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ADMISSIBLE RESTRICTION OF HOLOMORPHIC DISCRETE SERIES FOR EXCEPTIONAL GROUPS

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To Mischa Cotlar with respect

ABSTRACT. In this note, we give results about the restriction of a holomorphic discrete series of an exceptional simple Lie real group to a subgroup.

1. Introduction

A basic problem in representation theory of Lie groups is to derive "branching laws". By this we mean, for a given unitary irreducible representation of an ambient group G, consider its restriction to a fixed subgroup H and find the decomposition as a direct integral, and in particular compute the multiplicity of each irreducible factor of the restriction. There is a vast literature on this subject, and here we just direct the reader's attention to the extensive reviews of [13], [14] and references therein. In this note, we consider a *holomorphic discrete series* of a connected simple exceptional Lie group, and determine whether or not it has an admissible restriction to a given closed connected reductive subgroup $H \subset G$. Let us recall that a unitary representation of a topological group H is *admissible* if it is a discrete Hilbert sum of irreducible unitary sub-representations and each irreducible summand occurs with finite multiplicity.

Holomorphic discrete series are associated to Hermitian symmetric spaces. We consider a Hermitian symmetric space G/K, where G is a simple connected real Lie group G (which we shall assume for convenience, to minimize notations, with finite center), and K a maximal compact subgroup. For a Lie group we denote its Lie algebra by the corresponding German lower case letter. We write the Cartan decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Thus $\mathfrak{s}$, the tangent space of G/K at the origin, is provided with a complex structure $J \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s}, \mathfrak{s})$ corresponding to a choice of square root $\mathbf{i} \in \mathbb{C}$. To denote the complexification of a vector space, we add the subscript $\mathbb{C}$. We denote by $\mathfrak{s}^+$ and $\mathfrak{s}^-$ the eigenspaces of J in $\mathfrak{s}_{\mathbb{C}}$ with respective eigenvalues $\{+\mathbf{i}, -\mathbf{i}\}$: a linear form $f \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s}, \mathbb{C})$ is $\mathbb{C}$-linear if and only if its linear extension to $\mathfrak{s}_{\mathbb{C}}$ is zero on the subspace $\mathfrak{s}^-$. Moreover, $\mathfrak{s}_{\mathbb{C}} = \mathfrak{s}^+ \oplus \mathfrak{s}^-$, is the

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decomposition of $\mathfrak{s}_{\mathbb{C}}$ as a direct sum of two irreducible K-modules, dual to each other.

Recall (see [7]) that the center $\mathfrak{z}$ of $\mathfrak{k}$ is one dimensional, and that we can choose uniquely a basis (denoted by the same letter J) of $\mathfrak{z}$ whose adjoint action in $\mathfrak{s}$ is the complex structure J of the tangent space at the origin of G/K. We write $\mathfrak{k}_{ss} = [\mathfrak{k}, \mathfrak{k}]$. We have $\mathfrak{g} = \mathfrak{k}_{ss} \oplus \mathbb{R}J \oplus \mathfrak{s}$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{ss\mathbb{C}} \oplus \mathbb{C}J \oplus \mathfrak{s}^+ \oplus \mathfrak{s}^-$. Correspondingly, we have $K = K_{ss}Z$, where Z is isomorphic to $\mathrm{SO}(2, \mathbb{R})$, and $K_{ss} \cap Z$ is finite.

An irreducible unitary representation of G is called *holomorphic* if its underlying Harish-Chandra module has a non zero vector v which is annihilated by $\mathfrak{s}^-$. An irreducible irreducible unitary representation of G is called *a discrete series representation* if its coefficients are square integrable on G with respect to a given Haar measure.

The exceptional connected simple Lie groups whose quotient by a maximal compact subgroup carries an invariant complex structure has been classified by E. Cartan. They are the connected groups with Lie algebras $\mathfrak{e}_{6(-14)}$ and $\mathfrak{e}_{7(-25)}$. The respective complexified Cartan decompositions are :

$$\mathfrak{e}_6 = \mathfrak{e}_{6(-14)_{\mathbb{C}}} = \mathfrak{so}(10, \mathbb{C}) + \mathbb{C}J + (\mathfrak{s}^+ \oplus \mathfrak{s}^-).$$

Here, $\mathfrak{s}^{\pm}$ are the half spin 16-dimensional representations.

$$\mathfrak{e}_7 = \mathfrak{e}_{7(-25)_{\mathbb{C}}} = \mathfrak{e}_6 + \mathbb{C}J + (\varpi_1 \oplus \varpi_6).$$

Here, $\varpi_\star$ are the two fundamental representations of dimension twenty seven of the complex simple algebra $\mathfrak{e}_6$.

In this paper, for $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ and $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ we give list closed connected reductive subgroups H of G such that an holomorphic discrete series of G has an admissible restriction to H. In [6], we gave several results concerning restrictions of more general discrete series for more general reductive groups, in particular, we introduced a sufficient condition —we call it condition (C)— which implies admissibility of restriction, and allows to compute multiplicities of restrictions by mean of a Blattner-Kostant type formula involving a partition function. However, there exist many cases of admissibility where condition (C) is not satisfied —many examples are given in [6], all of them for compact groups H. One of our interests in studying precisely what happens for holomorphic discrete series of exceptional groups, besides our wish to understand the full picture, is to find other interesting examples. In particular, we give several non compact examples.

We would like to point out that in his Ph.D. thesis [21], S. Simondi has obtained the results on admissibility when rank of L is equal to rank of K, they follow from Theorem 1. His technique is different from the one is used in this note.

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2. Some general results

2.1. **A criterium for admissibility of restriction.** We recall some results which we will use in our proofs. Let G be a connected simple Lie group with finite center, choose a maximal compact group K, and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. We denote by $K_{\mathbb{C}}$ the corresponding complex group. Let H be a closed connected reductive subgroup. We assume that $L := H \cap K$ is a maximal compact subgroup of H.

In [6], we prove a result which reduces the problem of admissibility of restriction of discrete series to the case of compact subgroups :

Proposition 1. *Let (π, V) be a discrete series for G. Then its restriction to H is admissible if and only if its restriction to L is admissible.*

There are many criteria for admissibility of the restriction to a subgroup of an irreducible unitary representation (π, V) of G (see e. g. [13], [14]). When the subgroup is compact, we will use a criterium in term of the *associated variety* which we explain. We denote by (π, V_f) the representation of $\mathfrak{g}_C$ in the space of K-finite vectors of V. Vogan [26] defined the *associated variety* $\mathcal{V}(V_f)$, which is a Zariski-closed $K_{\mathbb{C}}$-invariant cone of dual $\mathfrak{s}_{\mathbb{C}}^*$ of $\mathfrak{s}_{\mathbb{C}}$. Let us denote by $\mathbb{C}[\mathcal{V}(V_f)]$ the ring of regular functions on $\mathcal{V}(V_f)$. The following criterium is known (see in particular Huang and Vogan [9], Kobayashi [12], Vergne [25]).

Proposition 2. *Let (π, V) be an irreducible unitary representation of G. Then its restriction to L is admissible if and only if $\mathbb{C}[\mathcal{V}(V_f)]^{L_{\mathbb{C}}} = \mathbb{C}$, that is the only $L_{\mathbb{C}}$ invariant regular functions on $\mathcal{V}(V_f)$ are the constant.*

Assume now that G/K is hermitian symmetric. The criterium is particularly pleasant for holomorphic discrete series (see [12], [6], [25]) :

Proposition 3. *Let (π, V) be a holomorphic discrete series of G. Then $\mathcal{V}(V_f)$ is the orthogonal of $\mathfrak{s}^-$ in $\mathfrak{s}_{\mathbb{C}}^*$. Thus its restriction to L is admissible if and only if $S[\mathfrak{s}^+]^{L_{\mathbb{C}}} = \mathbb{C}$.*

The most obvious example of proposition 3 is the group K. The restriction to K of an holomorphic discrete series is admissible (in fact it is true for any unitary irreducible representation of G), and we have also $S[\mathfrak{s}^+]^{K_{\mathbb{C}}} = \mathbb{C}$. Thus our problem of restriction is a particular case of a well known problem in invariant theory (see [22],[23]): Find pairs of connected reductive groups complex linear groups $A \subset B \subset \mathrm{GL}(\mathfrak{s}^+)$ such that $S[\mathfrak{s}^+]^A = S[\mathfrak{s}^+]^B$.

Remark 1. *If L is semi-simple, the condition $S[\mathfrak{s}^+]^{L_{\mathbb{C}}} = \mathbb{C}$ holds if and only if $L_{\mathbb{C}}$ has an open orbit in $\mathfrak{s}^+$.*

The subgroups Z and K_{ss} of K deserve a special attention. For completeness, we recall the following well known result (which can serve as an illustration of proposition 3)

Proposition 4. *Let (π, V) be a holomorphic discrete series of G. Its restriction to Z (and also to any closed subgroup $H \subset G$ which contains Z) is admissible.*

To study the restriction to K_{ss}, recall that Hermitian symmetric spaces G/K are divided in two categories: the *tube type*, and the *non tube type*. One of the many equivalent definitions of tube type is (see [7]):

The Hermitian symmetric spaces G/K *is of non tube type if and only if* $S[\mathfrak{s}^+]^{K_{ss\mathbb{C}}} = \mathbb{C}$.

We will also say that $\mathfrak{g}$ is of tube type. Hermitian symmetric spaces of tube type are related to simple Jordan algebras [7]; They are interesting because they have associated Zeta functions. However, from our point of view, non tube type is more interesting:

Proposition 5. *Let* (π, V) *be a holomorphic discrete series of* G*. Its restriction to* K_{ss} *is admissible if and only if* G/K *is not of tube type.*

The list of Hermitian symmetric spaces G/K of tube type is well known (see [7]). Among the two exceptional ones, $\mathfrak{e}_{6(-14)}$ is not of tube type, and $\mathfrak{e}_{7(-25)}$ is of tube type. Thus we have the following preliminary results, which explains why the case $\mathfrak{e}_{6(-14)}$ is richer.

Theorem 1. *Let* (π, V) *be a holomorphic discrete series of* G *with Lie algebra* $\mathfrak{e}_{6(-14)}$*. Its restriction to* K_{ss} *is admissible.*

Theorem 2. *Let* (π, V) *be a holomorphic discrete series of* G *with Lie algebra* $\mathfrak{e}_{7(-25)}$*. Its restriction to* K_{ss} *(and to any of its closed subgroups* L*) is not admissible.*

2.2. **Condition (C).** We recall what is condition (C) of [6] in the particular case of a holomorphic discrete series. We choose a Cartan subgroup T of K, and denote by $\Phi \subset \mathbf{i}\mathfrak{t}^*$ be set the roots of T in $\mathfrak{g}_{\mathbb{C}}$. We choose a positive system $\Psi \subset \Phi$ such that the set of non compact roots Ψ_n is exactly the set of roots of T in $\mathfrak{s}^+$. We denote by $\mathcal{C} \subset \mathbf{i}\mathfrak{t}^*$ the closed convex pointed cone generated by Ψ_n.

We assume that $U := L \cap T$ is a Cartan subgroup of L. Let $\mathfrak{u}^\perp \subset \mathfrak{t}^*_{\mathbb{C}}$ be the orthogonal of $\mathfrak{u}$. Here is condition (C):

$$(C): \quad \mathcal{C} \cap \mathfrak{u}^\perp = \{0\}.$$

We rephrase condition (C). Let $\mathcal{C}^\top \subset \mathfrak{t}$ the cone dual to $\mathcal{C}/\mathbf{i} \subset \mathfrak{t}^*$; It is a closed convex cone whose interior $\mathcal{C}^\top_{int}$ contains J. Then condition (C) is equivalent to condition (C'):

$$(C'): \quad \mathcal{C}^\top_{int} \cap \mathfrak{u} \neq \{0\}.$$

Condition (C) depends only on the maximal torus* U of L. We have:

Theorem 3. *Let* (π, V) *be a holomorphic discrete series of* G*. Let* $U \subset T$ *be a compact connected torus. Then the restriction of* (π, V) *to* U *is admissible if and only if condition (C) holds.*

*For non holomorphic discrete series, it is usually not true.

Proof. As a $K_{\mathbb{C}}$-module, V is isomorphic to $F \otimes S[\mathfrak{s}^+]$, where F is an irreducible representation of $K_{\mathbb{C}}$. Thus, as a T module, it is a finite direct sum of $\mathbb{C}_\mu \otimes S[\mathfrak{s}^+]$, where $\mathbb{C}_\mu$ is a one dimensional representation of T with weight $\mu \in \mathbf{i}\mathfrak{t}^*$. The weights of T in $S[\mathfrak{s}^+]$ are exactly the weights of T contained in $\mathcal{C}$, occurring with finite multiplicity. The theorem follows. □

If condition (C) is satisfied for a torus $U \subset T$, it is also satisfied for some one dimensional torus $U_1 \subset U$. Then U_1 satisfies condition (C) if and only if $\mathfrak{u}_1$ has a basis B which belongs to $\mathcal{C}_{int}^{\top}$. In particular, Z satisfies condition (C) (which is a way of proving proposition 4), and also all one dimensional torus U_1 not to far away from Z.

On the other hand, it is easy to see that condition (C) is never satisfied for $\mathfrak{l} \subset \mathfrak{k}_{ss}$. Thus, for $\mathfrak{g} = \mathfrak{e}_{6(-14)}$, the group K_{ss} is an easy example where there is admissibility and condition (C) does not hold.

2.3. **Formulation of the problem.** Let us explain more precisely what has to be done in general. We fix a compact connected semisimple group $D \subset K_{ss}$ with a Cartan subgroup $A \subset T$. Let B be the connected component group of the centralizer of D in T. Then AB is a Cartan subgroup of DB.

Consider a connected closed groups $L \subset K$ such that $L_{ss} = D$. Up to conjugation, it will be of the form $L = B_L D$, where B_L, the connected center of L, is a closed connected subgroup of B. Note that B contains the center Z of K. For clarity, we distinguish two cases.

First, assume there is admissible restriction of holomorphic discrete series of G to D — or equivalently, that $\mathfrak{s}^+$ is a D-prehomogeneous space. Then there will be admissible restriction to any subgroup H containing D.

We assume now that the restriction of holomorphic discrete series of G to D is not admissible. Since $\mathfrak{b}$ contains J, the restriction of a holomorphic discrete series (π, V) of G to the group BD is admissible. Let us choose a positive Weyl chamber $\Gamma \subset \mathbf{i}(\mathfrak{a}+\mathfrak{b})^*$ for the group BD. Let $\mathcal{C}_{D,V} \subset \Gamma$ the set of highest weights of the irreducible representations of BD which occur in V, and $\mathcal{C}_D \subset \Gamma$ the asymptotic cone of $\mathcal{C}_{D,V}$. It is known that $\mathcal{C}_D$ is a closed convex polyhedral cone, independent of V, contained in the projection on $\mathbf{i}(\mathfrak{a}+\mathfrak{b})^*$ of the cone $\mathcal{C} \subset \mathbf{i}\mathfrak{t}^*$. We identify the orthogonal $\mathfrak{a}^\perp$ of $\mathfrak{b}$ in $\mathfrak{a}+\mathfrak{b}$ to $\mathfrak{b}^*$. We consider the cone $\mathcal{C}_D^{\mathfrak{b}} = \mathcal{C}_D \cap \mathfrak{a}^\perp \subset \mathbf{i}\mathfrak{b}^*$. The fact that the restriction of holomorphic discrete series of G to D is not admissible is equivalent to the fact that *the cone* $\mathcal{C}_D^{\mathfrak{b}}$ *is not reduced to* $\{0\}$. We consider its dual cone $\mathcal{C}_D^{\mathfrak{b}\perp} \subset \mathfrak{b}$, and its interior $\mathcal{C}_{D,int}^{\mathfrak{b}\perp}$. Note that J belongs to $\mathcal{C}_{D,int}^{\mathfrak{b}\perp}$.

Theorem 4. *Let (π, V) be a holomorphic discrete series of G. Suppose that its restriction to D is not admissible. Then the restriction of (π, V) to L is admissible if and only if one of the two following equivalent conditions hold:*

$$(C_L): \quad \mathcal{C}_D^{\mathfrak{b}} \cap \mathfrak{b}_L^\perp = \{0\}.$$

or

$$(C'_L): \quad \mathcal{C}_{D,int}^{\mathfrak{b}\top} \cap \mathfrak{b}_L \neq \{0\}.$$

Thus, discrete series of G have admissible restriction to L if and only if L contains a closed subgroup $L_1 = B_1 D$, where $\mathfrak{b}_1 \subset \mathfrak{b}$ is a one-dimensional subspace which intersects $\mathcal{C}_{D,int}^{\mathfrak{b}\top}$.

Theorem 4 suggests a method to find all closed connected reductive groups $H \subset G$ for which there is admissible restriction of holomorphic discrete series.

1. For each closed connected semisimple subgroup $D \subset K_{ss}$, determine whether there is admissibility of restriction of holomorphic discrete series. This step is not too difficult, for instance this is never the case when $\mathfrak{g}$ is of tube type, and we will give below the complete answer for $\mathfrak{g} = \mathfrak{e}_{6(-14)}$.

2. When it is not the case, compute (with the notation as above) the algebra $\mathfrak{b}$ and the cone $\mathcal{C}_D^{\mathfrak{b}\perp} \subset \mathfrak{b}$. This will give the list of closed connected subgroup $L \subset K$ such that $L_{ss} = D$ for which there is admissibility of restriction of holomorphic discrete series.

For each particular D, this is probably a feasible task, and we give several examples. However, we do not know an useful statement for all D.

3. Given L as in 2, list the closed connected reductive subgroups $H \subset G$ such that $H \cap K = L$.

2.4. **Some cones.** We use the notations of the previous subsection. We assume moreover that T normalizes D, or, equivalently, that $T = AB$. This means that $\mathfrak{d}_{\mathbb{C}}$ is the sum of the root spaces for a certain subset Φ_D of roots, and of the space $\mathfrak{a}_{\mathbb{C}}$ generated by the corresponding coroots. We give some bounds on the cone $\mathcal{C}_D$.

For this we need to recall some important facts proven in [20]. Let r be the real rank of $\mathfrak{g}$. There exists a set $\{\gamma_1, \ldots, \gamma_r\} \subset \Psi_n$ of pairwise strongly orthogonal roots such that the highest weights of the representations of $K_{\mathbb{C}}$ occurring in $S[\mathfrak{s}^+]$ are exactly those which belong to the cone generated by $\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\gamma_2+\cdots+\gamma_r$. We recall that γ_1 is the highest weight of the $K_{\mathbb{C}}$-module $\mathfrak{s}^+$, that γ_2 is the maximal element (for a suitable order) among the roots orthogonal to γ_1, etc...

This means that $\mathcal{C}_{K_{ss}}$ is the cone generated by $\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\gamma_2+\cdots+\gamma_r$. Moreover, $\mathcal{C}_D$ is a polyhedral cone such that $\mathcal{C}_{K_{ss}} \subset \mathcal{C}_D \subset \mathcal{C}$.

For later use, we introduce some related notations. We will label the simple compact roots as $\alpha_1, \ldots, \alpha_d$, and the unique simple non compact root will be denoted by β. Note that γ_1 is the corresponding fundamental weight, and that $\gamma_1 = w_o\beta$, where w_o is the longest element of the Weyl group of $K_{\mathbb{C}}$.

2.5. **Non compact** H**.** For this subsection G denotes one of the groups $E_{6(-14)}$, $E_{7(-25)}$. We fix a holomorphic discrete series representation (π, V) for G. Then,

Theorem 5. *For a maximal connected reductive subgroup H of G, (π, V) restricted to H is admissible if and only if the center of K is a subgroup of H.*

When H is so that (G, H) a symmetric space, the theorem is a result of Kobayashi, [12], [15].

In [3] is shown that a maximal connected subgroup of G is either parabolic or reductive. For sake of completeness we list the maximal reductive subalgebras of

$\mathfrak{g}$. The classification of maximal connected subgroups of G was completed by [16]. Some of the subalgebras has a compact abelian one dimensional factor which may not be the center of $\mathfrak{k}$.

$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-25)}$
$\mathfrak{so}(10) \oplus \mathfrak{z}$	$\mathfrak{e}_6 \oplus \mathfrak{z}$
$\mathfrak{so}(2,8) \oplus \mathfrak{so}(2)$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$
$\mathfrak{so}^\star(10) \oplus \mathfrak{so}(2)$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
$\mathfrak{su}(5,1) \oplus \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{so}^\star(12) \oplus \mathfrak{su}(2)$
$\mathfrak{su}(2,4) \oplus \mathfrak{su}(2)$	$\mathfrak{su}^\star(8)$
$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(6,2)$
$\mathfrak{sp}(2,2)$	$\mathfrak{so}(10,2) \oplus \mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{g}_2 \oplus \mathfrak{su}(2,1)$	$\mathfrak{g}_2 \oplus \mathfrak{sp}(3,\mathbb{R})$
$\mathfrak{su}(2,1) \oplus \mathfrak{su}(2,1) \oplus \mathfrak{su}(3)$	$\mathfrak{f}_{4(-20)} \oplus \mathfrak{sl}(2,\mathbb{R})$

Proof. The subgroups H listed on the first seven lines corresponds to symmetric pairs (G,H). The result follows from Kobayashi [12]. Under his hypothesis, Kobayashi has shown that the multiplicity function is bounded. We do not know if this fact holds for other pairs (G,H).

For $\mathfrak{g}_2 \oplus \mathfrak{su}(2,1)$ we have that the center of the maximal compact subgroup of $e_{6(-14)}$ is contained in $\mathfrak{su}(2,1)$. Hence, owing to Proposition 4 there is admissible restriction to the subgroup. In fact, center of K is contained in SU(2,1). For this, we consider the usual imbedding $Spin(7) \times Spin(3)$ as a subgroup of $Spin(10)$. Then, $\mathfrak{s}^+$ restricted to $Spin(7)$ is equivalent to twice the spin representation of $Spin(7)$.
Counting dimensions, we get $\mathfrak{s}^+$ restricted to $G_2 \subset Spin(7)$ is equivalent to twice $V_7 \oplus \mathbb{C}$. Here, V_7 (resp. $\mathbb{C}$) is the seven dimensional (resp. one dimensional) irreducible representation for G_2. It follows from a computation that the Cartan decomposition of $\mathfrak{su}(2,1)$ is $spin(3) \oplus \mathfrak{z} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{p}^+$ is a subspace of the two copies of the trivial representation. From this we get that the center of $\mathfrak{k}$ is contained in $\mathfrak{su}(2,1)$.

The maximal compact subgroup of $G_2 \times Sp(3,\mathbb{R})$ is $G_2 \times U(3)$. We show the center of $U(3)$ is the center of K. In fact, in [16] is stated $G_2 \times SU(3)$ is a maximal subgroup of E_6. Hence, the projection of the center of $U(3)$ on the direction of E_6 is trivial. Proposition 4 yields (π, V) has admissible restriction to $G_2 \times Sp(3,\mathbb{R})$.

Next, we dealt with $\mathfrak{f}_{4(-20)} + \mathfrak{sl}_2$ in $\mathfrak{e}_{7(-25)}$. Let β denote the noncompact simple root for the holomorphic system in $e_{7(-25)}$ and let α be the compact simple root adjacent to β. We claim that of $\mathfrak{sl}_2 \cap \mathfrak{k}$ is spanned by $i(\Lambda_\beta - \Lambda_\alpha)$. The root system for the immersion of $\mathfrak{e}_{6(-14)}$ in $\mathfrak{e}_{7(-25)}$ is spanned by $\beta+\alpha$ and the five compact simple roots different from α. Hence, $i(\Lambda_\beta - \Lambda_\alpha)$ belongs to the centralizer of $\mathfrak{e}_{6(-14)}$. Since, [16], $\mathfrak{f}_{4(-20)}$ is a subalgebra of $\mathfrak{e}_{6(-14)}$, and the centralizer of $\mathfrak{f}_{4(-20)}$ in $\mathfrak{e}_7$ is $\mathfrak{sl}_2$ [1], the claim follows. Thus, a maximal compact subgroup of $F_{4(-20)} \times SL_2$ is $Spin(9) \times exp(\mathbb{R}i(\Lambda_\beta - \Lambda_\alpha))$. Moreover, the $\mathfrak{so}(2)$ factor of the immersion $spin(10) \oplus so(2)$ in $\mathfrak{e}_6$ is spanned by $i\Lambda_\alpha$.
Now, ϖ_1 restricted to $Spin(10)$ is equivalent to $\mathfrak{s}^+ \oplus \mathbb{C}^{10} \oplus \mathbb{C}Y_\beta$. Also, Λ_α takes

on the values $2, 1, 0$ on each irreducible factor. Hence $\Lambda_\beta - \Lambda_\alpha$ takes on the values $-1, 0, 1$. Therefore, $Spin(9) \times exp(\mathbb{R}i(\Lambda_\beta - \Lambda_\alpha))$ fix nonzero vectors in ϖ_1.

□

It may happen there is admissible restriction to a non compact subgroup which is not a maximal subgroup. In fact, we have,

Theorem 6. *A holomorphic discrete series for $E_{6(-14)}$ has admissible restriction to any of the subgroups*

$$SO^\star(10), \quad SO(2,8), \; SU(4,1) \times SU(2).$$

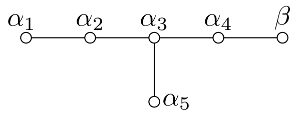
We notice that none of the subgroups listed above contain the center of K. A consequence of Theorem 2 and Theorem 8 is

Theorem 7. *If a holomorphic Discrete Series of an exceptional group has an admissible restriction to H, then center of L is a torus.*

Theorem 7 does not hold for classical groups because Proposition 3 yields that holomorphic Discrete series for $SU(2n, 1)$ has an admissible restriction to $Sp(n)$.

3. The case $\mathfrak{g} = \mathfrak{e}_{6(-14)}$

In this section, $\mathfrak{g} = \mathfrak{e}_{6(-14)}$. Following [8], we label the Dynkin diagram as follows.



The real rank is 2. We have $\gamma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \beta$, and $\gamma_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \beta$.

We provide $\mathbf{i}\mathfrak{t}^*$ with the invariant scalar product for which $(\alpha, \alpha) = 2$ for each root.This scalar product produces an isomorphism $\lambda \to H_\lambda$ from $\mathbf{i}\mathfrak{t}^*$ to $\mathbf{i}\mathfrak{t}$, and H_α is the coroot corresponding to α.

Let ϖ_β be the fundamental weight corresponding to β. We have $\varpi_\beta = \frac{1}{3}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 3\alpha_5 + 4\beta)$. We note that $\mathbf{i}H_{\varpi_\beta} = J$.

Since, by theorem 1, there is admissible restriction to K_{ss}, we consider proper maximal subgroups of K_{ss}. We show

Theorem 8. *Let (π, V) be a holomorphic discrete series representation for $E_{6(-14)}$ Then,*

i) (π, V) restricted to $K_{ss} = Spin(10)$ is admissible.

ii) Let $U(5) \to SO(10)$ denote the usual imbedding and let $\widehat{U(5)}$ denote the analytic subgroup of $Spin(10)$ associated to $\mathfrak{u}(5)$, then the restriction of π to $\widehat{U(5)}$ is admissible.

iii) For any other maximal subgroup L of K_{ss}, (π, V) restricted to L is not admissible.

iv) Let L be a closed proper subgroup of $\widehat{U(5)}$. Then π restricted to L is not an admissible representation.

Proof. To begin with, we recall the Cartan decomposition of $\mathfrak{e}_{6(-14)} = \mathfrak{so}(10) + \mathfrak{so}(2) \oplus \mathfrak{s}^+ \oplus \mathfrak{s}^-$, where $\mathfrak{s}^+, \mathfrak{s}^-$ are the two spin representations of $\mathfrak{so}(10)$. Let $\widehat{SU(5)}$ (resp. Z_5) denote the simple factor of $\widehat{U(5)}$ (resp. the center of $U(5)$).
Owing to Proposition 3, Theorem 8 follows from:

a) $S[\mathfrak{s}^+]^{Spin(10)} = \mathbb{C}$, b)$S[\mathfrak{s}^+]^{\widehat{U(5)}} = \mathbb{C}$,
c) $S[\mathfrak{s}^+]^L \neq \mathbb{C}$ for a maximal subgroup L of K_{ss} not locally isomorphic to $U(5)$.
d) For subgroup $L_1 \subseteq Z_5$ and maximal subgroup $L_2 \subseteq \widehat{SU(5)}$, or $L_1 = \{e\}$ and $L_2 = \widehat{SU(5)}$. Then $S[\mathfrak{s}^+]^{L_1 L_2} \neq \mathbb{C}$.

In [2], we find a proof of $S[\mathfrak{s}^+]^{Spin(10)} = \mathbb{C}$. Thus a) follows.
To continue, we fix an orthogonal basis $\epsilon_1, \cdots, \epsilon_5$ of $i\mathfrak{t}^\star$ so that a system of positive compact roots is $\{\epsilon_i \pm \epsilon_j, i < j\}$ and the weights of the representation $\mathfrak{s}^+$ are $\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5)$ with and odd number of $+$. The positive roots of $\mathfrak{u}(5)$ are $\{\epsilon_i - \epsilon_j, i < j\}$. Let J_5 denotes an infinitesimal generator of Z_5 chosen so that $J_5(\epsilon_1 + \cdots \epsilon_5) = i$. The $\mathfrak{u}(5)-$module $\mathfrak{s}^+$ decomposes as

$$V_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)} \oplus V_{\epsilon_1 - \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)} \oplus V_{\epsilon_1+\epsilon_2+\epsilon_3 - \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)}.$$

In [10] page 98, we find a proof that

$$(SL(2m+1), \Lambda^1(\mathbb{C}^{2m+1})^\star \oplus \Lambda^2(\mathbb{C}^{2m+1}))$$

is a prehomogeneous space. Hence, $SU(5)_{\mathbb{C}}$ has an open orbit in

$$V_{\epsilon_1} \oplus V_{\epsilon_1+\epsilon_2+\epsilon_3}$$

Therefore, $\widehat{U(5)}_{\mathbb{C}}$ has an open orbit in $\mathfrak{s}^+$. Hence, $S[\mathfrak{s}^+]^{\widehat{U(5)}} = \mathbb{C}$ and $b)$ follows. Since, $SU(5)$ acts trivially on the factor $V_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)}$, it follows $d)$ for $L_1 = \{e\}, L_2 = \widehat{SU(5)}$. Since, an element of $V_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)}$ times un element of any of the other irreducible factors is invariant under Z_5 we obtain d) for the other extreme case.
In order to show $c)$ we list, up to conjugation, the maximal connected closed subgroups of $SO(10)$. These subgroups have been classified by Dynkin in [5]. They are:

- $SO(r) \times SO(s)$ for $r + s = 10$, $U(5)$.
- $L \subseteq SO(10)$, for L a connected, simple subgroup so that $\mathbb{R}^{10}$ is an absolutely irreducible representation.

To continue, we assume $S[\mathfrak{s}^+]^L = \mathbb{C}$ for each maximal subgroup L of $Spin(10)$ not locally isomorphic to $U(5)$. From this we derive a contradiction.
As before, $\mathfrak{s}^+ : Spin(10) \longrightarrow Gl(\mathfrak{s}^+)$ denote the half spin representation. To begin with, we consider $\rho : L \longrightarrow Spin(10)$ an irreducible, simple, maximal subgroup. $Then(\mathfrak{s}^+ \circ \rho, \mathfrak{s}^+)$ decomposes as the sum irreducible $L-$modules

$$V_1 \oplus \cdots \oplus V_r.$$

We set ρ_j equal to the projection onto V_j followed by $\mathfrak{s}^+ \circ \rho$. Owing to our hypothesis, it follows $S[V_j]^{\rho_j(L)} = \mathbb{C}$ for $j = 1, \cdots, r$. In [2], [10], [11] [17] we find the list of triples (L, ρ_j, V_j) where: L is a simple algebraic group, ρ_j is an irreducible representation and $S[V_j]^{\rho_j(L)} = \mathbb{C}$. The list is:

$$(A_n, \Lambda_1, \mathbb{C}^{n+1}), \quad (A_{2n}, \Lambda_2, \mathbb{C}^{n(2n+1)}), \quad (C_n, \Lambda_1, \mathbb{C}^{2n}).$$

We first verify none of the V_j is equivalent to (C_n, Λ_1). Since the ten dimensional irreducible representation of SL_2 is symplectic, we have $n \geq 2$. For L of type $C_n, n \geq 2$ and $r = 1$ we obtain $n = 8$, a contradiction. For L of type $C_n, n \geq 2$ and $r \geq 2$ the symplectic form lead us to $S[\mathfrak{s}^+]^L \neq \mathbb{C}$, another contradiction.

For L of type $A_n, n \geq 2$, if at least one V_j is equivalent to (A_{2k}, Λ_2), then $n = 2k$ and $k(2k+1) \leq 16$, hence, L is one of A_2, A_4. $SL(3)$ has two irreducible representations of dimension ten whose highest weight are $(3, 0, 0) = 3\Lambda_1, -w_o(3\Lambda_1)$, neither of these two representations are orthogonal [3]. $SL(5)$ also has two ten dimensional representations of highest weight Λ_2 or Λ_3, [3] neither of them is orthogonal.

We are left to analyze the situation all V_j are equivalent to $(A_n, \Lambda_1), n \geq 2$. Since L is a subgroup of $Spin(10)$ we have $n \leq 5$. The case n even was analyzed in the previous paragraph. The ten dimensional irreducible representations of $SL(4)$ have highest weight $2\Lambda_1$ or $2\Lambda_3$ they are not orthogonal. $SL(6)$ has no irreducible representation of dimension ten.

To conclude the proof of c) we show

$$S[\mathfrak{s}^+]^{\mathfrak{so}(p) \oplus \mathfrak{so}(q)} \neq \mathbb{C} \text{ for } \ p \geq 1, \, q \geq 1, p + q = 10.$$

We recall the following facts, for a proof, see [1], [3] Table 1.

- A half spin representations $(s^{\pm})$ for $Spin(2k)$ restricted to $Spin(2k-1)$ is equivalent to the spin representation (s).
- The spin representation for $Spin(2k+1)$ restricted to $Spin(2k)$ is equivalent to the sum of the two half spin representations.
- An irreducible spin representation for $Spin(9), Spin(8), Spin(7)$ is orthogonal.
- An irreducible spin representation for $Spin(5), Spin(4)$ is symplectic.

For $p = 9, q = 1$, $\mathfrak{s}^+$ restricted to $Spin(9)$ is equivalent to the spin representation of $Spin(9)$. Since the spin representation of $Spin(9)$ is orthogonal, we obtain $S[\mathfrak{s}^+]^{\mathfrak{so}(9) \oplus \mathfrak{so}(1)} \neq \mathbb{C}$.

For $p = 8, q = 2$ $\mathfrak{s}^+_{|_{Spin(8)}} = s^+ \oplus \ s^-$, besides $Spin(2)$ acts on $s^{\pm}$ by $\pm\frac{1}{2}$. Let $b_{\pm}$ denote a $Spin(8)$ invariant quadratic form in $\mathfrak{s}^{\pm}$. Then $b_+ b_-$ is invariant under $Spin(8) \times Spin(2)$.

For $p = 7, q = 3$, $\mathfrak{s}^+_{|_{Spin(7)}} = s \oplus s$. Hence, $\mathfrak{s}^+_{|_{Spin(7) \times Spin(3)}} = s \boxtimes \mathbb{C}^2$. In [11] it is shown it is not an irreducible prehomogeneous vector space.

For $p = 6, q = 4$, $\mathfrak{s}^+_{|_{Spin(6)}} = (s_+ \oplus s_-) \oplus (s_+ \oplus s_-)$.

Here, $L = SL(4) \times SL(2)_+ \times SL(2)_-$ and the restriction of $\mathfrak{s}^+$ to L is equivalent to

$$\mathbb{C}^4 \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C} \oplus (\mathbb{C}^4)^{\star} \boxtimes \mathbb{C} \boxtimes \mathbb{C}^2$$

Hence, the restriction of $\mathfrak{s}^+$ to L is equivalent to

$$\mathbb{C}^{4\times 2} \oplus \mathbb{C}^{4\times 2}$$

with action

$$(T,A,B)^{-1}(X,Y) = (T^{-1}XA, T^tYB) \qquad T \in SL(4), A, B \in SL(2), X, Y \in \mathbb{C}^{4\times 2}$$

An invariant polynomial function is $p(X,Y) = det(Y^tX)$. By duality $S[\mathfrak{s}^+]^{SO(4)\times SO(6)}$ $\neq \mathbb{C}$. Actually, the invariant polynomial functions are the polynomial ring in p.

Finally we examine $p = q = 5$. Here, the restriction of $\mathfrak{s}^+$ to $Spin(5) \times Spin(5)$ is equivalent to $s \boxtimes s$. In [11] Apendix, it is shown that this representation is not a prehomogeneous vector space. Hence,
$S[\mathfrak{s}^+]^{Spin(5)\times Spin(5)} \neq \mathbb{C}$ and we have verified c).

We now show d). For this we recall the work of [5] on the maximal subgroups of $SU(5)$. Up to conjugation, the maximal connect subgroups of $SU(5)$ are among the subgroups

$$SO(5), \quad S(U(k) \times U(5-k))\, k = 1,2,3,4, \quad (SU(2), \rho)$$

Here, ρ is the five dimensional irreducible representation of $SU(2)$.
Either the representation of $SO(5)$ or $SU(2)$ is orthogonal, [3], hence, an invariant for Z_5 times one of these two groups, is given by an element of $V_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)}$ times the invariant quadratic form.
$\mathfrak{s}^+$ restricted to $SU(4) \times SU(1)$ is equivalent to

$$\mathbb{C} \oplus (\mathbb{C}^4 \oplus \mathbb{C}e_5) \oplus (\Lambda^3(\mathbb{C}^4) \oplus \Lambda^2(\mathbb{C}^4) \wedge \mathbb{C}e_5).$$

The representation of $S(U(4) \times U(1))$ in $\mathbb{C}^4 \oplus \Lambda^3(\mathbb{C}^4)$ is orthogonal, the action of J_5 in

$$V_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)} \oplus V_{\epsilon_1-\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)} \oplus V_{\epsilon_1+\epsilon_2+\epsilon_3-\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)}$$

is respectively

$$\frac{5}{6}i, \quad -\frac{3}{6}i, \quad \frac{1}{6}i.$$

Hence, after multiplying a suitable power of the invariant quadratic form times a power of an element in $V_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)}$ we obtain an invariant for the group $Z_5S(U(4) \times U(1))$.
The decomposition of $\mathfrak{s}^+$ under $SU(3) \times SU(2)$ is

$$\mathbb{C} \boxtimes \mathbb{C} \oplus (\mathbb{C}^3 \boxtimes \mathbb{C} \oplus \mathbb{C} \boxtimes \mathbb{C}^2) \oplus (\Lambda^3(\mathbb{C}^3) \boxtimes \mathbb{C} \oplus \Lambda^2(\mathbb{C}^3) \boxtimes \Lambda^1(\mathbb{C}^2) \oplus \Lambda^1(\mathbb{C}^3) \boxtimes \Lambda^2(\mathbb{C}^2)).$$

Applying duality of representations we find in $S[\mathbb{C}^2 \oplus \Lambda^1(\mathbb{C}^2)]$ an element of degree two $\sum_r X_rY_r$, which is $U(2)-$invariant. Also, in $S[\mathbb{C}^3 \oplus \Lambda^2(\mathbb{C}^3)]$ there is an invariant, under $U(3)$, of degree two $\sum_j Z_j W_j$. It readily follows that

$$\sum_r (1 \otimes X_r) \sum_j (Z_j \otimes 1)(W_j \otimes Y_r)$$

is invariant under $U(3) \times U(2)$. As for the previous case it follows there is an invariant under $Z_5 S(U(3) \times U(2))$. Thus, we have verified d) and we conclude the proof of Theorem 8.

□

Finally, we analyze the admissibility of (π, V) restricted to specific reductive subgroups of $E_{6(-14)}$. Let ρ_j denote the fundamental weight of $spin(10)$ associated to α_j. The centralizer of ρ_j in $spin(10)$ is equal to a semisimple Lie algebra $\mathfrak{r}_j$ plus the line spanned by H_{ρ_j}.
We fix a, b real numbers, j runs from 1 to 5
We define $\mathfrak{l}_{j,a,b}$ to be the subalgebra spanned by $\mathfrak{r}_j$ together with the vector $aJ + bH_{\rho_j}$. We only consider a, b such that the analytic subgroup associated to $\mathfrak{l}_{j,a,b}$ is compact. Either $\mathfrak{l}_{4,a,b}$, or $l_{5,a,b}$ is isomorphic to $\mathfrak{u}(5)$. $\mathfrak{l}_{4,0,1}, \mathfrak{l}_{5,0,1}$ are the usual two immersions of $\mathfrak{u}(5)$ in $spin(10)$. From now on, we write $T_{j,a,b}$ for the analytic subgroup of K associated to $\mathbb{R}(aJ + bH_{\rho_j})$.

Proposition 6. *A holomorphic discrete series for $E_{6(-14)}$ has an admissible restriction to the subgroups:*

$$T_{1,a,b} \text{ iff } |a| > |\frac{b}{2}|;\ T_{2,a,b} \text{ iff } |a| > |b|;$$

$$T_{3,a,b} \text{ iff } |a| > |\frac{3b}{2}|;\ T_{4,a,b} \text{ iff } (a - \frac{5b}{4})(a + \frac{3b}{4}) > 0;$$

$$T_{5,a,b} \text{ iff } (a + \frac{5b}{4})(a - \frac{3b}{4}) > 0;$$

$$L_{4,a,b} \text{ iff } (a - \frac{5b}{4}) \neq 0\,;$$

$$L_{5,a,b} \text{ iff } (a + \frac{5b}{4}) \neq 0\,.$$

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