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HYPERGEOMETRIC FUNCTIONS AND BINOMIALS

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Abstract. We highlight the role of primary decomposition of binomial ideals in a commutative polynomial ring, in the description of the holonomicity, the holonomic rank, and the shape of solutions of multivariate hypergeometric differential systems of partial differential equations.

En honor a Mischa Cotlar, con afecto y admiración

1. Introduction

There have been two main directions in the study of classical hypergeometric functions. The first of these is to study the properties of a particular series, going back to Euler [Eu1748] and Gauss [Gau1812]. The second one is to find a differential equation satisfied by the hypergeometric function, and to study all the solutions of that equation, going back to Kummer [Kum1836] and Riemann [Rie1857].

Recall that a (convergent) univariate series

$$F(x) = \sum_{n\geq 0} A_n x^n$$

is called hypergeometric if the ratio between two consecutive coefficients $R(n) := A_{n+1}/A_n$ is a rational function of n. If we write $R(n) = P(n)/Q(n+1)$ with P, Q polynomials, this is equivalent to the recurrence $Q(n+1)A_{n+1} - P(n)A_n = 0$. The main basic observation is that this recurrence can be expressed by the fact that the hypergeometric differential operator $Q(\Theta) - xP(\Theta)$ annihilates F. Here Θ denotes the operator $x\frac{d}{dx}$.

In the generalization to hypergeometric functions in several variables, these two points of view suggest different approaches. A natural definition of hypergeometric power series in several variables was proposed by Horn [Hor1889, Hor31]. A series

$$F(z) = \sum_{\alpha\in\mathbb{N}^m} a_\alpha z_1^{\alpha_1}\cdots z_m^{\alpha_m}$$

in m variables with complex coefficients is *hypergeometric in the sense of Horn* if there exist rational functions $R_1, R_2, \ldots, R_m$ in m variables such that

$$\frac{a_{\alpha+e_k}}{a_\alpha} = R_k(\alpha) \quad \text{for all } \alpha\in\mathbb{N}^m \text{ and } k = 1,\ldots,m, \tag{1.1}$$

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where $e_1, \dots, e_m$ denote the standard basis vectors of $\mathbb{N}^m$. As before, denote by Θ_i the operator $z_i \frac{\partial}{\partial z_i}$, $i = 1, \dots, m$. For all monomials z^α and all polynomials g it holds that $g(\Theta_1, \dots, \Theta_m) z^\alpha = g(\alpha_1, \dots, \alpha_m) z^\alpha$, which implies again the following relation between the behaviour of the ratios of coefficients and the existence of hypergeometric differentials operators annihilating F. If we write the rational functions R_k as

$$R_k(\alpha) = P_k(\alpha)/Q_k(\alpha + e_k) \quad k = 1, \dots m,$$

where P_k and Q_k are relatively prime polynomials and z_k divides Q_k, the series F satisfies the following *Horn hypergeometric system of differential equations*:

$$(Q_k(\Theta_1, \dots, \Theta_m) - z_k P_k(\Theta_1, \dots, \Theta_m))F = 0, \quad k = 1, \dots, m. \tag{1.2}$$

Gel′fand, Graev, Kapranov, and Zelevinsky [GGZ87, GKZ89, GKZ90] developed a highly interesting point of view, by "dressing" the hypergeometric functions and operators with "homogeneities". This allows to understand the properties of classical hypergeometric systems via tools in algebraic geometry and combinatorics. They associated to a configuration $A = \{a_1, \dots, a_n\} \subset \mathbb{Z}^d$ of lattice points spanning $\mathbb{Z}^d$ (which we encode as the columns of a $d \times n$ integer matrix, also called A) and a vector $\beta \in \mathbb{C}^d$, a *holonomic* left ideal in the Weyl algebra in n variables $D_n := \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ as follows.

Definition 1.1. The *A-hypergeometric system* (or GKZ-hypergeometric system) with exponent β is the left ideal $H_A(\beta)$ in the Weyl algebra D_n generated by the *toric operators* $\partial^u - \partial^v$, for all $u, v \in \mathbb{N}^n$ such that $Au = Av$, and the *Euler operators* $\sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$ for $i = 1, \dots, d$.

We refer the reader to [SST00] for an account of A-hypergeometric systems with emphasis on computations.

A local holomorphic function $F(x_1, \dots, x_n)$ is *A-hypergeometric of degree β* if it is annihilated by $H_A(\beta)$. A-hypergeometric systems are homogeneous (and complete) versions of classical hypergeometric systems in $m := n - d$ variables [GGR92, DMS05], in the following sense. Let $B \in \mathbb{Z}^{n \times (m)}$ be a matrix whose colums span the integer kernel $\mathcal{L}$ of A, and also call $B = \{b_1, \dots, b_n\} \subset \mathbb{Z}^m$ the *Gale dual* lattice configuration given by the rows of B. Consider the surjective open map

$$x^B : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m, \quad x \mapsto (\prod_{j=1}^n x_j^{b_{j1}}, \dots, \prod_{j=1}^n x_j^{b_{jm}}).$$

Let $U \subseteq (\mathbb{C}^*)^n$, $V = x^B(U)$ be simply connected open sets and denote by $y = (y_1, \dots, y_m)$ the coordinates in V. Given a holomorphic function $\psi \in \mathcal{O}(V)$, call $\varphi = x^c \psi(x^B) \in \mathcal{O}(U)$, where $c \in \mathbb{C}^n$. Then, $\left(\sum_{j=1}^n a_{kj} x_j \partial_{x_j}\right)(\varphi) = (A \cdot c^t)_k(\varphi)$, for $k = 1, \dots d$. Moreover, for any $u = B \cdot \lambda \in \mathcal{L}$, we have that $T_u(\varphi) = 0$ if and only if $H_u(\psi) = 0$, where T_u and H_u denote the following differential operators in n and m variables respectively:

$$T_u = \prod_{u_i > 0} \left(\frac{\partial}{\partial x_i}\right)^{u_i} - \prod_{u_i < 0} \left(\frac{\partial}{\partial x_i}\right)^{-u_i}, \tag{1.3}$$

$$H_u = \prod_{u_i>0} \prod_{l=0}^{u_i-1} (b_i \cdot \theta_y + c_i - l) - y^\lambda \prod_{u_i<0} \prod_{l=0}^{|u_i|-1} (b_i \cdot \theta_y + c_i - l), \tag{1.4}$$

where $b_i \cdot \theta_y = \sum_{j=1}^m b_{ij} y_j \frac{\partial}{\partial_{y_j}}$ Note that system (1.2) corresponds to the choices $\lambda = e_1, \dots, e_m$ ($u = Be_1, \dots, Be_m$). On the A-side, this amounts to considering only codimension many binomial operators (corresponding to the columns of B, that is to a $\mathbb{Z}$-basis of $\mathcal{L}$, and not the whole system of toric binomials in (1.1)).

Although the isomorphism sending ψ to φ is only at the level of local holomorphic solutions and not at the level of D-modules, it preserves many of the pertinent features, including the dimensions of the spaces of local holomorphic solutions and the structure of their series expansions. This point of view allowed us in [DMS05] to study the holonomicity, the holonomic rank and the persistence of Puiseux polynomial solutions to Horn systems for $m = 2$ via the translation into a multihomogeneous setting and the study of primary components of binomial ideals in a commutative polynomial ring. This work was widely generalized in the recent article [DMM06], where we propose the definition of a *binomial D-module* (see Definition 3.1 below) which comprises the notion of A-hypergeometric system. In this note we illustrate our perspective and results via examples and extended comments.

2. Examples: Holonomic rank and binomials

Before giving more precise statements and definitions, we focus on classical examples under a modern "binomial" view.

2.1. **Horn G_3 and Appel F_1 hypergeometric systems in two variables.** We consider two classical bivariate Horn systems of hypergeometric partial differential equations as in (1.2):

- The system associated to Horn G_3 hypergeometric series is:

  $$\big(y_1(2\theta_{y_1} - \theta_{y_2} + a')(2\theta_{y_1} - \theta_{y_2} + a' + 1) - (-\theta_{y_1} + 2\theta_{y_2} + a)\theta_{y_1}\big) f = 0\ ,$$
  $$\big(y_2(-\theta_{y_1} + 2\theta_{y_2} + a)(-\theta_{y_1} + 2\theta_{y_2} + a + 1) - (2\theta_{y_1} - \theta_{y_2} + a')\theta_y\big) f = 0\ ,$$

  where a, a' are generic parameters.
- The system associated to Appell series F_1 is:

  $$\big(y_1(\theta_{y_1} + \theta_{y_2} + a)(\theta_{y_1} + b) - \theta_{y_1}(\theta_{y_1} + \theta_{y_2} + c - 1)\big) f = 0\ ,$$
  $$\big(y_2(\theta_{y_1} + \theta_{y_2} + a)(\theta_{y_2} + b') - \theta_{y_2}(\theta_{y_1} + \theta_{y_2} + c - 1)\big) f = 0,$$

  where a, b, b' and c are generic parameters.

Recall that the holonomic rank of a system of linear differential operators with polynomial (or holomorphic coefficients) is the vector space dimension of the space of local holomorphic solutions around a point which is not in the singular locus of the associated D-module. Both systems are defined by two linear operators in two variables, so a first guess for the holonomic rank is 4 in both cases. This is the right answer for Horn G_3 system, but Appell F_1 system has holonomic rank equal to 3. Moreover, Erdélyi noted in [Erd50] that, in a neighborhood of a given point, three linearly independent solutions of Horn G_3 system can be obtained through contour integral methods. He also found a fourth linearly independent solution: the Puiseux monomial $x^{-(a+2a')/3}y^{-(2a+a')/3}$. He remarked that the existence of this elementary solution was puzzling, and offered no explanation for its occurrence. Via the translation to a homogeneous system (which is part of an A-hypergeometric system, as in the Introduction), we gave in [DMS05] an explanation for these facts.

Look at the binomials

$$q_1 = \partial_1^2\partial_4^0 - \partial_2^1\partial_3^1, \quad q_2 = \partial_1^1\partial_4^0 - \partial_2^2$$

in the commutative polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_4]$. The exponents are read from the (integer) coefficients of the 4 linear forms in $(\Theta_{y_1}, \Theta_{y_2})$ occuring in the system. Its zero set in $\mathbb{C}^4$ has two irreducible components. One is the toric variety which is the closure of the solutions in the torus $(\mathbb{C}^*)^4$ of the binomials q_1, q_2, which has degree 3 and corresponds to the 3 fully supported solutions described by Erdélyi (we refer the reader to [Stu96] for background on toric varieties). The other component $\{\partial_1 = \partial_2 = 0\}$ lies "at infinity", that is, in the union of the coordinate hyperplanes. Its multiplicity equals the intersection multiplicity μ_0 at the origin of the system of 2 binomials in 2 variables

$$p_1 = \partial_1^{b_{11}} - \partial_2^{b_{12}},\, p_2 = \partial_1^{b_{21}} - \partial_2^{b_{22}},\ b_{11} = b_{22} = 2, b_{12} = b_{21} = 1.$$

This multiplicity can be computed as ([DSt02])

$$\mu_0 = \min\{b_{11} \cdot b_{22}, b_{12} \cdot b_{22}\} = 1.$$

Since the determinant of the 2×2 matrix (b_{ij}) is non zero, Theorem 2.5 in [DMS05] asserts that this multiplicity equals the dimension of the space of solutions to the Horn system which have finite support. Further results in [DMS05] explain the fact that in this case one gets a (non zero) Puiseux monomial solution.

The same result explains the holonomic degree 4 for Appell F_1 system. The translation to the homogeneous setting gives the following two binomials:

$$q_1' = \partial_1^1\partial_3^1 - \partial_2^1\partial_4^1, \quad q_2' = \partial_1\partial_5 - \partial_2\partial_6$$

in the commutative polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_4]$. In this case we have 6 variables since there are 6 different integer linear forms in $(\Theta_{y_1}, \Theta_{y_2})$ occuring in the two differential operators. The zero set of q_1', q_2' in $\mathbb{C}^2$ has again two components. One is the toric variety which is the closure of the solutions in the torus $(\mathbb{C}^*)^4$ of the binomials q_1', q_2', which has degree 3 and gives 3 fully supported linearly independent solutions. The other component $\{\partial_1 = \partial_2 = 0\}$ lies "at infinity", and it has multiplicity $\mu_0' = \min\{1.1, 1.1\} = 1$. But this time the determinant of the matrix with rows $(1,1), (-1,-1)$ is zero, and so for generic parameters a, b, b', c the differential system is holonomic and this component does not contribute any new solution.

For particular choices of parameters, Appell F_1 Horn system is not holonomic. Consider the matrix

$$B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix},$$

and a vector of parameters $c = (c_1, \ldots, c_6)$. Denote by ∂_i the partial derivative with respect to the variable x_i and let $\Theta_i = x_i \partial_i$, for $i = 1, \ldots, 6$. We can choose the following matrix A, with Gale dual B:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Set $\beta = Ac^t$. The corresponding Horn system (1.2) looks in its "binomial incarnation" as follows:

$$\langle \partial_1\partial_3 - \partial_2\partial_4, \partial_1\partial_5 - \partial_2\partial_6, E_1 - \beta_1, E_2 - \beta_2, E_3 - \beta_3, E_4 - \beta_4 \rangle,$$

where $E_1 = \Theta_1 + \Theta_2$, $E_2 = \Theta_3 + \Theta_4$, $E_3 = \Theta_5 + \Theta_6$, $E_4 = \Theta_1 + \Theta_4 + \Theta_6$. If $\beta_1 = 0$, then any (local holomorphic) function $f(x_3, x_4, x_5, x_6)$ annihilated by the operators $\Theta_3 + \Theta_4 - \beta_2, \Theta_5 + \Theta_6 - \beta_3, \Theta_4 + \Theta_6 - \beta_4$ is a solution, for instance all monomials $x_3^{\gamma_3} x_4^{\gamma_4} x_5^{\gamma_5} x_6^{\gamma_6}$ with $(\gamma_3, \gamma_4, \gamma_5, \gamma_6) \in \mathbb{C}^4$ satisfying $\gamma_3 + \gamma_4 = \beta_2, \gamma_5 + \gamma_6 = \beta_3, \gamma_4 + \gamma_6 = \beta_4$. So, the space of such functions is infinite-dimensional; in fact, it has uncountable dimension. Again, this phenomenon is explained in general in [DMS05].

2.2. **Mellin hypergeometric system for algebraic functions.** Given coprime integers $0 < k_1 < \ldots < k_m < n$, set

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & k_1 & \ldots & k_m & n \end{pmatrix},$$

and $\beta = (0, -1)$. The local roots $t(z)$ of the generic sparse polynomial

$$f(z;t) := z_0 + z_{k_1}\, t^{k_1} + \cdots + z_{k_m}\, t^{k_m} + z_n\, t^n ,$$

viewed as functions of the coefficients, are algebraic solutions to the associated A-hypergeometric system [Ma37, Bir27, Stu00, CDD99, PT04]. In particular, any solution to this A-hypergeometric system has a double homogeneity property and can therefore be considered as a function of m variables. This implies that we can arbitrarily prescribe the values of any two nonzero coefficients in $f(z;t)$ without losing any essential information on the general solution to this equation. If we divide by $-z_0$ and then set $y = (-z_n/z_0)^{1/n} t$, we obtain an algebraic equation of the form

$$y^n + x_m y^{k_m} + \ldots + x_1 y^{k_1} - 1 = 0. \tag{2.1}$$

A classical result of Mellin from 1921 (see [Mel21]) states that if $y(x)$ is a local root of (2.1), then it satisfies the following system of m partial differential equations:

$$\prod_{k=0}^{k_j-1}(k_1\theta_1+\ldots+k_m\theta_m+nk+1)\prod_{k=0}^{m_j^{'}-1}(k_1^{'}\theta_1+\ldots+k_m^{'}\theta_n+nk-1)y(x)=$$

$$(-1)^{k_j}n^n\frac{\partial^n y(x)}{\partial x_{m-j}^n},\quad j=1,\ldots,n, \tag{2.2}$$

where $\theta_j = x_j\frac{\partial}{\partial x_j}$ and $k_j^{'} = n-k_j$.

In [DS07] we studied the solutions of the Mellin system via homogenization and translation to the binomial setting. Mellin not only observed in [Mel21] that the roots $y(x)$ of the algebraic equation (2.1) satisfy the Mellin system (2.2), but he also made the following remark. The solution $y_{pr}(x)$ around the origin which satisfies $y_{pr}(0)=1$, is given by

$$y_{pr}(x)=\sum_{\nu_1,\ldots,\nu_m\geq 0}\frac{(-1)^{|\nu|}}{n^{|\nu|}}\,\frac{\prod_{\mu=1}^{|\nu|-1}(k_m\nu_1+\ldots+k_1\nu_m-n\mu+1)}{\nu_1!\ldots\nu_m!}\,x_1^{\nu_1}\ldots x_n^{\nu_n}. \tag{2.3}$$

Here $|\nu|=\nu_1+\ldots+\nu_n$ and the empty product is defined to be 1. All other solutions around the origin have the form

$$\eta\, y_{pr}(\eta^{k_1}x_1,\ldots,\eta^{k_m}x_m),$$

where η runs through the n-roots of 1. It is also clear that for any choice of $I=(i_1,\ldots,i_m)\in\mathbb{N}^m$, the function $y_I(x)=y_{pr}(\varepsilon^{i_1}x_1,\ldots,\varepsilon^{i_m}x_m)$ is a root of the algebraic equation

$$y^n+\varepsilon^{i_1}x_1y^{k_1}+\ldots+\varepsilon^{i_m}x_my^{k_m}-1=0, \tag{2.4}$$

where we denote $\varepsilon=e^{2\pi i/n}$. It happens that the holonomic rank of Mellin system (2.2) equals n^m. In the effort of getting m equations such that in each of them a partial derivative of $y(x)$ with respect to each of the variables $x_1,\ldots,x_m$ is expressed in terms of other derivatives, the Mellin system has not only the roots of the algebraic equation (2.1) as solutions, but also the roots of the associated equations (2.4).

When we add homogeneities, we are led to an isomorphism between the solutions of the Mellin system and the solutions of a Horn system in m variables associated to the following matrix $B\in\mathbb{Z}^{(m+2)\times m}$:

$$B:=\begin{pmatrix} -k_m & -k_{m-1} & \ldots & -k_1\\ n & 0 & \ldots & 0\\ 0 & n & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & n\\ -k_m' & -k_{m-1}' & \ldots & -k_1'\end{pmatrix}, \tag{2.5}$$

Let $c=(-1/n,0\ldots,0,1/n)$ and consider the Horn system $\langle H_{Be_1},\ldots,H_{Be_m}\rangle$, given by the operators defined in (1.4). The colums of B generate a non saturated lattice of order n^{m-1} and the binomials read in its columns

$$z_0^{k_j} z_n^{k'_j} - z_{k_j}^n, \quad j = 1, \dots, m$$

define a complete intersection variety X with n^{m-1} components not at infinity, given by torus translates of the toric variety X_A associated to the matrix A (which correspond to the associate algebraic equations (2.4). The degree of X_A (and hence of all the components) is n, so by [DMM06, Theorem 6.10], the holonomic rank of the Horn system equals $n^{m-1}.n = n^m$, which explains the holonomic rank of the Mellin system. Moreover, we describe in [DS07, Theorem 4.3] the dimension of the space of algebraic solutions and the ocurrence of explicit non algebraic logarithmic solutions. This behaviour is related to the interactions among the different primary components of X. The general pattern is given in [DMM06, Theorem 6.8].

3. Main definitions and results

3.1. **What is a binomial D-module?** We give now the main general definition of a binomial D-module, which contains the previous hypergeometric systems as special cases. The building blocks in the world of binomial D-modules are the A-hypergeometric systems (1.1) for different A.

We consider an integer matrix $A \in \mathbb{Z}^{d\times n}$ such that the cone generated by the columns $a_1, \dots, a_n$ of A contains no lines, all of the a_i are nonzero, and $\mathbb{Z}A = \mathbb{Z}^d$. The matrix A induces a $\mathbb{Z}^d$-grading of the polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n] = \mathbb{C}[\partial]$ by setting $\deg(\partial_i) = -a_i$. An ideal of $\mathbb{C}[\partial]$ is *A-graded* if it is generated by elements that are homogeneous for the A-grading.

A *binomial ideal* is an ideal generated by *binomials* $\partial^u - \lambda\partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$. A binomial ideal is A-graded precisely when it is generated by binomials $\partial^u - \lambda\partial^v$ each of which satisfies either $Au = Av$ or $\lambda = 0$.

The Weyl algebra $D = D_n$ of linear partial differential operators is also A-graded by additionally setting $\deg(x_i) = a_i$. For each $i \in \{1, \dots, d\}$, the i-th *Euler operator* is defined as

$$E_i = a_{i1}\theta_1 + \cdots + a_{in}\theta_n.$$

Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, \dots, E_d - \beta_d$. Note that these operators are A-homogeneous of degree 0. A (local) holomorphic function $f(x)$ is annihilated by $E - \beta$ precisely when f is A-homogeneous of homogeneity β, that is

$$f(\lambda^{a_1}x_1, \dots, \lambda^{a_n}x_n) = \lambda^\beta f(x_1, \dots, x_n),$$

where $\lambda \in (\mathbb{C}^*)^d$, with all λ_i close to 1.

Definition 3.1. Given an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_A(I, \beta)$ the left ideal $I + \langle E - \beta\rangle$ in the Weyl algebra D. The binomial D-module associated to I is the quotient $D/H_A(I, \beta)$.

Thus, a *binomial D-module* is a quotient by a left D-ideal generated by an A-graded binomial ideal I with constant coefficients plus Euler operators of order 1 associated to the row span of A, which prescribe A-homogeneity infinitesimally. As we pointed out in the classical case, the binomial differential operators annihilate a (multivariate Puiseux) series if and only if the coefficients of the series satisfy (special) linear recurrences.

When I equals the toric ideal $I_A = \langle \partial^u - \partial^v,\, u, v, \in \mathbb{N}^n,\, Au = Av\rangle$, the associated binomial D-module is just the A-hypergeometric system (1.1). On the other side, when I is generated by binomials $p_i, i = 1, \ldots, m$, $p_i = \partial^{u_i} - \partial^{v_i}$, with $u_1 - v_1, \ldots, u_m - v_m$ linearly independent over $\mathbb{Q}$ (in this case I is called a *l*attice basis ideal), the corresponding binomial D-module is a Horn system (in binomial version).

3.2. **Main questions and answers about binomial D-modules.** We summarize the main questions concerning binomial D-modules:

- For which parameters does the space of local holomorphic solutions around a nonsingular point of a binomial Horn system have finite dimension as a complex vector space?
- What is a combinatorial formula for the minimum holonomic rank, over all possible choices of parameters?
- Which parameters are generic, in the sense that the minimum dimension is attained?
- How do (the supports of) series solutions centered at the origin of a binomial Horn system look, combinatorially?
- When is $D/H_A(I, \beta)$ a holonomic D-module?
- When is $D/H_A(I, \beta)$ a *regular* holonomic D-module?

We now summarize the main answers to these questions. We refer to [DMM06] for precise definitions, statements and proofs. We explicitly describe and classify all primary components of I (in particular, all monomials that are present), their multiplicities, their behaviour with respect to the grading (which splits the components into *toral* (when they don't have any further homogeneities, as in the component at infinity of the Horn G_3 system, and *Andean* (when they admit a new homogeneity as in the component at infinity in the Appell F_1 system), and their holonomic rank. Moreover, we explicitly define three subspace arrangements associated to the Andean components *(Andean arrangement)*, to the pairwise intersections of two components *(primary arrangement)* and to the rank-jumping parameters where the holonomic rank increases *(jump arrangement)*, as the Zariski closure of parameters for which the corresponding piece in certain local cohomology modules is non zero, which account for non generic behaviour of the complex parameter β.

Concretely,

- The dimension is finite exactly for $-\beta$ not in the Andean arrangement.
- Given $J \subset \{1, 2, \ldots, n\}$, denote by A_J the submatrix of A which consists only of the columns a_j of A for $j \in J$ and denote by L a sublattice of $\mathbb{Z}^J \cap \ker(A)$. We refer to the Introduction of [DMM06] for complete definitions and explanations, in particular for the definition of the multiplicities $\mu(L, J)$ and the notion of *toral* associated sublattice. The generic (minimum) holonomic rank is $\sum \mu(L, J) \cdot \mathrm{vol}(A_J)$, the sum being over all toral associated sublattices

with $\mathbb{C}A_J = \mathbb{C}^d$, where $\mathrm{vol}(A_J)$ is the volume of the convex hull of A_J and the origin, normalized so a lattice simplex in $\mathbb{Z}A_J$ has volume 1.
- The minimum rank is attained precisely when $-\beta$ lies outside of an explicit affine subspace arrangement determined by certain local cohomology modules, containing the Andean arrangement.
- When the Horn system is regular holonomic and β is general, there are $\mu(L, J) \cdot \mathrm{vol}(A_J)$ linearly independent solutions supported on (translates of) the L-bounded classes (for the definition, see [DMM06, Subsection 1.6]), with hypergeometric recursions determining the coefficients.
- Only $g \cdot \mathrm{vol}(A)$ many Gamma series solutions have full support, where we denote $g = |\ker(A)/\mathbb{Z}B|$ the index of $\mathbb{Z}B$ in its saturation $\ker(A)$.
- Holonomicity is equivalent to finite dimension of the (local holomorphic) solutions spaces.
- Holonomicity is equivalent to regular holonomicity when I is standard $\mathbb{Z}$-graded—i.e., the row-span of A contains the vector $[1 \cdots 1]$. Conversely, if there exists a parameter β for which $D/H_A(I, \beta)$ is regular holonomic, then I is $\mathbb{Z}$-graded.

3.3. **Main tools.** We briefly discuss the main tools in the proofs of the previous results. As we said in the introduction, these results are generalizations of the beautiful theory of A-hypergeometric systems developed by Gel′fand, Kapranov, and Zelevinsky in [GKZ89, GKZ90], see also [Ado94, SST00] for further developments of the basic theory. They defined the systems in terms of binomials and Euler operators, proved that they are holonomic, computed the holonomic rank for generic parameters, described the singular locus (in terms of sparse discriminants), and constructed bases of solutions for generic parameters in terms of Gamma series.

We also use the results in [DMM08], which give a precise description of the combinatorial and commutative algebra of primary components of binomial ideals in semigroup rings, based on the basic work of Eisenbud and Sturmfels on binomial ideals in [ES96]. In particular we have characterized the monomials in each primary component.

In characteristic zero, the primary decomposition of arbitrary binomial ideals is controled by the geometry and combinatorics of lattice point congruences, which also governs the D-module theoretic properties of binomial D-modules. In order to make this translation, we need to extend the results on Euler-Koszul homology from [MMW05], which functorially translate the commutative algebra of A-graded primary decomposition into the D-module setting. The Euler-Kozsul homology functor is used to pull apart the primary components of binomial ideals, thereby isolating the contribution of each to the solutions of the corresponding binomial D-module. This allows us to integrate results from [GKZ89, GKZ90, Ado94, Hot91, SchW08, DMS05]. In particular, we prove that for parameters β outside the (finite) primary subspace arrangement, the binomial D-module $D/H_A(I, \beta)$ decomposes as a direct sum over the toral primary components of I.

4. Further examples: binomials and the shape of the solutions

The basic blocks in all these descriptions, besides the prime binomial ideals (i.e. the toric ideals), are the zero dimensional binomial ideals. We illustrate in this section how the information is assembled to give bases of series solutions for Horn binomial D-modules. We refer the reader to [DMM06, Section t] for general definitions and results.

4.1. **Finding polynomial solutions of square binomial ideals.** Let M be a $m \times m$ square matrix such that each column has at least one negative and one positive entry. Such an M defines an infinite graph G_M with vertices in the points $u \in \mathbb{N}^m$ and edges (u, v) for all pairs such that $\pm(u - v)$ is a column of M. Consider the system of constant coefficient binomial differential operators $I(M) = \langle H_1, \ldots, H_m \rangle$, where the operator H_j is associated to the j-th column of M as follows:

$$H_j = \prod_{m_{ij}>0} \left(\frac{\partial}{\partial x_i}\right)^{m_{ij}} - \prod_{m_{ij}<0} \left(\frac{\partial}{\partial x_i}\right)^{-m_{ij}}, j = 1, \ldots, m.$$

The number of bounded connected components of the graph G_M corresponds precisely to the irreducible supports of polynomial solutions to $I(M) \subset \mathbb{C}[\partial]$. This number equals the dimension of the local quotient by $I(M)$ at the origin. Moreover, let P_M be the set of all the monomials occuring in a polynomial solution of $I(M)$. Then, P_M equals precisely all monomials in the complement of a monomial ideal $J(M)$, which is the biggest monomial ideal (in the commutative ring generated by the partial derivatives) which lies inside $I(M)$.

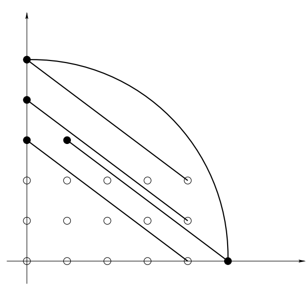
Consider for instance the following 2×2 matrix

$$M = \begin{pmatrix} 4 & 5 \\ -3 & -5 \end{pmatrix}$$

The system $I(M)$ is defined by the operators

$$\frac{\partial^4}{\partial x_1^4} - \frac{\partial^3}{\partial x_2^3}, \qquad \frac{\partial^5}{\partial x_1^5} - \frac{\partial^5}{\partial x_2^5}.$$

It has 15 linearly independent polynomial solutions, with the following minimal supports: The coefficients of these polynomial solutions are prescribed (up to con-



stant) by the recurrence imposed by the binomial operators of the system. For instance, the bounded connected component $\{(4,2),(0,5),(5,0),(1,3)\}$ gives rise to the quatrinomial solution of $I(M)$;

$$p = 5x_1^4x_2^2 + 2x_2^5 + 2x_1^5 + 40x_1x_2^3. \quad (4.1)$$

The set P_M has 22 monomials, precisely those monomials which do not belong to the monomial ideal $J(M) = \langle \frac{\partial^6}{\partial x_1^6}, \frac{\partial^6}{\partial x_2^6}, \frac{\partial^5}{\partial x_1 \partial x_2^4}, \frac{\partial^5}{\partial x_1^2 \partial x_2^3} \rangle$, which is the biggest monomial ideal inside $I(M)$. Note that in this case, the matrix M is invertible, so its "Gale dual" matrix A would be empty.

4.2. **Description of solutions to complete intersection Horn binomial D-modules.** Consider the matrices:

$$B = \begin{bmatrix} 4 & 5 & 0 \\ -3 & -5 & 0 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 7 & 6 & 5 & 10 & 0 \end{bmatrix}$$

We can associate a Horn binomial D-module $H(B,\beta)$ to the columns of B and a parameter vector $\beta \in \mathbb{C}^2$:

$$H(B,c) = \langle p_1(\partial), p_2(\partial), p_3(\partial), \sum_{i=1}^{5} \Theta_i - \beta_1, 7\Theta_1 + 6\Theta_2 + 5\Theta_3 + 10\Theta_4 - \beta_2 \rangle,$$

where p_1, p_2, p_3 denote the binomials:

$$p_1 = \partial_1^4 - \partial_2^3\partial_4,\ p_2 = \partial_1^5\partial_5 - \partial_2^5\partial_3,\ p_3 = \partial_3^2 - \partial_4\partial_5.$$

The variety $(p_1 = p_2 = p_3 = 0)$ has a primary component $I_{MN\hat{B}}$ "at infinity" associated to the block decomposition of the matrix B given by the following submatrices $M, N, \hat{B}$:

$$M = \begin{bmatrix} 4 & 5 \\ -3 & -5 \end{bmatrix}; \quad N = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \hat{B} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

Note that $\gcd\{2,-1,-1\} = 1$, so there is only one associated primary component coming from this decomposition, with associated prime $\langle \partial_1, \partial_2, \partial_3^3 - \partial_4\partial_5 \rangle$. Solutions to the binomial D-module associated to $I_{MN\hat{B}}$ can be constructed from the polynomial solutions to $I(M)$ and solutions to the $\hat{A}$-hypergeometric system associated to the matrix $\hat{A} = [\begin{smallmatrix} 1 & 1 & 1 \\ 5 & 10 & 0 \end{smallmatrix}]$, which, up to a rescaling in the homogeneities, equals the $\hat{A}'$-hypergeometric system associated to the matrix $\hat{A}' = [\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{smallmatrix}]$, whose columns span $\mathbb{Z}^2$.

For instance, consider the quatrinomial p in (4.1). Let f be any $\hat{A}$-hypergeometric function of $\hat{A}$-degree $\gamma := \beta - (3,20)$. Then, the following function is a solution of $H(B,\beta)$:

$$5x_1^4x_2^2(\partial_4^2\partial_5)(f) + 2x_2^5(\partial_4\partial_5)(f) + 2x_1^5(\partial_3\partial_4)(f) + 40x_1x_2^3(\partial_3)(f).$$

Note that all these solutions depend polynomially on x_1, x_2.

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