to provide the "backbone" of the structure (see Figure 1(a)). It has been widely recognized that practical precision requirements can only be achieved through the development of new high-fidelity mathematical models and corresponding numerical tools.

In this paper we study the dynamics of a basic truss component consisting of two RI beams connected through a joint (see Figure 1(b)). One of the more important characteristics of such space systems is their response to changing thermal loads, as they move in and out of the Earth's shadow. In this paper we study the thermoelastic behavior of a two-beam truss element subject to solar heating. The beams are fabricated as thin-walled circular cylinders.

Key words and phrases. Truss structures, Euler-Bernoulli beams, thermoelastic system.
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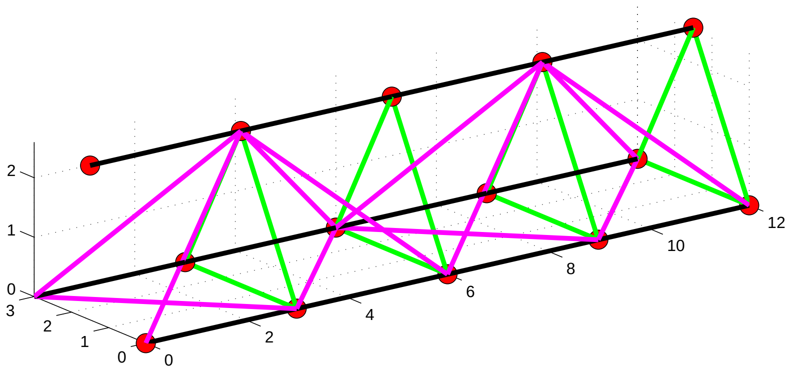
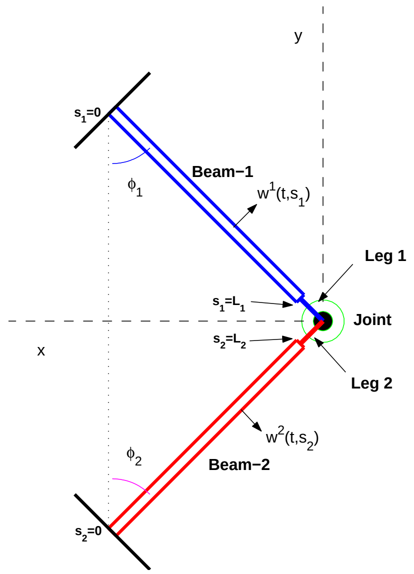


(a) Rigidizable-Inflatable truss structure



(b) Basic structure of the joint-legs-beams system

FIGURE 1.1. Truss (a) and basic structure of the joint-legs-beams system (b).

2. THERMOELASTIC MODEL

The equations of motion for the Joint-Leg-Beam system depicted in Figure 1(b) are the following (see [1] for details):

$$\rho_i A_i \frac{\partial^2 u^i(t,s_i)}{\partial t^2} = E_i A_i \frac{\partial^2 u^i(t,s_i)}{\partial s_i^2}, \qquad \rho_i A_i \frac{\partial^2 w^i(t,s_i)}{\partial t^2} = -E_i I_i \frac{\partial^4 w^i(t,s_i)}{\partial s_i^4}, \quad (2.1)$$

$$\mathbf{M}\frac{d^2}{dt}\left[\begin{array}{c} x(t) y(t) \theta_1(t) \theta_2(t) \end{array}\right]^T = \mathbf{C}\left[\begin{array}{c} M_1(t) N_1(t) M_2(t) N_2(t) F_1(t) F_2(t) \end{array}\right]^T \quad (2.2)$$

for time $t > 0$ and spatial variable $s_i \in [0, L_i]$, where M and C are 4×4 and 4×6 matrices give by

$$\mathbf{M} = \left[\begin{array}{cccc} m & 0 & -m_1 d_1 \cos\varphi_1 & m_2 d_2 \cos\varphi_2 \\ 0 & m & m_1 d_1 \sin\varphi_1 & m_2 d_2 \sin\varphi_2 \\ -m_1 d_1 \cos\varphi_1 & m_1 d_1 \sin\varphi_1 & I_{1\ell} + m_1 d_1^2 & 0 \\ m_2 d_2 \cos\varphi_2 & m_2 d_2 \sin\varphi_2 & 0 & I_{2\ell} + m_2 d_2^2 \end{array}\right], \quad (2.3)$$

$$\mathbf{C} = \left[\begin{array}{cccccc} 0 & -\cos\varphi_1 & 0 & \cos\varphi_2 & \sin\varphi_1 & \sin\varphi_2 \\ 0 & \sin\varphi_1 & 0 & \sin\varphi_2 & \cos\varphi_1 & -\cos\varphi_2 \\ 1 & \ell_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell_2 & 0 & 0 \end{array}\right], \quad (2.4)$$

and the other functions and parameters are as follows (here the supra or sub-index i, $i = 1, 2$ will always refer to beam or leg i): $u^i(t, s_i), w^i(t, s_i)$ longitudinal and transversal displacement of the beam; $x(t), y(t)$ horizontal and vertical displacement of the joint's tip; $\theta_i(t)$ rotation angle of the leg; $\rho_i, A_i, L_i, E_i, I_i$ mass density, cross section area, length, Young's modulus, moment of inertia of the beam; $m_i, d_i, \ell_i, I^i_\ell$ mass, center of mass, length, moment of inertia of the leg; m_p mass of the joint, $m = m_1 + m_2 + m_p$; φ_1 initial angle of leg 1 with positive y axis; φ_2 initial angle of leg 2 with negative y axis; $F_i(t)$ extensional force of beam at the end $s_i = L_i$; $N_i(t)$ shear force of beam at the end $s_i = L_i$; $M_i(t)$ bending moment of beam at the end $s_i = L_i$.

Each beam is clamped at the end $s_i = 0$. Thus the boundary conditions at $s_i = 0$ are

$$u^i(t, 0) = w^i(t, 0) = \frac{\partial w^i}{\partial s_i}(t, 0) = 0, \qquad i = 1, 2. \quad (2.5)$$

At the other end of each beam several obvious geometric compatibility conditions must be imposed. These conditions can be written in the form:

$$\left[\begin{array}{c} -\frac{\partial}{\partial s_1} w^1(t, L_1) \\ w^1(t, L_1) \\ -\frac{\partial}{\partial s_2} w^2(t, L_2) \\ w^2(t, L_2) \\ -u^1(t, L_1) \\ -u^2(t, L_2) \end{array}\right] = \left[\begin{array}{c} \theta_1(t) \\ -x(t)\cos\varphi_1 + y(t)\sin\varphi_1 + \ell_1\theta_1(t) \\ \theta_2(t) \\ x(t)\cos\varphi_2 + y(t)\sin\varphi_2 + \ell_2\theta_2(t) \\ x(t)\sin\varphi_1 + y(t)\cos\varphi_1 \\ x(t)\sin\varphi_2 - y(t)\cos\varphi_2 \end{array}\right] = \mathbf{C}^T \left[\begin{array}{c} x(t) \\ y(t) \\ \theta_1(t) \\ \theta_2(t) \end{array}\right]. \quad (2.6)$$

In [1], system (2.1)-(2.6) was re-cast as an abstract second-order ODE in an appropriate Hilbert space. Semigroup theory was then used to prove that the system is well-posed. Moreover, it was shown that if Kelvin-Voigt damping to both transverse and longitudinal motions is added, then the corresponding semigroup is analytic and exponentially stable. The spectrum of the infinitesimal generator of this semigroup was also characterized. The case of local damping was analyzed in [4] where it was shown that if only one of the beams is damped, then only polynomial stability is obtained even if additional rotational damping is assumed in the joint. Numerical approximations and several numerical results are shown in [2].

3. Thermal Dynamics

The external heat flux in the space normal to the beam's surface is given by (see [10])

$$S_i \doteq S_0 \cos\left(\xi_i - \frac{\partial w^i}{\partial s_i}\right), \tag{3.1}$$

where S_0 denotes the solar flux and ξ_i the angle of orientation of the solar vector with respect to the beam. In this equation we shall neglect the contribution of $\frac{\partial w^i}{\partial s_i}$ since we are assuming it is small. We denote by $T^i(t, s_i, \phi_i)$ the deviation of the temperature of the thin-walled circular beam i with respect to a reference temperature T_0^i at time t at the point on the beam corresponding to axial coordinate s_i and circumferential coordinate ϕ_i (here $\phi_i = 0$ corresponds to the top of the beam while $\phi_i = \pi$ corresponds to the bottom). Conservation of energy for a small segment of circular cylinder including longitudinal and circumferential conduction in the cylinder wall and radiation from the cylinder's surface yields the following equation for T^i:

$$\rho_i c_i \frac{\partial T^i}{\partial t} - \frac{k_c^i}{R_i^2}\frac{\partial^2 T^i}{\partial \phi_i^2} - k_a^i \frac{\partial^2 T^i}{\partial s_i^2} + \frac{\sigma \epsilon_i}{h_i}(T_0^i + T^i)^4 = \frac{\alpha_s^i}{h_i} S_i \cos(\phi_i)\,\delta(\phi_i) \tag{3.2}$$

where k_a^i and k_c^i are the axial and circumferential thermal conductivity coefficients, respectively, c_i is the specific heat, R_i the radius of the cylinder, h_i is the thickness of the wall, ϵ_i is the surface emissivity and α_s^i is the surface absorptivity, σ is the Stefan-Boltzmann constant, δ is a function defined on $[-\pi, \pi]$ by $\delta(\phi_i) = 1$ for $\phi_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\delta(\phi_i) = 0$ for $\phi_i \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$. The heat flux distribution on the RHS of equation (3.2) can be written as

$$S_i \cos(\phi_i)\,\delta(\phi_i) \;=\; S_i\left(\frac{1}{\pi} + g(\phi_i)\right) \;=\; \frac{S_i}{\pi} + S_i\, g(\phi_i) \tag{3.3}$$

where $g(\phi_i) \doteq \cos(\phi_i)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\phi_i) - \frac{1}{\pi}$ (here χ denotes the characteristic function). Clearly $g(\phi_i)$ is continuous and it has zero average in $[-\pi, \pi]$.

For each beam, the temperature distribution is separated into two parts, namely:

$$T^i(t, s_i, \phi_i) = T^i(t, s_i) + T^{m,i}(t, s_i)\, g(\phi_i), \tag{3.4}$$

where $T^i(t, s_i)$ is independent of ϕ_i and corresponds to the uniform part of the flux, $\frac{S_i}{\pi}$, in (3.3), and $T^{m,i}(t, s_i)\, g(\phi_i)$ amounts for the circumferential variation of the flux in (3.3). Note that for every $s_i \in [0, L_i]$ and $t \geq 0$ one has that $T^{m,i}(t, s_i) = T^i(t, s_i, 0) - T^i(t, s_i, \pi) \;=\; T^i(t, s_i, 0) - T^i(t, s_i, \phi)$ for any $\phi \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence, $T^{m,i}(t, s_i)$ can be thought of as the thermal gradient between the top and the bottom of the beam at the axial location s_i.

Also, we approximate the thermal radiation term $(T_0^i + T^i(t, s_i, \phi_i)\,)^4$ in (3.2) by linearizing $T(t, s_i, \phi_i)$ around $T(t, s_i, \phi_i) = T_s^i$ (where T_s^i, to be determined later, is the steady-state constant temperature increment produced on the undeformed beam i by the solar flux S_i), i.e., we approximate $(T_0^i + T^i(t, s_i, \phi_i)\,)^4$ by $(T_0^i + T_s^i)^4 +$

$4(T_0^i+T_s^i)^3\left(T^i(t,s_i)-T_s^i+T^{m,i}(t,s_i)g(\phi_i)\right)$. Hence equation (3.2) is replaced by

$$\begin{aligned}
&\rho_i c_i\frac{\partial T^i(t,s_i)}{\partial t}+\rho_i c_i\frac{\partial T^{m,i}(t,s_i)}{\partial t}\,g(\phi_i)-\frac{k_c^i}{R_i^2}T^{m,i}(t,s_i)\,g''(\phi_i)\\
&-k_a^i\frac{\partial^2 T^i(t,s_i)}{\partial s_i^2}-k_a^i\frac{\partial^2 T^{m,i}(t,s_i)}{\partial s_i^2}\,g(\phi_i)\\
&+\frac{\sigma\epsilon_i}{h_i}\left[(T_0^i+T_s^i)^4+4(T_0^i+T_s^i)^3\left(T^i(t,s_i)-T_s^i+T^{m,i}(t,s_i)g(\phi_i)\right)\right]\\
&=\frac{\alpha_s^i S_i}{h_i}\left[\frac{1}{\pi}+g(\phi_i)\right]. \qquad (3.5)
\end{aligned}$$

Since g has zero average, integration of equation (3.5) over the cylinder's cross sectional area yields

$$\begin{aligned}
&\rho_i c_i\frac{\partial T^i(t,s_i)}{\partial t}-k_a^i\frac{\partial^2 T^i(t,s_i)}{\partial s_i^2}+\frac{4\sigma\epsilon_i(T_0^i+T_s^i)^3}{h_i}[T^i(t,s_i)-T_s^i]\\
&=\left[\frac{\alpha_s^i S_i}{\pi h_i}-\frac{\sigma\epsilon_i(T_0^i+T_s^i)^4}{h_i}\right]\doteq f_i\,. \qquad (3.6)
\end{aligned}$$

Since $g'(\phi_i)$ is discontinuous at $\phi_i=\pm\frac{\pi}{2}$ the integration of $g''(\phi_i)$ above must be performed in the distributional sense. The value of T_s^i is now determined by setting the RHS, f_i, equals to zero. By doing so we obtain

$$T_s^i=\left(\frac{\alpha_s^i S_i}{\pi\sigma\epsilon_i}\right)^{\frac{1}{4}}-T_0^i \qquad (3.7)$$

Note that with this value of T_s^i corresponds to the steady-state $T^i(t,s_i)=T_s^i$ for the case of homogeneous Neumann boundary conditions and, since usually $T^{m,i}(t,s_i)$ is small compared to T_0^i, the linearization of the thermal radiation term performed above, is justified near the steady state solution.

Now multiplying (3.5) by $g(\phi_i)$ and integrating over the cylinder's cross sectional area, we obtain for $T^{m,i}$ the following equation:

$$\begin{aligned}
&\rho_i c_i\frac{\partial T^{m,i}(t,s_i)}{\partial t}-k_a^i\frac{\partial^2 T^{m,i}(t,s_i)}{\partial s_i^2}\\
&\quad+\left(\frac{k_c^i\pi^2}{R_i^2(\pi^2-4)}+\frac{4\sigma\epsilon_i(T_0^i+T_s^i)^3}{h_i}\right)T^{m,i}(t,s_i)=\frac{\alpha_s^i S_i}{h_i}. \qquad (3.8)
\end{aligned}$$

Thermally induced vibrations in the system is taken into account by considering Hooke's law for the stress-strain relation in the form $\epsilon_{11}^i=\frac{1}{E_i}\sigma_{11}^i+\alpha_i T^i$, where α_i is the thermal expansion coefficient, and T^i is, as before, the deviation from the reference temperature T_0^i. Note that at $T^i=0$ thermal strain vanishes, so that T_0^i is interpreted as the (uniform) temperature of beam i in the unstressed, rest-state. By the standard derivation of Euler-Bernoulli beam equation, we modify the Joint-Leg-Beam system (2.1) as follows:

$$\rho_i A_i\frac{\partial^2 u^i(t,s_i)}{\partial t^2} = E_i A_i\frac{\partial}{\partial s_i}\left(\frac{\partial u^i(t,s_i)}{\partial s_i}-\alpha_i T^i(t,s_i)\right), \qquad (3.9)$$

$$\rho_i A_i \frac{\partial^2 w^i(t,s_i)}{\partial t^2} = -E_i I_i \frac{\partial^2}{\partial s_i^2}\left(\frac{\partial^2 w^i(t,s_i)}{\partial s_i^2} + \frac{\alpha_i}{2R_i} T^{m,i}(t,s_i)\right) \tag{3.10}$$

The above beam equations are coupled to the heat equations modified from equations (3.6) and (3.8) and with T_s^i chosen as in equation (3.7) (so that $f_i = 0$ in (3.6)), that is:

$$\begin{aligned}\rho_i c_i \frac{\partial T^i(t,s_i)}{\partial t} &= k_a^i \frac{\partial^2 T^i(t,s_i)}{\partial s_i^2} \\ &\quad - \frac{4\sigma\epsilon_i (T_0^i + T_s^i)^3}{h_i}\left(T^i(t,s_i) - T_s^i\right) - \alpha_i E_i T_0^i \frac{\partial^2}{\partial s_i \partial t} u^i(t,s_i)\,,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\rho_i c_i \frac{\partial T^{m,i}(t,s_i)}{\partial t} &= k_a^i \frac{\partial^2 T^{m,i}(t,s_i)}{\partial s_i^2} - \left[\frac{k_c^i \pi^2}{R_i^2(\pi^2-4)} + \frac{4\sigma\epsilon_i (T_0^i + T_s^i)^3}{h_i}\right] T^{m,i}(t,s_i) \\ &+ \frac{\alpha_i E_i I_i T_0^i}{2R_i A_i} \frac{\partial^3}{\partial s_i^2 \partial t} w^i(t,s_i) + \frac{\alpha_s^i S_i}{h_i},\end{aligned} \tag{3.12}$$

We impose Robin type boundary conditions for the temperature at both ends of each beam, i.e.

$$\frac{\partial}{\partial s_i} T^i(t,L_i,\phi_i) = \lambda_R^i \left(T^* - T_0^i - T^i(t,L_i,\phi_i)\right), \quad \frac{\partial}{\partial s_i} T^i(t,0,\phi_i) = \lambda_L^i \left(T_0^i + T^i(t,0,\phi_i) - T^*\right),$$

$\forall t \geq 0$, $\phi_i \in [-\pi,\pi]$, $i = 1,2$, where T^* is the temperature of the surrounding medium and λ_L^i, λ_R^i, $i = 1,2$, are nonnegative constants. By writing $T^i(t,s_i,\phi_i)$ in terms of the decomposition given in (3.4) these equations take the form:

$$\begin{aligned}&\frac{\partial}{\partial s_i} T^i(t,L_i) + \frac{\partial}{\partial s_i} T^{m,i}(t,L_i) g(\phi_i) = \lambda_R^i \left(T^* - T_0^i - T^i(t,L_i) - T^{m,i}(t,L_i) g(\phi_i)\right),\\ &\frac{\partial}{\partial s_i} T^i(t,0) + \frac{\partial}{\partial s_i} T^{m,i}(t,0) g(\phi_i) = \lambda_L^i \left(T_0^i + T^i(t,0) + T^{m,i}(t,0) g(\phi_i) - T^*\right).\end{aligned}$$

Since these equations must hold for all $\phi_i \in [-\pi,\pi]$ it follows that

$$\frac{\partial}{\partial s_i} T^i(t,L_i) = \lambda_R^i \left(T^* - T_0^i - T^i(t,L_i)\right), \quad \frac{\partial}{\partial s_i} T^i(t,0) = \lambda_L^i \left(T_0^i + T^i(t,0) - T^*\right) \tag{3.13}$$

and

$$\frac{\partial}{\partial s_i} T^{m,i}(t,L_i) = -\lambda_R^i T^{m,i}(t,L_i), \quad \frac{\partial}{\partial s_i} T^{m,i}(t,0) = \lambda_L^i T^{m,i}(t,0), \tag{3.14}$$

for all $t \geq 0$, $i = 1,2$. So, in the same way that the dynamics for the temperature distribution (3.5) decouples into equations (3.11) and (3.12) for T^i and $T^{m,i}$, respectively, we observe that the boundary conditions also decouple. Note however in equation (3.13) that the boundary conditions for the axial component of the temperature, $T^i(t,s_i)$, are non-homogeneous. By defining $\tilde{T}^i(t,s_i) \doteq$

$T^i(t,s_i) - (T^* - T_0^i)$, equation (3.11) can be written in the form

$$\rho_i c_i \frac{\partial \tilde{T}^i(t,s_i)}{\partial t} = k_a^i \frac{\partial^2 \tilde{T}^i(t,s_i)}{\partial s_i^2} - \frac{4\sigma\epsilon_i (T_0^i + T_s^i)^3}{h_i}\left(\tilde{T}^i(t,s_i) + T^* - T_0^i - T_s^i\right) - \alpha_i E_i T_0^i \frac{\partial^2}{\partial s_i \partial t} u^i(t,s_i), \tag{3.15}$$

while the boundary conditions (3.13) now take the form

$$\frac{\partial}{\partial s_i}\tilde{T}^i(t,L_i) = -\lambda_R^i\, \tilde{T}^i(t,L_i), \qquad \frac{\partial}{\partial s_i}\tilde{T}^i(t,0) = \lambda_L^i\, \tilde{T}^i(t,0), \tag{3.16}$$

Observe now that these boundary conditions are exactly the same as those in (3.14) for the circumferential component of the temperature. Finally, note also that in equation (3.9), $T^i(t,s_i)$ can be replaced by $\tilde{T}^i(t,s_i)$ without any changes.

System (3.9)-(3.12) (or equivalently (3.9), (3.10), (3.12), (3.15)), together with the joint-leg dynamics described by equation (2.2) constitute the thermoelastic Joint-Leg-Beam equations with the external solar heat source. The extensional forces, shear forces and bending moments of the beams at $s_i = L_i$ are now given by:

$$F_i(t) = E_i A_i \left(\frac{\partial u^i}{\partial s_i}(t,s_i) - \alpha_i T^i(t,s_i) \right)\Bigg|_{s_i=L_i}, \tag{3.17}$$

$$N_i(t) = E_i I_i \frac{\partial}{\partial s_i} \left(\frac{\partial^2 w^i}{\partial s_i^2}(t,s_i) + \frac{\alpha_i}{2R_i} T^{m,i}(t,s_i) \right)\Bigg|_{s_i=L_i}, \tag{3.18}$$

$$M_i(t) = E_i I_i \left(\frac{\partial^2 w^i}{\partial s_i^2}(t,s_i) + \frac{\alpha_i}{2R_i} T^{m,i}(t,s_i) \right)\Bigg|_{s_i=L_i}. \tag{3.19}$$

4. Well-posedness

In this section, we consider the well-posedness of the Joint-Leg-Beam system with solar heat flux, i.e., equations (3.9), (3.10), (3.12), (3.15) subject to the geometric beam-leg interface compatibility conditions (2.6), the dynamic boundary conditions (3.17), (3.18), (3.19) and the boundary conditions (2.5), (3.14), (3.16). We first rewrite the system as a first order evolution equation in an appropriate Hilbert space. Well-posedness is then obtained by using semigroup theory. Since the corresponding system without thermal effects has been studied in [1], we will follow the notation used there as much as possible for consistency. Numerical results for that case are reported in [2].

First, we define the following Hilbert spaces with their corresponding inner products:

$$\begin{cases} \mathcal{H}_z = L^2(0,L_1) \times L^2(0,L_2) \times L^2(0,L_1) \times L^2(0,L_2), \\ \langle z_1, z_2 \rangle_{\mathcal{H}_z} \doteq \displaystyle\sum_{i=1}^{2} \rho_i A_i \left[\langle w_1^i, w_2^i \rangle + \langle u_1^i, u_2^i \rangle \right]; \end{cases}$$

$$\begin{cases} \mathcal{H}_b = [\ker(C)]^{\perp} = \mathrm{range}(C^T), \\ \langle b_1\,, b_2\rangle_{\mathcal{H}_b} = \langle b_1\,, (C^T M^{-1} C)^{\dagger} b_2\rangle_{\mathbb{R}^6}; \end{cases}$$

$$\begin{cases} \mathcal{H}_\zeta = L^2(0,L_1)\times L^2(0,L_2)\times L^2(0,L_1)\times L^2(0,L_2), \\ \langle \zeta_1\,,\zeta_2\rangle_{\mathcal{H}_\zeta} \doteq \displaystyle\sum_{i=1}^{2} \frac{\rho_i c_i A_i}{T_0^i}\left[\langle T_1^{m,i}, T_2^{m,i}\rangle + \langle \tilde{T}_1^i, \tilde{T}_2^i\rangle\right]; \end{cases}$$

where $z_j \doteq \left(w_j^1, w_j^2, u_j^1, u_j^2\right)^T$, $\zeta_j \doteq \left(T_j^{m,1}, T_j^{m,2}, \tilde{T}_j^1, \tilde{T}_j^2\right)^T$, and $(C^T M^{-1} C)^{\dagger}$ denotes the Moore-Penrose generalized inverse of $C^T M^{-1} C$. We also define the operators $\mathcal{A}_z : \mathcal{H}_z \to \mathcal{H}_z$ and $\mathcal{B}_z : \mathcal{H}_\zeta \to \mathcal{H}_z$ by

$$\mathrm{dom}(\mathcal{A}_z) \doteq H_\ell^2 \cap H^4(0,L_1)\times H_\ell^2\cap H^4(0,L_2)\times H_\ell^1\cap H^2(0,L_1)\times H_\ell^1\cap H^2(0,L_2),$$

$$\mathcal{A}_z \doteq \begin{pmatrix} \frac{E_1 I_1}{\rho_1 A_1} D^4 & 0 & 0 & 0 \\ 0 & \frac{E_2 I_2}{\rho_2 A_2} D^4 & 0 & 0 \\ 0 & 0 & -\frac{E_1}{\rho_1} D^2 & 0 \\ 0 & 0 & 0 & -\frac{E_2}{\rho_2} D^2 \end{pmatrix},$$

$$\mathrm{dom}(\mathcal{B}_z) \doteq H^2(0,L_1)\times H^2(0,L_2)\times H^1(0,L_1)\times H^1(0,L_2),$$

$$\mathcal{B}_z \doteq \begin{pmatrix} -\frac{\alpha_1 E_1 I_1}{2R_1\rho_1 A_1} D^2 & 0 & 0 & 0 \\ 0 & -\frac{\alpha_2 E_2 I_2}{2R_2\rho_2 A_2} D^2 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_1 E_1}{\rho_1} D & 0 \\ 0 & 0 & 0 & -\frac{\alpha_2 E_2}{\rho_2} D \end{pmatrix}.$$

where $D^n \doteq \frac{d^n}{ds_i^n}$ and for $n \in \mathbb{N}$, $H_\ell^n(0,L)$ denotes the space of functions in $H^n(0,L)$ that vanish, together with all derivatives up to the order $n-1$, at the left boundary. With this notation, equations (3.9)-(3.10) can now be written as the following abstract second order ODE in $\mathcal{H}_z$:

$$\ddot{z}(t) + \mathcal{A}_z z(t) - \mathcal{B}_z \zeta(t) = 0. \tag{4.1}$$

Next we define the operators $\mathcal{A}_\zeta : \mathcal{H}_\zeta \to \mathcal{H}_\zeta$ and $\mathcal{B}_\zeta : \mathcal{H}_z \to \mathcal{H}_\zeta$ by

$$\mathrm{dom}(\mathcal{A}_\zeta) \doteq H_{rb}^2(0,L_1)\times H_{rb}^2(0,L_2)\times H_{rb}^2(0,L_1)\times H_{rb}^2(0,L_2),$$

$$\mathcal{A}_\zeta\,\zeta = \mathcal{A}_\zeta \begin{pmatrix} T^{m,1} \\ T^{m,2} \\ \tilde{T}^1 \\ \tilde{T}^2 \end{pmatrix} \doteq \begin{pmatrix} -\frac{k_a^1}{\rho_1 c_1} D^2 T^{m,1} + \left[\frac{k_c^1 \pi^2}{\rho_1 c_1 R_1^2(\pi^2-4)} + \frac{4\sigma\epsilon_1 (T_0^1+T_s^1)^3}{\rho_1 c_1 h_1}\right] T^{m,1} \\ -\frac{k_a^2}{\rho_2 c_2} D^2 T^{m,2} + \left[\frac{k_c^2 \pi^2}{\rho_2 c_2 R_2^2(\pi^2-4)} + \frac{4\sigma\epsilon_2 (T_0^2+T_s^2)^3}{\rho_2 c_2 h_2}\right] T^{m,2} \\ -\frac{k_a^1}{\rho_1 c_1} D^2 \tilde{T}^1 + \frac{4\sigma\epsilon_1 (T_0^1+T_s^1)^3}{\rho_1 c_1 h_1} \tilde{T}^1 \\ -\frac{k_a^2}{\rho_2 c_2} D^2 \tilde{T}^2 + \frac{4\sigma\epsilon_2 (T_0^2+T_s^2)^3}{\rho_2 c_2 h_2} \tilde{T}^2 \end{pmatrix},$$

$$\mathrm{dom}(\mathcal{B}_\zeta) \doteq H^2(0,L_1)\times H^2(0,L_2)\times H^1(0,L_1)\times H^1(0,L_2),$$

$$\mathcal{B}_\zeta z \doteq \begin{pmatrix} \frac{\alpha_1 E_1 I_1 T_0^1}{2R_1\rho_1 c_1 A_1} D^2 & 0 & 0 & 0 \\ 0 & \frac{\alpha_2 E_2 I_2 T_0^2}{2R_2\rho_2 c_2 A_2} D^2 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_1 E_1 T_0^1}{\rho_1 c_1} D & 0 \\ 0 & 0 & 0 & -\frac{\alpha_2 E_2 T_0^2}{\rho_2 c_2} D \end{pmatrix}.$$

where $H^2_{rb}(0,L)$ denotes the space of functions in $H^2(0,L)$ satisfying the Robin boundary conditions (3.14) or equivalently (3.16). With this notation, equations (3.12), (3.15), can now be written as the following abstract first order ODE in $\mathcal{H}_\zeta$:

$$\dot{\zeta}(t) - \mathcal{B}_\zeta \dot{z}(t) + \mathcal{A}_\zeta \zeta(t) \;=\; S \tag{4.2}$$

where

$$S \doteq \left(\tfrac{\alpha_s^1}{\rho_1 c_1 h_1} S_1,\ \tfrac{\alpha_s^2}{\rho_2 c_2 h_2} S_2,\ \tfrac{4\sigma\epsilon_1 (T_0^1+T_s^1)^3}{\rho_1 c_1 h_1}(T_s^1+T_0^1-T^*),\ \tfrac{4\sigma\epsilon_2 (T_0^2+T_s^2)^3}{\rho_2 c_2 h_2}(T_s^2+T_0^2-T^*) \right)^T.$$

We also define three boundary projection operators P_1^B, P_2^B from $\mathcal{H}_z$ into $\mathbb{R}^6$ and P_3^B from $\mathcal{H}_\zeta$ into $\mathbb{R}^6$ by

$$\begin{aligned} \mathrm{dom}(P_1^B) &\doteq H^2(0,L_1)\times H^2(0,L_2)\times H^1(0,L_1)\times H^1(0,L_2),\\ \mathrm{dom}(P_2^B) &\doteq H^4(0,L_1)\times H^4(0,L_2)\times H^2(0,L_1)\times H^2(0,L_2),\\ \mathrm{dom}(P_3^B) &\doteq H^2(0,L_1)\times H^2(0,L_2)\times H^1(0,L_1)\times H^1(0,L_2), \end{aligned}$$

$$P_1^B\begin{pmatrix} w^1\\ w^2\\ u^1\\ u^2\end{pmatrix} \doteq \begin{pmatrix} -\frac{\partial}{\partial s_1} w^1(L_1)\\ w^1(L_1)\\ -\frac{\partial}{\partial s_2} w^2(L_2)\\ w_2(L_2)\\ -u^1(L_1)\\ -u^2(L_2)\end{pmatrix}, \quad P_2^B\begin{pmatrix} w^1\\ w^2\\ u^1\\ u^2\end{pmatrix} \doteq \begin{pmatrix} \frac{\partial^2}{\partial s_1^2} w^1(L_1)\\ \frac{\partial^3}{\partial s_1^3} w^1(L_1)\\ \frac{\partial^2}{\partial s_2^2} w^2(L_2)\\ \frac{\partial^3}{\partial s_2^3} w^2(L_2)\\ \frac{\partial}{\partial s_1} u^1(L_1)\\ \frac{\partial}{\partial s_2} u_2(L_2)\end{pmatrix}, \quad P_3^B\begin{pmatrix} T^{m,1}\\ T^{m,2}\\ \tilde{T}^1\\ \tilde{T}^2\end{pmatrix} \doteq \begin{pmatrix} T^{m,1}(L_1)\\ \frac{\partial}{\partial s_1} T^{m,1}(L_1)\\ T^{m,2}(L_2)\\ \frac{\partial}{\partial s_2} T^{m,2}(L_2)\\ \tilde{T}^1(L_1)\\ \tilde{T}^2(L_2)\end{pmatrix}.$$

Now, by using the geometric compatibility conditions (2.6) and the dynamic boundary conditions (3.17)-(3.19), the equation for the leg-joint dynamics (2.2) can be written as the following abstract second order ODE in $\mathcal{H}_b$:

$$\frac{d^2}{dt^2}\left(P_1^B z(t)\right) - C^T M^{-1} C E \left(P_2^B z(t) \,+\, \Lambda P_3^B \zeta(t)\right) = \tilde{R} \tag{4.3}$$

where $E \doteq \mathrm{diag}(E_1 I_1, E_1 I_1, E_2 I_2, E_2 I_2, E_1 A_1, E_2 A_2)$, $\Lambda \doteq \mathrm{diag}(\frac{\alpha_1}{2R_1}, \frac{\alpha_1}{2R_1}, \frac{\alpha_2}{2R_2}, \frac{\alpha_2}{2R_2}, -\alpha_1, -\alpha_2)$ and $\tilde{R} \doteq C^T M^{-1} C \left(0,\ 0,\ 0,\ 0,\ E_1 A_1 \alpha_1 (T^* - T_0^1),\ E_2 A_2 \alpha_2 (T^* - T_0^2)\ \right)^T$. Next we define the Hilbert space $\mathcal{H}_{zb} \doteq \mathcal{H}_z \times \mathcal{H}_b$ with the usual inner product inherited from those in $\mathcal{H}_z$ and $\mathcal{H}_b$. In this Hilbert space we define the elastic operator $\mathcal{A}_{zb}$ by

$$\mathrm{dom}(\mathcal{A}_{zb}) \doteq \left\{ \begin{pmatrix} z\\ b\end{pmatrix} \in \mathrm{dom}(\mathcal{A}_z)\times\mathcal{H}_b \;:\; P_1^B\, z = b \right\}$$

$$\text{and } \mathcal{A}_{zb}\begin{pmatrix} z\\ b\end{pmatrix} \doteq \begin{pmatrix} \mathcal{A}_z z\\ -\,C^T M^{-1} C E\, P_2^B z\end{pmatrix}.$$

Furthermore, we define $B_{zb} : \mathcal{H}_\zeta \to \mathcal{H}_{zb}$ by $\mathrm{dom}(\mathcal{B}_{zb}) \doteq H^2(0,L_1) \times H^2(0,L_2) \times H^1(0,L_1) \times H^1(0,L_2)$ and $\mathcal{B}_{zb}\zeta \doteq \begin{pmatrix} \mathcal{B}_z\zeta \\ C^T M^{-1} C E \Lambda P_3^B \zeta \end{pmatrix}$. Thus, equations (4.1) and (4.3) can be combined as

$$\frac{d^2}{dt^2}\begin{pmatrix} z(t) \\ b(t) \end{pmatrix} + \mathcal{A}_{zb}\begin{pmatrix} z(t) \\ b(t) \end{pmatrix} - \mathcal{B}_{zb}\zeta(t) = R \qquad \text{on } \mathcal{H}_{zb}, \tag{4.4}$$

where $R \doteq (0, \tilde{R})^T$. It has been proved in [1] that the operator $\mathcal{A}_{zb}$ is self-adjoint and strictly positive. Thus, we can define the state space $\mathcal{H} \doteq \mathrm{dom}(\mathcal{A}_{zb}^{1/2}) \times \mathcal{H}_{zb} \times \mathcal{H}_\zeta$ with the inner product $\left\langle \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \right\rangle_{\mathcal{H}} \doteq \langle \mathcal{A}_{zb}^{1/2} X_1, \mathcal{A}_{zb}^{1/2} Y_1 \rangle_{\mathcal{H}_{zb}} + \langle X_2, Y_2 \rangle_{\mathcal{H}_{zb}} + \langle X_3, Y_3 \rangle_{\mathcal{H}_\zeta}$. Finally, we define operator $\mathcal{A}$ on $\mathcal{H}$ by $\mathrm{dom}(\mathcal{A}) \doteq \left\{ \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \mathcal{H} \,\middle|\, X_1 \in \mathrm{dom}(\mathcal{A}_{zb}), X_2 \in \mathrm{dom}(\mathcal{A}_{zb}^{1/2}),\ X_3 \in \mathrm{dom}(\mathcal{A}_\zeta) \right\}$, $\mathcal{A}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \doteq \begin{pmatrix} 0 & I & 0 \\ -\mathcal{A}_{zb} & 0 & \mathcal{B}_{zb} \\ 0 & (\mathcal{B}_\zeta, 0) & -\mathcal{A}_\zeta \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$. Then, equations (4.2) and (4.4) can be rewritten as a first order nonhomogeneous evolution equation

$$\dot{X}(t) = \mathcal{A}X(t) + G \qquad \text{on } \mathcal{H} \tag{4.5}$$

where $X \doteq \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \quad X_1 \doteq \begin{pmatrix} z \\ b \end{pmatrix}, \quad X_2 \doteq \dot{X}_1, \quad X_3 \doteq \zeta \quad \text{and} \quad G \doteq \begin{pmatrix} 0 \\ R \\ S \end{pmatrix}.$

Theorem 4.1. (Well-posedness): *Let $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ be as defined above. Then $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $\mathcal{S}(t)$ on $\mathcal{H}$ and hence, for any initial condition $X_0 = X(0) \in$ dom$(\mathcal{A})$, system* (4.5) *has a unique global solution $X(t)$ given by*

$$X(t) = \mathcal{S}(t)\, X_0 + \int_0^t S(t-s) G \, ds.$$

Proof: It can be shown that $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$). Since $\mathrm{dom}(\mathcal{A})$ is dense in $\mathcal{H}$, it then follows from Theorem 1.2.4 in [8] that $\mathcal{A}$ generates a strongly continuous semigroup of contractions $S(t)$ on $\mathcal{H}$. The existence and uniqueness of solutions for system (4.5) for any initial condition $X_0 = X(0) \in \mathrm{dom}(\mathcal{A})$ finally follows from Corollary 2.10 in [9]. For more details see [3]. ■

5. Exponential Stability

We now turn our attention to the stability of system (4.5). It is well known that the semigroup associated with longitudinal and transversal motion of a thermoelastic Euler beam is exponentially stable ([5], [8]). System (4.5) consists of two thermoelastic beam equations plus the equations for the joint-leg dynamics. This type of system is often referred to as "*hybrid system*". It is certainly an interesting problem to determine whether the thermal damping is strong enough by itself to induce exponential stability of this kind of system. We shall prove this in the affirmative.

The following result by Huang [6] will be used:

Theorem 5.1. *Let H be a Hilbert space, $A : H \to H$ a closed, densely defined linear operator. Assume that A generates a C_0-semigroup of contractions $T(t)$ on H. Then $T(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \cap \sigma(A) = \emptyset, \tag{5.1}$$

$$\lim_{\beta\to\infty} \|(\boldsymbol{i}\beta - A)^{-1}\| < \infty. \tag{5.2}$$

Theorem 5.2. *The C_0-semigroup of contractions $S(t)$ generated by $\mathcal{A}$ (see Theorem 4.1) is exponentially stable.*

Proof: If (5.2) is false then there exists a sequence $\{\beta_n\} \subset \mathbb{R}$ with $\beta_n \to \infty$ and a sequence $\{X_n\} \subset D(\mathcal{A})$ with $\|X_n\|_{\mathcal{H}} = 1\ \forall\, n$ such that

$$\lim_{n\to\infty} \|(\boldsymbol{i}\beta_n - \mathcal{A})X_n\|_{\mathcal{H}} = 0. \tag{5.3}$$

Using the components related to the thermoelastic beam equations it can be show that (5.3) yields the contradiction $\|X_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Similarly, if the condition (5.1) is false, then there exist $\beta \in \mathbb{R}$ and a sequence $\{X_n\} \subset D(\mathcal{A})$ with $\|X_n\|_{\mathcal{H}} = 1\ \ \forall n$, such that

$$\lim_{n\to\infty} \|(\boldsymbol{i}\beta - \mathcal{A})X_n\|_{\mathcal{H}} = 0. \tag{5.4}$$

By repeating the same arguments we get the contradiction $\|X_n\|_{\mathcal{H}} \to 0$. For complete details on these proofs, we refer the reader to [3]. Hence $\mathcal{A}$ satisfies conditions (5.1) and (5.2) and therefore, the C_0-semigroup of contractions $S(t)$ generated by $\mathcal{A}$ is exponentially stable. ∎

6. Conclusions

In this article we considered a system of two thermoelastic Euler-Bernoulli beams coupled to a joint through two legs. By means of semigroup theory the well posedness of the system was proved and its exponential stability was derived. It is certainly of much interest to develop numerical approximations for our state-space model (4.5). Such numerical schemes will be useful in simulation and identification studies to predict and better understand the structural and thermal responses of space-borne observation systems. Efforts in this direction are already under way.

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