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A COMPACT TRACE THEOREM FOR DOMAINS WITH EXTERNAL CUSPS

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ABSTRACT. This paper deals with the compact trace theorem in domains $\Omega \subset \mathbb{R}^3$ with external cusps. We show that if the power sharpness of the cusp is bellow a critical exponent, then the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ exists and it is compact.

1. INTRODUCTION

Up to now, Lipschitz domains make up the most general class of domains where a rich function theory can be developed. However, domains with external cusps could appear at several branches of mathematics and applications. In obstacle problems, for example, the free boundary with external cusps may enter into corner points of the fixed boundary (*e.g.* [8]). Therefore, it is important to know what kind of results in the theory of Sobolev spaces remain valid in cuspidal domains.

Key tools in harmonic analysis and numerical application are the Rellich's theorem and the compact trace theorem. This paper deals with the compact trace theorem in domains $\Omega \subset \mathbb{R}^3$ with external cusps. We show that if the power sharpness α of the cusp is bellow a critical exponent α_c, then the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ exists and it is compact. For cuspidal models in $\mathbb{R}^2$, $\alpha_c = 2$ (see [1]).

Several classical results of harmonic analysis can be extended in this context, to begin with the divergence theorem, for example, or the characterization of the spaces $H^{1/2}(\partial\Omega)$ via the Steklov eigenfunction expansions [3, 4]. In several branch of harmonic analysis, the compacity of the operators $H^1(\Omega) \subset L^2(\Omega)$ and $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ are key tools.

It is worth to remark here that certain classical counterexamples of analysis in cuspidal domains, like those of Friedrichs related to Korn inequality [5], have cusps of power sharpness equal to the critical exponent.

In [7] the authors characterize the traces of the Sobolev spaces $W^{1,p}(\Omega), 1 \leq p < \infty$, by using some weighted norm on the boundary. In [1] a different kind of trace result was obtained by introducing a weighted Sobolev space in Ω , such that the restriction to the boundary of functions in that space are in $L^p(\partial\Omega)$. We extended the arguments in this work to domains in $\mathbb{R}^3$ with some slight modification in the

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trace estimate, which is more useful in order to prove the compacity of the trace operator.

We shall consider a family of *standard models* or especial domains Ω in $\mathbb{R}^3$ which have cusps of power sharpness $\alpha > 1$. We follow the standard notation for the Sobolev spaces H^s and Sobolev norms. For simplicity, we consider only the case $p = 2$, and we do not consider other Sobolev spaces $W^{1,p}$.

1.1. $\mathcal{Q}_{k,\alpha}$ MODELS.

Definition 1. *Let $k = 1, 2$. We shall say that Ω is a $\mathcal{Q}_{k,\alpha}$ cusp if*

$$\Omega = \phi(\Psi_k)$$

where the map ϕ is defined by

$$\overline{x} = z^{\alpha}x,\ \overline{y} = z^{(k-1)\alpha}y,\ \overline{z} = z,$$

$$\Psi_1 = \{(x, y, z) \in \mathbb{R}^3 | -1 < x < 1; -1 < y < 1; 0 < z < 1\}$$

and

$$\Psi_2 = \Gamma \times (0, 1), \quad \Gamma \subset \mathbb{R}^2$$

is a bounded connected region with Lipschitz boundary such that $(0, 0) \in \Gamma$.

The *Jacobian* of the desingularizing map is $J\phi(x, y, z) = z^{k\alpha}$.

Trace theorems for domains with external cusps could be obtained in weighted Sobolev spaces [1]. For $u \in C^1(\overline{\Omega})$, we define

$$||\, u\, ||_{2,\alpha} := ||\, u\, z^{-\frac{\alpha}{2}}\, ||_{L^2(\Omega)},$$

and we introduce the weighted Sobolev space $H^1_\alpha(\Omega)$ as the closure of $C^1(\overline{\Omega})$ in the norm

$$||\, u\, ||^2_{H^1_\alpha(\Omega)} := ||\, u\, ||^2_{2,\alpha} + ||\, \nabla u\, ||^2_{L^2(\Omega)}.$$

In what follows, we use the letter C to denote a generic constant which depends only on Ω.

Theorem 2. *Let Ω be a $\mathcal{Q}_{k,\alpha}$ model. Then, there exists a constant C such that for any $u \in H^1_\alpha(\Omega)$, the trace function γu is in $L^2(\partial\Omega)$ and*

$$||\, \gamma u\, ||_{L^2(\partial\Omega)} \leq C\, ||\, u\, ||^{1/2}_{2,\alpha}\, ||\, u\, ||^{1/2}_{H^1_\alpha(\Omega)}. \tag{1}$$

The proof of this theorem will be given later in the last section. We shall first explore some consequences of this result.

Let $\nu = (\alpha - 1)k$. In the next theorem we will make use of the inclusion

$$H^1(\Omega) \subset L^{2q}(\Omega) \qquad \text{for} \qquad 1 \leq q \leq \frac{\nu + 3}{\nu + 1}, \tag{2}$$

which is a particular case of the results given in [2].

We can obtain the inclusion $H^1(\Omega) \subseteq H^1_\alpha(\Omega)$ under appropriate assumptions on the values of α and k.

Definition 3. *The $\mathcal{Q}_{k,\alpha}$ cusp Ω satisfies **Condition A1** if*

$$\begin{aligned} \alpha < \sqrt{2} \qquad &\text{for} \qquad k = 1,\\ \alpha < 2 \qquad &\text{for} \qquad k = 2. \end{aligned}$$

Theorem 4. *If Ω satisfies **Condition A1**, then $H^1(\Omega) = H^1_\alpha(\Omega)$.*

Proof. We shall follow the arguments in [1]. By Hölder's inequality with an exponent q to be chosen below

$$\int_\Omega |\, u\,|^2\, z^{-\alpha} \leq \left(\int_\Omega |\, u\,|^{2q} \right)^{\frac{1}{q}} \left(\int_\Omega z^{-\alpha \frac{q}{q-1}} \right)^{\frac{q-1}{q}}. \tag{3}$$

From (2), if $1 \leq q \leq \frac{\nu+3}{\nu+1}$ we have

$$\left(\int_\Omega |\, u\,|^{2q} \right)^{\frac{1}{q}} \leq C\, ||\, u\,||^2_{H^1(\Omega)}.$$

On the other hand, $\left(\int_\Omega z^{-\alpha \frac{q}{q-1}} \right)^{\frac{q-1}{q}}$ is bounded if $-\alpha \frac{q}{q-1} + k\alpha > -1$.

If $k = 2$, we must take q such that

$$\frac{2\alpha + 1}{\alpha + 1} < q \leq \frac{2\alpha + 1}{2\alpha - 1},$$

and this is possible only if $\alpha < 2$.

For $k = 1$, we have

$$1 + \alpha < q \leq \frac{\alpha + 2}{\alpha}.$$

Hence, $\alpha < \sqrt{2}$. □

Corollary 5. *If Ω satisfies **Condition A1**, then the trace function γu is in $L^2(\partial\Omega)$ for any $u \in H^1(\Omega)$. Furthermore, the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is compact.*

Proof. It only remains to show that γ is compact. Then, let $\{u_n\}$ be a bounded sequence in $H^1(\Omega)$ and, since we know that the inclusion $H^1(\Omega) \subset L^2(\Omega)$ is compact [6], we can also assume that $\{u_n\}$ is a Cauchy sequence in $L^2(\Omega)$. We shall see now that $\{u_n\}$ is a Cauchy sequence in the $||\cdot||_{2,\alpha}$ norm.

For $r \geq 2$, let $\Omega_r := \{(x, y, z) \in \Omega \,|\, z < 1/r\}$. By (3), we have

$$\int_{\Omega_r} |\, u_n\,|^2\, z^{-\alpha} \leq \left(\int_{\Omega_r} |\, u_n\,|^{2q} \right)^{\frac{1}{q}} \left(\int_{\Omega_r} z^{-\alpha \frac{q}{q-1}} \right)^{\frac{q-1}{q}} \leq C \left(\int_{\Omega_r} z^{-\alpha \frac{q}{q-1}} \right)^{\frac{q-1}{q}} \qquad \forall n.$$

Since

$$\int_{\Omega_r} z^{-\alpha \frac{q}{q-1}} < \infty,$$

given $\epsilon > 0$, we can chose r such that

$$\int_{\Omega_r} |\, u_n\,|^2\, z^{-\alpha} \leq \frac{\epsilon}{3} \quad \forall n.$$

On the other hand,

$$\int_{\Omega\setminus\Omega_r} |\, u_{n+m} - u_n\,|^2\, z^{-\alpha} \leq r^{\alpha} \int_{\Omega\setminus\Omega_r} |\, u_{n(\epsilon)+m} - u_{n(\epsilon)}\,|^2 < \frac{\epsilon}{3} \quad \forall m,$$

if $n(\epsilon)$ is chosen such that

$$\int_{\Omega} |\, u_{n(\epsilon)+m} - u_{n(\epsilon)}\,|^2 < \frac{\epsilon}{3r^{\alpha}} \quad \forall m.$$

Then,

$$\begin{aligned}
\int_{\Omega} |\, u_{n(\epsilon)+m} - u_{n(\epsilon)}\,|^2\, z^{-\alpha} &= \int_{\Omega_r} |\, u_{n(\epsilon)+m} - u_{n(\epsilon)}\,|^2\, z^{-\alpha} + \int_{\Omega\setminus\Omega_r} |\, u_{n(\epsilon)+m} - u_{n(\epsilon)}\,|^2\, z^{-\alpha} \\
&\leq \int_{\Omega_r} |\, u_{n(\epsilon)+m}\,|^2\, z^{-\alpha} + \int_{\Omega_r} |\, u_{n(\epsilon)}\,|^2\, z^{-\alpha} + \frac{\epsilon}{3} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall m.
\end{aligned}$$

Now, the result follows easily by estimate (1). □

Remark 6. *In the bidimensional case, Theorem 4 for $\alpha-$cups was obtained in [1]. The compacity of the trace operator follows by the same arguments given above. The key tool is estimate (1) in this appropriate form.*

2. ALMOST LIPSCHITZ DOMAINS WITH EXTERNAL CUSPS

Let denote I^3 the open cube $(-1,1)\times(-1,1)\times(-1,1)$.

Definition 7. *A bounded domain $\Omega \subset \mathbb{R}^3$ satisfies **Condition A2** if and only if:*

(i) There exists a finite family of open subsets $\{U_1, ..., U_m\}$ of $\mathbb{R}^3$ such that $\partial\Omega \subset \cup_{i=1}^{m} U_i$.

(ii) A Lipschitz diffeomorhism $F_i : I^3 \to U_i$ such that one of the two possibilities occurs:

*(iii) $U_i \cap \overline{\Omega}$ is the image of a standard cusp $\mathcal{Q}_{k,\alpha}$ in I^3 which satisfies **Condition A1**.*

(iv) There exists a Lipschitz map $f_i : (-1,1)\times(-1,1) \to (-1,1)$ such that $f_i(0,0) = 0$ and

$$F_i^{-1}(U_i \cap \overline{\Omega}) = \{(x,y,z) \in I^3 \mid f(x,y) \leq z\}$$

When **Condition A2** holds, there is an outward unit normal ν defined at σ *a.e.*point of $\partial\Omega$, where σ represents Hausdorff 2-dimensional measure and functions in $\partial\Omega$ are integrated with respect to this measure . Furthermore, by a partition of unity argument we can obtain the following result.

Theorem 8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain which satisfies **Condition A2.** Then, the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$.*

3. PROOF OF THE TRACE THEOREM

We proceed first with case $k = 1$. Thus, ϕ is defined by

$$\overline{x} = z^\alpha x;\ \overline{y} = y;\ \overline{z} = z,$$

and the Jacobian of ϕ is $J\phi(x, y, z) = z^\alpha$. Let $\overline{u} \in C^1(\overline{\Omega})$ and $u = \overline{u} \circ \phi$. Then,

$$\int_{\Psi_1} u^2 = \int_{\Psi_1} \left(u\, z^{-\frac{\alpha}{2}}\right)^2 z^\alpha = \int_\Omega \left(\overline{u}\, \overline{z}^{-\frac{\alpha}{2}}\right)^2 = ||\, \overline{u}\, \overline{z}^{-\frac{\alpha}{2}}\, ||^2_{L^2(\Omega)}.$$

On the other hand,

$$\frac{\partial u}{\partial x} = \frac{\partial \overline{u}}{\partial \overline{x}} z^\alpha$$

and

$$\int_{\Psi_1} \left(\frac{\partial u}{\partial x}\right)^2 = \int_{\Psi_1} \left(\frac{\partial \overline{u}}{\partial \overline{x}}\right)^2 z^{2\alpha} \le \int_\Omega \left(\frac{\partial \overline{u}}{\partial \overline{x}}\right)^2$$

Now, let $\partial\Omega_1 := \phi(\{(x, y, z) \in \Psi_1 : x = 1\})$. Then, $\partial\Omega_1$ is parametrized by

$$X(y, z) = z^\alpha \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}.$$

Thus,

$$\frac{\partial X}{\partial y} = \mathbf{j}, \quad \frac{\partial X}{\partial z} = \alpha z^{(\alpha-1)}\mathbf{i} + \mathbf{k}, \quad \frac{\partial X}{\partial y} \times \frac{\partial X}{\partial z} = \mathbf{i} + \alpha z^{(\alpha-1)}\mathbf{k}.$$

and it follows that

$$\int_{\partial\Omega_1} \overline{u}^2\, dS \le C \int_{\{x=1\}} u^2\, dydz.$$

Let $\omega : [-1, 1] \to \mathbb{R}_+$ be a C^1 function such that $\omega \equiv 0$ in $[-1, 0]$ and $\omega \equiv 1$ in $[0, 1]$, and define $\widetilde{u}$ by $\widetilde{u}(x, y, z) = \omega(y)\, u(x, y, z)$.

Setting

$$u^2(1, y, z) = 2 \int_{-1}^1 \widetilde{u}\, \frac{\partial \widetilde{u}}{\partial x}\, dx,$$

by Hölder's inequality we have

$$\int_{\partial\Omega_1} \overline{u}^2\, dS \le C\, ||\, \widetilde{u}\, ||^{1/2}_{L^2(\Psi_1)}\, ||\, \widetilde{u}_x\, ||^{1/2}_{L^2(\Psi_1)}.$$

Now, it is clear that $||\, \widetilde{u}\, ||^{1/2}_{L^2(\Psi_1)} = ||\, u\, ||^{1/2}_{2,\alpha}$. On the other hand,

$$|\, \widetilde{u}_x|^2 \le C(\,|\, u\,|^2 + |\, u_x\,|^2).$$

From this, we can easily obtain that

$$\|\widetilde{u}\|_{L^2(\Psi_1)}^{1/2}\,\|\widetilde{u}_x\|_{L^2(\Psi_1)}^{1/2} \leq C\,\|u\|_{2,\alpha}^{1/2}\,\|u\|_{H^1_\alpha(\Omega)}^{1/2}$$

and the result follows. The proof for $\partial\Omega_{-1} := \phi(\{(x,y,z) \in \Psi_1 : x = -1\})$ is the same.

Case $k = 2$:

We shall explain the main arguments for the curve $\Gamma = S^1 \subset \mathbb{R}^2$. It will be clear from the proof that the general case follows along the same lines via a partition of unity.

We consider S^1 parametrized by $(\cos(\theta), \sin(\theta))$ and the parametrization $X : [0, 2\pi] \times [0, 1] \to \partial\Omega$ given by

$$X(\theta, z) = z^\alpha \cos(\theta)\,\mathbf{i} + z^\alpha \sin(\theta)\,\mathbf{j} + z\,\mathbf{k}.$$

It follows that

$$\frac{\partial X}{\partial \theta} = -z^\alpha \sin(\theta)\,\mathbf{i} + z^\alpha \cos(\theta)\,\mathbf{j}, \quad \frac{\partial X}{\partial z} = \alpha z^{(\alpha-1)} \cos(\theta)\,\mathbf{i} + \alpha z^{(\alpha-1)} \sin(\theta)\,\mathbf{j} + \mathbf{k},$$

$$\frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial z} = z^\alpha(\cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} + \alpha z^{(\alpha-1)}\,\mathbf{k}.$$

Thus,

$$\int_{\partial\Omega} \overline{u}^2\, dS \leq C \int_0^1 \left(\int_0^{2\pi} u^2\, d\theta \right) z^\alpha\, dz.$$

For $z \in (0, 1)$, we want now to estimate $\int_0^{2\pi} \overline{u}^2\, d\theta$ with the same arguments as above. We introduce polar coordinates in (x, y) and we define

$$\widetilde{u}(r, \theta, z) := \omega(r)\, u(r, \theta, z)$$

where $\omega \in C^1[0, 1]$ such that $\omega(r) = 1$ for $1/3 \leq r \leq 1$ and $\omega(r) = 0$ for $0 \leq r \leq 1/3$.

Thus,

$$\begin{aligned}
\int_0^1 \left(\int_0^{2\pi} u^2\, d\theta \right) z^\alpha\, dz &\leq C \int_0^1 \int_0^{2\pi} \int_{1/3}^1 \widetilde{u} \frac{\partial \widetilde{u}}{\partial r} r\, dr d\theta\, z^\alpha\, dz \\
&\leq C \int_{\Psi_2} \left(\widetilde{u} z^{\alpha/2} \right) \left(\frac{\partial \widetilde{u}}{\partial x} z^{\alpha/2} + \frac{\partial \widetilde{u}}{\partial y} z^{\alpha/2} \right) \\
&\leq C\, \|\widetilde{u} z^{\alpha/2}\|_{L^2(\Psi_2)}^{1/2}\, \|\nabla \widetilde{u}\, z^{\alpha/2}\|_{L^2(\Psi_2)}^{1/2}.
\end{aligned}$$

First, we have

$$\begin{aligned}
\int_{\Psi_2} \widetilde{u}^2 z^\alpha &\leq \int_{\Psi_2} \left(u\, z^{-\alpha/2} \right)^2 z^{2\alpha} \\
&= \int_\Omega \left(\overline{u}\, z^{-\alpha/2} \right)^2 \\
&= \|\overline{u}\|_{2,\alpha}^2.
\end{aligned}$$

To complete the proof, we must take into account that

$$\frac{\partial u}{\partial x} = \frac{\partial \overline{u}}{\partial \overline{x}} z^{\alpha} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial \overline{u}}{\partial \overline{y}} z^{\alpha}.$$

Then, calculating for the first derivative, we get

$$\left| \frac{\partial \widetilde{u}}{\partial x} \right| \leq C \left(\left| \frac{\partial u}{\partial x} \right| + |\, u \,| \right).$$

Hence,

$$\begin{aligned} \int_{\Psi_2} \left| \frac{\partial \widetilde{u}}{\partial x} \right|^2 z^{\alpha} &\leq C \left(\int_{\Psi_2} \left| \frac{\partial u}{\partial x} z^{-\alpha/2} \right|^2 z^{2\alpha} + \int_{\Psi_2} |\, u z^{-\alpha/2} \,|^2 z^{2\alpha} \right) \\ &\leq C \left(\int_{\Omega} \left| \frac{\partial \overline{u}}{\partial \overline{x}} \right|^2 + \int_{\Omega} |\, u z^{-\alpha/2} \,|^2 \right). \end{aligned}$$

The same inequality is valid for the second derivative and we get

$$||\, \nabla \widetilde{u}\, z^{\alpha/2} \,||^2_{L^2(\Psi_2)} \leq C \,||\, \overline{u} \,||^2_{H^1_\alpha(\Omega)}.$$

Considering these facts together, it is easy to see that we have concluded the proof of the theorem.

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