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A SURVEY ON HYPER-KÄHLER WITH TORSION GEOMETRY

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Abstract. Manifolds with special geometric structures play a prominent role in some branches of theoretical physics, such as string theory and supergravity. For instance, it is well known that supersymmetry requires target spaces to have certain special geometric properties. In many cases these requirements can be interpreted as restrictions on the holonomy group of the target space Riemannian metric. However, in some cases, they cannot be expressed in terms of the Riemannian holonomy group alone and give rise to new geometries previously unknown to mathematicians. An example of this situation is provided by hyper-Kähler with torsion (or HKT) metrics, a particular class of metrics which possess a compatible connection with torsion whose holonomy lies in $Sp(n)$.

A survey on recent results on HKT geometry is presented.

1. Introduction

A hyper-Hermitian structure on a $4n$-dimensional manifold M is given by a hypercomplex structure $\{J_\alpha\}$, $\alpha = 1, 2, 3$ (a triple of complex structures satisfying the imaginary quaternion relations) and a Riemannian metric g with respect to which J_α is skew-symmetric, for any α. The hyper-Hermitian manifold $(M, \{J_\alpha\}, g)$ is said to be *hyperkähler with torsion* (HKT for short) [19] if there exists a hyper-Hermitian connection ∇^B whose torsion tensor is a 3-form, that is,

$$\nabla^B g = 0, \quad \nabla^B J_\alpha = 0,\ \alpha = 1, 2, 3, \quad c(X, Y, Z) = g(X, T^B(Y, Z)) \text{ is a 3-form,} \tag{1}$$

where T^B is the torsion of ∇^B. It follows from the definition that the holonomy of ∇^B lies in $Sp(n)$. HKT geometry is a generalization of hyper-Kähler geometry. In fact, when the 3-form c associated to an HKT structure vanishes, then the connection ∇^B coincides with the Levi-Civita connection ∇^g and the metric g is hyper-Kähler.

An HKT structure is called *strong* or *weak* depending on whether the 3-form c is closed or not.

In [16] it was proved that the HKT condition is equivalent to

$$J_1 d\omega_1 = J_2 d\omega_2 = J_3 d\omega_3, \tag{2}$$

where ω_α are the associated Kähler forms

$$\omega_\alpha(X, Y) = g(J_\alpha X, Y), \quad \alpha = 1, 2, 3. \tag{3}$$

Also, a holomorphic characterization has been given in [16], where the authors proved that (2) is equivalent to

$$\partial_{J_1}(\omega_2 + i\,\omega_3) = 0. \tag{4}$$

More recently, in [24], it has been shown that if $(M, \{J_\alpha\}, g)$ is almost hyper-Hermitian, then condition (2) implies the integrability of $J_\alpha,\ \alpha = 1, 2, 3$.

Given a Hermitian manifold (M, J, g), there exists a unique Hermitian connection ∇ such that

$$\nabla g = 0, \quad \nabla J = 0, \quad c(X, Y, Z) = g(X, T(Y, Z)) \text{ is a 3-form,}$$

where T is the torsion of ∇. Such a connection is called in Hermitian geometry the Bismut connection [7] (KT connection in the physics literature). In the case of an HKT manifold, the three Bismut connections associated to the Hermitian structures (J_α, g) coincide and this connection is said to be an HKT connection. In contrast to the case of complex structures, not every hypercomplex structure on a manifold admits a compatible HKT metric. In fact, there exist hypercomplex manifolds of dimension ≥ 8 which do not admit any HKT metric compatible with the hypercomplex structure [10, 4]. These manifolds are nilmanifolds, that is, they are compact quotients of nilpotent Lie groups by co-compact discrete subgroups. We point out that in 4 dimensions every hyper-Hermitian metric is HKT. This fact, which has been first proved in [12], also follows from (4).

The study of hyper-Hermitian connections satisfying (1) is motivated by the fact that these structures appear in some branches of theoretical physics, such as string theory, in the context of certain supersymmetric sigma models [11, 19, 20, 28]. These connections are also present in supergravity theories. For instance, it has been shown in [13] that the geometry of the moduli space of a class of black holes in five dimensions is hyper-Kähler with torsion (see also [27]).

Many examples of HKT manifolds have been obtained. A twistor construction of HKT manifolds was proposed in [19] and HKT reduction has been studied in [15] in order to construct new examples. A large family of strong HKT manifolds is given by compact Lie groups with the hypercomplex structure constructed in [31] and independently by Joyce in [22], which was generalized in [26] to the case of homogeneous spaces. On the other hand, there are partial results concerning HKT structures on solvable Lie groups, where weak examples abound [4, 9]. Strong HKT structures on Lie groups with compact Levi factor have been obtained in [5]. Using results of [29], it was shown in [16] that $S^1 \times S^{4k-1}$ carries inhomogeneous weak HKT structures. Also, inhomogeneous examples of compact HKT manifolds which are not locally conformal hyper-Kähler can be obtained by considering the total space of a hyperholomorphic bundle over a compact HKT manifold [34].

Some geometrical and topological properties have been investigated. Differential geometric properties of HKT manifolds and their twistor spaces have been studied in [21] and it was proved in [1] that, in analogy to the hyper-Kähler case, locally any HKT metric admits an HKT potential. A simple characterization of HKT geometry in terms of the intrinsic torsion of the $Sp(n)Sp(1)$-structure was obtained in [24]. A version of Hodge theory for HKT manifolds has been given in [33] by exploiting a remarkable analogy between the de Rham complex of a Kähler manifold and the Dolbeault complex of an HKT manifold. More recently, in [35] balanced HKT metrics were studied, showing that the HKT metrics are precisely the quaternionic Calabi-Yau metrics defined in terms of the quaternionic Monge-Ampère equation. Moreover, by [35] a balanced HKT manifold has Obata connection with holonomy in $SL(n,\mathbb{H})$.

2. Hodge theory on HKT manifolds

We review in this section some fundamental facts from the theory developed by Verbitsky in [33] that will be relevant to explain the main result obtained in [4] (see §4).

We recall first the properties of the de Rham algebra of a Kähler manifold (see, for instance, [17]). Let (M,J,g) be a Hermitian manifold, that is, J is a complex structure on M and g is a Riemannian metric such that $g(JX,JY)=g(X,Y)$ for all vector fields X,Y on M.

The Kähler form ω is defined as in (3) and (M,J,g) is Kähler if and only if ω is closed. J acts on differential forms as

$$J(\eta_1\wedge\cdots\wedge\eta_r)=J(\eta_1)\wedge\cdots\wedge J(\eta_r),\quad \eta_k\in\Lambda^1(M),$$

with $J(\eta)(X)=-\eta(JX)$, $\eta\in\Lambda^1(M)$, $X\in\mathfrak{X}(M)$. Let d^c be the following differential operator acting on forms:

$$d^c=-JdJ,$$

so that

$$\partial=\frac{1}{2}(d+id^c),\qquad \overline{\partial}=\frac{1}{2}(d-id^c).$$

Using the Kähler form ω, it is possible to define the following linear operators:

$$L_\omega\eta=\omega\wedge\eta,\qquad \Lambda_\omega=*L_\omega*,\qquad H_\omega=[L_\omega,\Lambda_\omega],\tag{5}$$

where $*$ is the Hodge-star operator. When (M,J,g) is Kähler, the operators

$$L_\omega,\quad \Lambda_\omega,\quad H_\omega,\quad d,\quad d^c,\tag{6}$$

satisfy the Kodaira relations (see [17]). For instance, one has

$$[\Lambda_\omega,d]=*d^c*,\qquad [\Lambda_\omega,d^c]=-*d*,$$

and, moreover,

$$[H_\omega,L_\omega]=-2L_\omega,\qquad [H_\omega,\Lambda_\omega]=2\Lambda_\omega,$$

that is, when (M,J,g) is Kähler, L_ω, Λ_ω, H_ω induce an action of $\mathfrak{sl}(2,\mathbb{C})$ on the complex cohomology of M.

There are some cohomological restrictions imposed by the existence of a Kähler metric on a compact manifold. One necessary condition is that the odd Betti numbers must be even. The following classical result gives another cohomological condition satisfied by compact Kähler manifolds.

Hard Lefschetz Theorem. (See [17]) *Let M^{2n} be a compact Kähler manifold with Kähler form ω. Then, for any $j = 0, 1, \dots, n$, the map*

$$L_\omega{}^j : H^{n-j}(M) \to H^{n+j}(M)$$

is an isomorphism, where $L_\omega([\gamma]) = [\omega \wedge \gamma]$.

Given an HKT manifold $(M, \{J_\alpha\}, g)$, it is shown in [33] that the Dolbeault differential graded algebra $(\Lambda^{*,0}(M, J_1), \partial)$ is an analogue of the de Rham algebra of a Kähler manifold. The roles of the de Rham differential d and of the Kähler form ω are played by ∂ and by the $(2,0)$-form $\Omega \in \Lambda^{2,0}(M, J_1)$ defined in (7) below, respectively. One can associate with Ω three operators $L_\Omega,\ \Lambda_\Omega,\ H_\Omega$ as in (5), thereby obtaining an action of $\mathfrak{sl}(2,\mathbb{C})$ on $\Lambda^{*,0}(M, J_1)$.

Let $(\{J_\alpha\}, g)$ be a hyper-Hermitian structure on a $4n$-dimensional manifold M and consider the following $(2,0)$-form with respect to J_1:

$$\Omega := \frac{1}{2}(\omega_2 + i\omega_3). \tag{7}$$

Using Ω it is possible to construct three linear operators $L_\Omega,\ \Lambda_\Omega,\ H_\Omega$ as in (5). We denote by $\Lambda^{p,q}(M, J_1)$ the forms of type (p,q) with respect to J_1. Let

$$\partial : \Lambda^{p,q}(M, J_1) \to \Lambda^{p+1,q}(M, J_1) \tag{8}$$

be the Dolbeault operator with respect to the complex structure J_1 and

$$\partial_{J_2} : \Lambda^{p,q}(M, J_1) \to \Lambda^{p+1,q}(M, J_1), \qquad \partial_{J_2} = -J_2 \overline{\partial} J_2. \tag{9}$$

It was shown in [33] that on an HKT manifold, ∂_{J_2} plays the role of d^c on a Kähler manifold. In fact, it follows from [33, Corollary 7.2] that on an HKT manifold, the operatots

$$L_\Omega, \quad \Lambda_\Omega, \quad H_\Omega, \quad \partial, \quad \partial_{J_2}, \tag{10}$$

satisfy the same identities which hold for the operators (6).

The next result, which is a particular case of [33, Theorem 10.2] and is one of the main steps in the proof of Theorem 4.2, is an analogue of the Hard Lefschetz Theorem for the Dolbeault cohomology of HKT manifolds.

Theorem 2.1 ([33]). *Let $(M, \{J_\alpha\}, g)$ be a compact $4n$-dimensional HKT manifold with $(2,0)$-form Ω as in (7) and assume that the canonical bundle $\Lambda^{2n,0}(M, J_1)$ is holomorphically trivial. Then, for any $j = 0, 1, \dots, 2n$:*

$$L_\Omega^j : H_\partial^{2n-j,0}(N, J_1) \to H_\partial^{2n+j,0}(N, J_1)$$

is an isomorphism, where $L_\Omega([\gamma]) = [\Omega \wedge \gamma]$.

3. HKT structures on Lie groups

An HKT structure $(\{J_\alpha\}, g)$ on a Lie group G is called *left-invariant* when left translations L_x, $x \in G$, are isometries and holomorphic maps with respect to J_α for any α. In this case, it has been shown in [9] that the HKT condition is equivalent to:

$$\begin{aligned} &g([J_1X, J_1Y], Z) + g([J_1Y, J_1Z], X) + g([J_1Z, J_1X], Y) \\ &= g([J_2X, J_2Y], Z) + g([J_2Y, J_2Z], X) + g([J_2Z, J_2X], Y) \\ &= g([J_3X, J_3Y], Z) + g([J_3Y, J_3Z], X) + g([J_3Z, J_3X], Y). \end{aligned} \tag{11}$$

for any $X, Y, Z \in \mathfrak{g}$, the Lie algebra of G.

A large family of HKT manifolds is provided by $G = U(1)^k \times K$ with the hypercomplex structure obtained in [31] (see also [22]), where K is a compact semisimple Lie group. In this case, the restriction of the HKT metric to K is the opposite of the Killing-Cartan form and the Bismut connection is the canonical affine connection ∇ on G defined by

$$\nabla_X Y = 0, \qquad X, Y \in \mathfrak{g}. \tag{12}$$

Conversely, if (12) is the Bismut connection of some left invariant KT metric, then G is isomorphic to a direct product of an abelian Lie group by a compact semisimple Lie group (Corollary 3.1). This fact is a consequence of a classical result due to Milnor [25].

Lemma 3.1. *Let G be a connected Lie group with a left invariant metric g. Then*

$$\tau(X, Y, Z) = g(X, [Y, Z]), \qquad X, Y, Z \in \mathfrak{g}, \tag{13}$$

is a 3-form if and only if G is isomorphic to $\mathbb{R}^k \times K$, where K is a compact connected Lie group.

Proof. We observe that τ is a 3-form if and only if ad_Y is skew-symmetric for any $Y \in \mathfrak{g}$. It follows from [25, Lemmas 7.2 and 7.5] that this occurs if and only if G is as in the statement. □

Corollary 3.1. *Let G be a connected Lie group and ∇ the canonical connection* (12) *on G. If ∇ is the KT connection associated to some left invariant Hermitian metric g on G, then G is isomorphic to $\mathbb{R}^k \times K$, where K is a compact connected Lie group.*

Proof. We observe that the torsion of ∇ is $T(U, V) = -[U, V]$, $U, V \in \mathfrak{g}$. Therefore, if ∇ is the Bismut connection of g, we must have that $c = -\tau$ is a 3-form and the corollary follows from Lemma 3.1. □

As a consequence of the above corollary, one has that if the canonical connection ∇ defined in (12) is the Bismut connection of an HKT structure on G, then G is as in Corollary 3.1.

In [5] strong HKT structures have been constructed on non-semisimple Lie groups starting with a compact Lie group K acting on $\mathbb{H}^n$ by quaternionic linear maps which are isometries of the Euclidean metric.

A left invariant complex (resp. hypercomplex) structure on G is called *abelian* (see [2]) when $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{g}$ (resp. $[J_\alpha X, J_\alpha Y] = [X, Y]$, $\alpha = 1, 2, 3$). Observe that in this case (11) is automatically satisfied for any hyper-Hermitian metric g, that is, given an abelian hypercomplex structure, any hyper-Hermitian metric is HKT. Moreover, it was shown in [9, Proposition 2.1] that left-invariant HKT structures arising from abelian hypercomplex structures are always weak.

It was shown in [9, Theorem 3.1] that for 2-step nilpotent Lie groups every left-invariant HKT structure arises from an abelian hypercomplex structure. We proved in [4] that this theorem still holds for k-step nilpotent Lie groups admitting lattices, for arbitrary k (see Theorem 4.2).

We point out that the left-invariant complex structure J on G gives rise to a decomposition

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0}$, $\mathfrak{g}^{0,1}$ are the eigenspaces of the induced complex structure on $\mathfrak{g}$. It turns out that J is abelian if and only if $\mathfrak{g}^{1,0}$ is an abelian subalgebra of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

4. HKT structures on nilmanifolds

4.1. **Generalities on nilmanifolds.** A nilmanifold (see[23]) is a quotient $\Gamma\backslash G$ of a simply connected nilpotent Lie group by a lattice Γ (a discrete co-compact subgroup). It is well known that

- G admits lattices if and only if $\mathfrak{g}$ has a rational form.

Moreover, there is a one-to-one correspondence:

$$\{ \text{ lattices in } G\} \longleftrightarrow \{ \text{ rational forms of } \mathfrak{g}\}.$$

Let $N = \Gamma\backslash G$ be a nilmanifold and assume that G is equipped with a left-invariant complex structure J. Then J induces a complex structure on N. A complex structure J on N is called *abelian* if it is induced from a left-invariant abelian complex structure on G.

The first example of symplectic non-Kähler manifold was described by Thurston [32]: it is the nilmanifold $S^1 \times \Gamma_1\backslash H_3$, where

$$H_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

is the 3-dimensional Heisenberg group and Γ_1 is the subgroup of matrices in H_3 with integer entries.

For each $k \in \mathbb{N}$ one can define a lattice Γ_k in H_3 as follows (compare with [14]):

$$\Gamma_k = \left\{ \begin{pmatrix} 1 & a & c/k \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

It follows that:

- $\Gamma_i \subset \Gamma_j$ if and only if i divides j.
- $\Gamma_1 \backslash H_3$ covers $\Gamma_k \backslash H_3$ for any $k > 1$.
- $\Gamma_k / [\Gamma_k, \Gamma_k] \cong \mathbb{Z}^2 \oplus \mathbb{Z}_k$.

The nilmanifolds $S^1 \times \Gamma_k \backslash H_3$ have fundamental group isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}_k$, in particular, they are not homeomorphic. These are examples of symplectic non-Kähler manifolds. In the 80's, many authors (Abbena, Cordero, Fernández, Gray, de León, among others) obtained families of symplectic non-Kähler manifolds as generalizations of the previous example. Later, in 1988, the following remarkable theorem was proved by Benson-Gordon (see also [18], where the author showed that a minimal model of a nilmanifold is formal if and only if it is a torus):

Theorem 4.1 ([6, Theorem A]). *If $N = \Gamma \backslash G$ is a Kähler nilmanifold, then G is abelian and N is diffeomorphic to a torus.*

The main ingredients in the proof of the above theorem are:

- The de Rham cohomology of $\Gamma \backslash G$ can be identified with the Lie algebra cohomology of $\mathfrak{g}$ due to a result of Nomizu.
- It is proved that if $\mathfrak{g}$ is nilpotent, the Hard Lefschetz Theorem implies that $\mathfrak{g}$ is abelian.

More precisely, Benson-Gordon show that if $\mathfrak{g}$ is non-abelian nilpotent, then the map

$$L_\omega{}^{n-1} : H^1(\mathfrak{g}) \to H^{2n-1}(\mathfrak{g})$$

is not surjective ([6, Lemma 2.11]), which contradicts Hard Lefschetz.

For the case of HKT nilmanifolds, we proved in [4] the following analogue of Theorem 4.1:

Theorem 4.2 ([4]). *Let $N = \Gamma \backslash G$ be a $4n$-dimensional nilmanifold endowed with an HKT structure $(\{J_\alpha\}, g)$ induced by a left-invariant HKT structure on G. Then $\{J_\alpha\}$ is abelian.*

Sketch of proof. The aim is to show that $\mathfrak{g}^{1,0}$ is abelian (equivalently, J_1 is abelian). We observe that the canonical bundle of (N, J_1) is trivial [8], therefore Theorem 2.1 applies.

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\Lambda^{1,0}(\mathfrak{g}_{\mathbb{C}}, J_1) & \xrightarrow{\partial} & \Lambda^{2,0}(\mathfrak{g}_{\mathbb{C}}, J_1) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \Lambda^{2n,0}(\mathfrak{g}_{\mathbb{C}}, J_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Lambda^1\mathfrak{g}^{1,0} & \xrightarrow{d} & \Lambda^2\mathfrak{g}^{1,0} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda^{2n}\mathfrak{g}^{1,0}
\end{array}$$

where the vertical arrows are the natural identifications. The Lie algebra $\mathfrak{g}^{1,0}$ is nilpotent. If we assume that $\mathfrak{g}^{1,0}$ is not abelian, one can apply the same argument in [6, Lemma 2.11] to the bottom row of the previous diagram to obtain that

$$L_\Omega^{n-1} : H^1(\mathfrak{g}^{1,0}) \to H^{2n-1}(\mathfrak{g}^{1,0}) \text{ is not surjective.}$$

Equivalently,

$$L_\Omega^{n-1} : H_\partial^{1,0}(N, J_1) \to H_\partial^{2n-1,0}(N, J_1) \text{ is not surjective,}$$

which contradicts Theorem 2.1.

□

The following question was posed in [16]:

- Given a compact manifold M with a hypercomplex structure, is it always possible to find a compatible HKT metric?

A negative answer was given in [10] by exhibiting 2-step nilmanifolds with non-abelian hypercomplex structures. In view of Theorem 4.2, any non-abelian hypercomplex structure on a k-step nilmanifold admits no compatible HKT metric. Therefore, Theorem 4.2 provides a useful tool for obtaining many examples where the answer to the above question is negative. To illustrate this situation, we exhibit next a family of hypercomplex k-step nilmanifolds, for arbitrary k, admitting no compatible HKT metric (see [4]).

4.2. **A family of examples.** Let A be a finite dimensional associative algebra and let $\mathfrak{aff}(A)$ be the Lie algebra $A \oplus A$ with Lie bracket:

$$[(a, b), (a', b')] = (aa' - a'a, ab' - a'b), \qquad a, b, a', b' \in A.$$

This class of Lie algebras has first been considered in [3].

We observe that:

- $\mathfrak{aff}(A)$ is a nilpotent Lie algebra if and only if A is nilpotent as an associative algebra.

Let J_1 be the endomorphism of $\mathfrak{aff}(A)$ defined by:

$$J_1(a, b) = (b, -a), \qquad a, b \in A.$$

It has been shown in [3] that J_1 is a complex structure on $\mathfrak{aff}(A)$. If, moreover, A is a complex associative algebra, we can define another complex structure J_2 on $\mathfrak{aff}(A)$ by:

$$J_2(a, b) = (-ia, ib), \qquad a, b \in A.$$

Since $J_1J_2 = -J_2J_1$, setting $J_3 = J_1J_2$ we obtain a hypercomplex structure on $\mathfrak{aff}(A)$.

Remark 4.1. *$\{J_\alpha\}$ is abelian if and only if A is conmutative.*

Let T_k be the algebra of $(k+1)\times(k+1)$ strictly upper triangular matrices with complex entries and consider the simply connected Lie group $\mathrm{Aff}(T_k)$ with Lie algebra $\mathfrak{aff}(T_k)$, which is k-step nilpotent. The structure constants of $\mathfrak{aff}(T_k)$ with respect to the standard basis are integers, hence $\mathrm{Aff}(T_k)$ admits a lattice Γ_k and we obtain:

- The hypercomplex k-step nilmanifold $N_k = \Gamma_k\backslash\mathrm{Aff}(T_k)$ does not admit a compatible HKT metric.

Remark 4.2. *We point out that for $k \geq 4$ the Lie algebra $\mathfrak{aff}(T_k)$ is not two-step solvable, hence it does not admit abelian hypercomplex structures (see* [30]*). Therefore, Theorem implies that any left-invariant hypercomplex structure on Aff(T_k), $k \geq 4$, induces on the nilmanifold N_k a hypercomplex structure admitting no compatible HKT metric.*

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