REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 49, Número 2, 2008, Páginas 133–142

THE HILBERT TRANSFORM AND SCATTERING

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To the memory of Mischa Cotlar, my teacher and my friend

ABSTRACT. Through the prism of abstract scattering, and the invariant forms acting in them, we discuss the Hilbert transform in weighted L^p spaces in one and several dimensions.

1. Introduction

It all started with the study of the Hilbert transform in terms of scattering...

In the late seventies, Cotlar and I began a systematic study of algebraic scattering systems, and the invariant forms acting on them.

In the late eighties we started working in multidimensional scattering–although many did not consider such approach as relevant.

In the late nineties our outlook was finally vindicated. Multidimensional abstract scattering systems appeared as couterparts of the input-output conservative linear systems.

2. The Hilbert transform

The Hilbert transform operator

$$H : f \mapsto Hf = f * k$$

is given by convolution with the singular kernel

$$k(x) = p.v. \frac{1}{x}$$

for $x \in \mathbb{R}$. (Analogues in $\mathbb{T}$ and $\mathbb{R}^n, n > 1$.)

The basic result of Marcel Riesz (1927) is

$$H \text{ is bounded on } L^p,\ 1 < p < \infty.$$

Similar boundedness properties are valid in the "weighted" cases, both for H and for the iterated $H = H_1 H_2 ... H_n$,

$$H : L^p(\mu) \to L^p(\nu),\ 1 < p < \infty$$

2000 *Mathematics Subject Classification.* Primary: 42B30, Secondary: 47B35, 32A37.

where μ, ν are general measures.

3. The scattering property of the analytic projector

Given $f \in L^2$, we can decompose f as $f_1 + f_2$, where f_1 is analytic and f_2 is antianalytic.

Under this decomposition the Hilbert transform can be written as

$$Hf = -i\, f_1 + i\, f_2.$$

The **analytic projector** P, associated with the Hilbert tranform operator H, is defined as

$$Pf = P(f_1 + f_2) = f_1, \qquad P = \frac{1}{2}\,(I + iH).$$

The crucial observation is that P supports the **shift operator** $S : f(x) \mapsto e^{ix}\, f(x)$.

Then, the **range** of P is the set W_1 of analytic functions, and its **kernel** is the set W_2 of antianalytic functions:

$$S\, W_1 \subset W_1 \quad \text{and} \quad S^{-1}\, W_2 \subset W_2.$$

The "scattering property" of the 1-dimensional Hilbert transform provides the framework for a theory of invariant forms in scattering systems, leading to two-weight L^2-boundedness results for H.

The scattering properties are also essential to providing the two-weight L^2-boundedness of the **product** Hilbert transform in product spaces, where the analytic projectors supporting the n-dimensional shifts are at the basis of the lifting theorems in abstract scattering structures.

Notice that this fact, valid for the product Hilbert transforms, is not valid for the n-dimensional Calderón–Zygmund singular integrals, which do not share the scattering property.

4. H is bounded on L^2

In fact, H is an **isometry** on L^2, since

$$\|Hf\|_2 = \|f\|_2$$

and this follows easily from the Plancherel Theorem for the Fourier transform.

The result can also be obtained through the Cotlar Lemma on Almost Orthogonality, which extends the Hilbert transforms into ergodic systems.

These are two different ways to deal with the boundedness of H in L^2. The same happens for $p \neq 2$.

5. Two ways for $H : L^1$ to L^p, $1 < p < \infty$

Start checking that H is **weakly bounded** in L^1, and apply the Marcinkiewicz **Interpolation** Theorem between $p = 1$ and $p = 2$, and then, by duality, pass from $p = 2$ to $p < \infty$.

The "Magic Identity" for H, given by

$$(Hf)^2 = f^2 + 2\,H\,(f.Hf) \quad (*)$$

is valid for all functions f, smooth and with compact support.

Using **extrapolation**, since for $f \in L^2$, $(*)$ implies $Hf \in L^2$, then $f \in L^4$ implies $Hf \in L^4$, and $f \in L^{2^k}$ implies $Hf \in L^{2^k}$, $\forall\, k \geq 1$.

The boundedness of H in $L^{p'}$, $1/p + 1/p' = 1$, $p = 2^k$, follows by duality, and interpolation gives the boundedness of H in L^p, $1 < p < \infty$.

By polarization, the Magic Identity for an operator T becomes

$$T(f.Tg + Tf.g) = Tf.Tg - f.g.$$

The Magic Identity, and similar ones have been used extensively in harmonic analysis, in particular by Coifman and Meyer. Cotlar and Sadosky, and Rubio de Francia, used the Identity in dealing with the weighted Hilbert transform in Banach lattices.

Gian-Carlo Rota used three different "magic indentities" in his work in combinatorics, and his school encompassed all particular cases in a general inequality.

Gohberg and Krein showed that the polarized Magic Inequality holds in the space $\mathcal{S}^2$, and deduced the theorem of Krein and Macaev in a way similar to the passage from L^2 to L^p described before.

The "magic indentities" hold in a variety of non-commutative situations, starting with the non-commutative Hilbert transforms in von Neumann algebras, and that theory has been developed in the last years.

6. H is bounded in $L^2(\omega)$, $0 \leq \omega \in L^1$

6.1. **Helson-Szegő theorem (1960).**

$$\omega = e^{u\,+\,H\,v},$$

where u, v are real-valued bounded functions, such that $\|v\|_\infty < \pi/2$

is equivalent to

$$log\,\omega = u + Hv \in BMO$$

(with a special BMO norm).

6.2. **Hunt-Muckenhoupt-Wheeden theorem (1973).**

$$\omega \in A_2$$

is equivalent to

$$\left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \frac{1}{\omega}\right) \leq C, \quad \forall I \quad \text{interval}$$

6.3. **Remark.** Take note that, although both conditions are necessary and sufficient, the first one is good to **produce** such weights, while the second one is good at **checking** them.

7. Definition of u-boundedness

An operator T acting in a Banach lattice X, $T : X \to L^0(\Omega)$ is u-bounded if

$$\forall f \in X,\ \|f\| \leq 1,\ \exists g \in X,\ g \geq |f|,\ \|g\| + \|Tg\| \leq C.$$

The u-boundedness of operators is considerably weaker than boundedness. For example,

$$T \text{ is } u\text{-bounded on } L^\infty \text{ if and only if } T\,1 \in L^\infty$$

.

Now we can translate another equivalence for the Helson-Szegő theorem for p=2:

There exist $w \sim \omega$, real-valued functions, such that

$$|H\,w(x)| \leq C\,w(x)\ a.e.$$

(Here $w \sim \omega$ means $c\,\omega(t) \leq w(t) \leq C\,\omega(t)$, $\forall\, t$)

is equivalent to

$$T_\omega : f \mapsto \omega^{-1}\,H(\omega\, f)$$

is u-bounded in L^∞.

8. H is bounded in $L^p(\mathbb{T};\omega)$, $1 < p < \infty$

8.1. **Hunt-Muckenhoupt-Wheeden Theorem (1973).**

$$\omega \in A_p$$

is equivalent to

$$(\frac{1}{|I|}\int_I \omega)(\frac{1}{|I|}\int_I \omega^{-1/(p-1)})^{p-1} \leq C\,,\ \forall\, I \text{ interval}$$

8.2. **Cotlar-Sadosky Theorem (1982).**

$$\omega \in A_p,$$

defined by

$$T_\omega = \omega^{-2/p}\, H(\omega^{2/p}\, f), \quad p \geq 2,$$
$$T_\omega = \omega^{2/p}\, H(\omega^{-2/p}\, f), \quad p < 2,$$

is u-bounded in L^{p^*} where $1/p^* = |\,1 - 2/p\,|$.

9. H is bounded in weighted $L^2(\mathbb{T}^2;\omega)$

The following are equivalent:

1. The double Hilbert transform $H = H_1\, H_2$ is bounded in $L^2(\mathbb{T}^2;\omega)$
2. $$\omega = e^{u_1 + H_1\, v_1} = e^{u_2 + H_2\, v_2},\ u_1,\ u_2,\ v_1,\ v_2,$$

are real-valued bounded functions, $\|v_i\|_\infty < \pi/2$, $i = 1, 2$

3. $\log \omega \in bmo$ (with a special bmo norm)
4. $\exists\, w_1, w_2,\ w_1 \sim \omega \sim w_2$, such that
$$|H_1\, w_1(x)| \leq C\, w_1(x),\ |H_2\, w_2(x)| \leq C\, w_2(x)\ a.e.$$

5. $$T_1 : f \mapsto \omega^{-1}\, H_1(\omega\, f)$$

and

$$T_2 : f \mapsto \omega^{-1}\, H_2(\omega\, f)$$

are simultaneously u-bounded in $L^\infty(\mathbb{T}^2)$

6.

$$\omega \in A_2^*$$

10. H bounded in weighted $L^p(\mathbb{T}^2;\omega)$, $1 < p < \infty$

The following are equivalent:
1. The double Hilbert transform $H = H_1\, H_2$ is bounded in $L^p(\mathbb{T}^2;\omega)$

2. T_1 and T_2 are simultaneously u-bounded in L^{p^*}, where for $i = 1, 2$,

$$T_i : f \mapsto \omega^{-2/p}\, H_i(\omega^{2/p}\, f), \text{ if } 2 \le p < \infty$$

$$T_i : f \mapsto \omega^{2/p}\, H_i(\omega^{-2/p}\, f), \text{ if } 2 \le p < \infty$$

are simultaneously u-bounded in $L^{p^*}(\mathbb{T}^2)$.

3.

$$\omega \in A_p^*$$

11. The Lifting Theorem for invariant forms in algebraic scattering structures

Let V be a vector space, and σ be a linear isomorphism in V.

The subspaces of V, W_+, and W_-, are linear subspaces satisfying

$$\sigma\, W_+ \subset W_+, \quad \sigma^{-1}\, W_- \subset W_-.$$

Let $B_1, B_2 : V \times V \to \mathbb{C}$, be positive σ-invariant forms, such that

$\forall\, B_0 : W_+ \times W_- \to \mathbb{C} \ni \forall\,(x, y) \in W_+ \times W_-$

$$B_0(\sigma\, x, y) = B_0(x, \sigma^{-1}\, y),$$

$$|B_0(x, y)| \le\; B_1(x, x)^{1/2}\, B_2(y, y)^{1/2}$$

Then,

$$\exists\, B' : V \times V \to \mathbb{C} \ni \forall\,(x, y) \in V \times V$$

such that

$$B'(\sigma\, x, \sigma\, y) = B'(x,\, y),$$

or, equivalently,

$$B'(\sigma\, x, y) = B'(x, \sigma^{-1}\, y).$$

Furthermore,

$$|B'(x, y)| \le\; B_1(x, x)^{1/2}\, B_2(y, y)^{1/2}$$

such that

$$B'(x, y) = B_0(x, y),\, \forall (x, y) \in W_+ \times W_-.$$

12. The Lifting Theorem for trigonometric polynomials

Assume now that $V = \mathcal{P}$ is the set of trigonometric polynomials on $\mathbb{T}$, and $\mathcal{P}_1$, $\mathcal{P}_2$ are the sets of analytic and antianalytic polynomials in $\mathcal{P}$, where $\sigma = S$ is the shift operator.

The Herglotz-Bochner theorem then translates to

B is positive and S-invariant in $\mathcal{P}\times\mathcal{P}$ if and only if it exists a measure $\mu \geq 0$, such that

$$B(f,g) = \int f\,\bar{g}\,d\mu \quad \forall f,\, g \in \mathcal{P}.$$

Since $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, the domain of B splits in four pieces $\mathcal{P}_i \times \mathcal{P}_j$ for $i,j = 1,2$.

A weaker concept of S-invariance is

$$B(Sf,\, Sg) = B(f,\, g),$$

for all $(f,\, g)$ in each quarter $\mathcal{P}_i \times \mathcal{P}_j$, $i, j = 1, 2$.

Then the Lifting theorem asserts that B is positive in $\mathcal{P}\times\mathcal{P}$ and S-invariant in each quarter if and only if there exists $\mu = (\mu_{ij}) \geq 0$ such that, for all $f_1,\, f_2 \in \mathcal{P}_\infty \times \mathcal{P}_\in$,

$$B(f_1 + f_2,\, g_1 + g_2) = \sum_{i,j=1,2} \int f_i\,\bar{g}_j\,d\mu_{ij}.$$

Here $(\mu_{ij}) \geq 0$ means that the (complex) measures satisfy $\mu_{11} \geq 0$, $\mu_{22} \geq 0$, $\mu_{21} = \overline{\mu_{12}}$ and

$$|\mu_{12}(D)|^2 \leq \mu_{11}(D)\,\mu_{22}(D), \quad \forall\, D \subset \mathbb{T}.$$

13. The Lifting Theorem in $\mathcal{P}\times\mathcal{P}$

Let B be an S-invariant in $\mathcal{P}\times\mathcal{P}$.

Then

$$B \geq 0 \Longleftrightarrow \mu \geq 0,$$

but, when B is S-invariant in each $\mathcal{P}_i \times \mathcal{P}_j$,

$$B \geq 0 \iff \sum_{i,j} \int f_i \bar{f}_j\,d\mu_{ij} \geq 0$$

only for $f_1 \in \mathcal{P}_1$, $f_2 \in \mathcal{P}_2$, which is far less than

$$\mu \geq 0 \iff \sum_{i,j} \int f_i \bar{f}_j\,d\mu_{ij} \geq 0$$

for $f_1,\, f_2 \in \mathcal{P}$.

Let $B_1 = B|\mathcal{P}_1 \times \mathcal{P}_1$, $B_2 = B|\mathcal{P}_2 \times \mathcal{P}_2$, $B_0 = B|\mathcal{P}_1 \times \mathcal{P}_2$.

Then B_1, B_2 are positive, while B_0 is not positive but bounded,

$$|B_0(f_1, f_2)| \leq B_1(f_1, f_1)^{1/2} B_2(f_2, f_2)^{1/2}.$$

Since $\mu_{11} \geq 0$, $\mu_{22} \geq 0$,

$$|B_0(f_1, f_2)| \leq \|f_1\|_{L^2(\mu_{11})} \|f_2\|_{L^2(\mu_{22})}.$$

By the definition of B_0 it follows

$$B_0(f_1, f_2) = \int f_1 \bar{f}_2 \, d\mu_{12}$$

only for $f_1 \in \mathcal{P}_1$, $f_2 \in \mathcal{P}_2$.

And since

$$|\mu_{12}(D)|^2 \leq \mu_{11}(D) \, \mu_{22}(D), \quad \forall D \subset \mathbb{T},$$

defining

$$B'(f_1, f_2) := \int f_1 \bar{f}_2 \, d\mu_{12}, \forall f_1, f_2 \in \mathcal{P}$$

it is $\|B'\| = \|B_0\|$.

13.1. H **is bounded in** $L^2(\mathbb{T}; \nu, \mu)$**.** Let the operator H be defined in $\mathcal{P}$ as

$$H(f_1 + f_2) = -i \, f_1 + i \, f_2.$$

The two-weight inequality for H is

$$\int_{\mathbb{T}} |Hf|^2 d\mu \leq M^2 \int_{\mathbb{T}} |f|^2 d\nu$$

or, equivalently,

$$(\star) = \sum_{i,j=1,2} \int_{\mathbb{T}} f_i \bar{f}_j \, d\rho_{ij} \geq 0, \quad f_1 \in \mathcal{P}_1, \, f_2 \in \mathcal{P}_2$$

where

$$\rho_{11} = \rho_{22} = M^2\nu - \mu, \; \rho_{12} = \overline{\rho_{21}} = M^2\nu + \mu.$$

Defining by $B(f, g) = B(f_1 + f_2, \, g_1 + g_2) = (\star)$, B is S-invariant in each quarter and is also non-negative.

By the Lifting theorem, there exists (μ_{ij}), $i, j = 1, 2$, such that

$$\hat{\rho}_{ii}(n) = \hat{\mu}_{ii}(n), \forall n \in \mathbb{Z},$$

while $\hat{\rho}_{12}(n) = \hat{\mu}_{12}(n)$, is only for $n < 0$.

By the F. and M. Riesz theorem,

$$\mu_{11} = \mu_{22} = M^2\nu - \mu, \; \mu_{12} = \overline{\mu_{21}} = M^2\nu + \mu - h,$$

and $h \in H^1(\mathbb{T})$.

Then, the necessary and sufficient condition for boundedness in $L^2(\mathbb{T};\nu,\,\mu)$, with norm M, is that for all $0 \subset \mathbb{T}$ Borel sets,

$$|(M^2\nu+\mu)(D) - \int_D h\,dt\,| \leq \,(M^2\nu-\mu)(D)$$

where $h \in H^1(\mathbb{T})$.

In particular, μ is an absolutely continuous measure, $d\mu = w\,dt$ for some $0 \leq w \in L^1$.

When $\mu = \nu = \omega\,dt$ we return to the previous case:

The Hilbert transform H is a bounded operator in $L^2(\omega)$ with norm M

if and only if

$$|(M^2+1)\,\omega(t) - h(t)| \leq (M^2-1)\,\omega(t),\ a.e.\,\mathbb{T},$$

for $h \in H^1(\mathbb{T})$, which is the source to all the equivalences we mentioned above.

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Recibido: 8 de abril de 2008
Aceptado: 20 de diciembre de 2008