REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 50, Número 1, 2009, Páginas 47–59

THE SUBVARIETY OF Q-HEYTING ALGEBRAS GENERATED BY CHAINS

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ABSTRACT. The variety $\mathcal{QH}$ of Heyting algebras with a quantifier [14] corresponds to the algebraic study of the modal intuitionistic propositional calculus without the necessity operator. This paper is concerned with the subvariety $\mathcal{C}$ of $\mathcal{QH}$ generated by chains. We prove that this subvariety is characterized within $\mathcal{QH}$ by the equations $\nabla(x \wedge y) \approx \nabla x \wedge \nabla y$ and $(x \to y) \vee (y \to x) \approx 1$. We investigate free objects in $\mathcal{C}$.

1. Introduction and Preliminaries

Distributive lattices with a quantifier were considered as algebras for the first time by Cignoli in [7] who studied them under the name of Q-distributive lattices. A Q-distributive lattice is an algebra $(L; \vee, \wedge, 0, 1, \nabla)$ of type $(2, 2, 0, 0, 1)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation ∇ satisfies the following conditions, for any a, $b \in L$: $\nabla 0 = 0$, $a \wedge \nabla a = a$, $\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$ and $\nabla(a \vee b) = \nabla a \vee \nabla b$. These conditions were introduced by Halmos [9] as an algebraic counterpart of the logical notion of an existential quantifier.

Various further investigations have been carried out since [7] (see R. Cignoli [8], H. Priestley [13], M. Adams and W. Dziobiak [4], M. Abad and J. P. Díaz Varela [2] and A. Petrovich [11]). As a natural generalization, the operation of quantification was considered for Heyting algebras in [3] and [15]. A Heyting algebra is an algebra $(H; \vee, \wedge, \to, 0, 1)$ of type $(2, 2, 2, 0, 0)$ for which $(H; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and for a, $b \in H$, $a \to b$ is the relative pseudocomplement of a with respect to b, i.e., $a \wedge c \leq b$ if and only if $c \leq a \to b$. It is known that the class of Heyting algebras forms a variety. An important subvariety of Heyting algebras is the class of linear Heyting algebras [5]. A linear Heyting algebra is a Heyting algebra that satisfies the equation $(x \to y) \vee (y \to x) \approx 1$. Throughout this paper $\mathcal{H}$ will denote the category of Heyting algebras and Heyting algebra homomorphisms and $\mathcal{H}_L$ will denote the subcategory of linear Heyting algebras.

A Q-Heyting algebra is an algebra $(H; \nabla)$ such that H is an object of $\mathcal{H}$ and ∇ is a quantifier on H, that is, ∇ is a unary operation defined as for Q-distributive lattices. Monadic Boolean algebras are the simplest examples of Q-Heyting algebras.

The class of Q-Heyting algebras forms a variety, which we denote $\mathcal{QH}$. The subvariety of $\mathcal{QH}$ characterized within $\mathcal{QH}$ by the equation $(x \to y) \vee (y \to x) \approx 1$, that is, the subvariety of linear Q-Heyting algebras will be denoted by $\mathcal{Q}H_L$. Q-Heyting algebras were first introduced in [14] and have been investigated in [14, 15, 3].

In this paper we investigate the subvariety $\mathcal{C}$ of the variety of Q-Heyting algebras generated by chains. We characterize $\mathcal{C}$ by identities in Section 2 and we investigate free objects in this variety in Section 3.

We will usually use the same notation for a variety and for the algebraic category associated with it. And, similarly we will use the same notation for a structure and for its universe.

Recall that *Heyting algebras* are algebraic models of the intuitionistic propositional logic and that the study of extensions of Intuitionistic Propositional Calculus (IPC) reduces to the study of subvarieties of the variety $\mathcal{H}$. The language of intuitionistic modal logic (MIPC) is the language of IPC enriched with two modal unary operators of necessity $\Box$ and of posibility $\Diamond$. The algebraic models of MIPC are the monadic Heyting algebras.

Now, in MIPC the operators $\Box$ and $\Diamond$ are independent from each other, that is $\Box p \leftrightarrow \neg\Diamond\neg p$ and $\Diamond p \leftrightarrow \neg\Box\neg p$ are not theorems in MIPC. Hence, the set of theorems of the propositional calculus without of the necessity operator $\Box$, called the $\Box$-*free fragment of* MIPC is different from that of MIPC. Similarly, the set of theorems of the propositional calculus without of the possibility operator $\Diamond$, called the $\Diamond$-*free fragment* of MIPC is different from that of MIPC.

It turns out that the behaviour of the $\Diamond$-free fragment of MIPC is very much similar to that of MIPC. However, surprisingly enough, the $\Box$-free fragment of MIPC behaves pretty diferent from MIPC.

Q-Heyting algebras are the algebraic models of the $\Box$-free fragment of MIPC, that is, Q-Heyting algebras are the $\Box$-free reducts of monadic Heyting algebras.

For a poset X and $Y \subseteq X$, let $(Y] = \{u : u \leq v \text{ for some } v \in Y\}$ and $[Y) = \{u : u \geq v \text{ for some } v \in Y\}$. We write $[u)$, $(u]$ instead of $[\{u\})$, $(\{u\}]$ respectively. We say that Y is *decreasing* if $Y = (Y]$, *increasing* if $Y = [Y)$ and *convex* if $Y = (Y] \cap [Y)$. A mapping φ is *order preserving* if $\varphi(u) \leq \varphi(v)$ whenever $u \leq v$.

In order to describe the dual category of $\mathcal{QH}$ we recall that a *Priestley space* is a triple $(X; \leq, \tau)$ such that $(X; \leq)$ is a partially ordered set, $(X; \tau)$ is a compact topological space, and the triple is *totally order-disconnected* (that is, for u, $v \in X$, if $u \not\leq v$ then there exists a clopen increasing $U \subseteq X$ such that $u \in U$ and $v \notin U$). Priestley showed that the category of bounded distributive lattices and lattice homomorphisms is dually equivalent to the category of Priestley spaces and order preserving continuous functions (see the survey paper [12]).

A *Q-Heyting space* $(X; E)$ (see [7, 14, 15]) is a Priestley space $(X; \leq, \tau)$ together with an equivalence relation E defined on X such that (i) $(Y]$ is clopen for every convex clopen $Y \subseteq X$, (ii) $\nabla_E U \in D(X)$ for each $U \in D(X)$, where $\nabla_E U = \{v : vEu$ for some $u \in U\}$ and $D(X)$ is the lattice of clopen increasing subsets of X, and (iii) the blocks of E are closed in X. For $a \in H$, let $\sigma(a) \subseteq X = X(H)$ denote the clopen increasing set that represents a, where $X(H)$ is the set of prime filters of H, ordered by set inclusion and with the topology having as a sub-basis the sets $\sigma(a) = \{P \in X(H) \ : \ a \in P\}$ and $X(H) \setminus \sigma(a)$ for $a \in H$. If $a, b \in H$ then, under the duality, $a \to b$ corresponds to the clopen increasing set $X \setminus (\sigma(a) \setminus \sigma(b)]$.

For Q-Heyting spaces $(X; E)$ and $(Y; E')$, a Q-Heyting morphism is a continuous order-preserving mapping $\varphi : X \to Y$ such that $\varphi([u)) = [\varphi(u))$ and $\nabla_E \varphi^{-1}(V) = \varphi^{-1}(\nabla_{E'} V)$, for each $V \in D(Y)$.

It can be proved in the usual way that the category of Q-Heyting algebras and homomorphisms is dually equivalent to the category of Q-Heyting spaces and Q-Heyting morphisms [14, 15]. For each Q-Heyting algebra $(H; \nabla)$ the corresponding Q-Heyting space is $(X(H); E_\nabla)$, where $E = E_\nabla = \{(P, Q) \in X(H)^2 : P \cap \nabla(H) = Q \cap \nabla(H)\}$. Conversely, if $(X; E)$ is a Q-Heyting space, the corresponding Q-Heyting algebra is $(D(X); \nabla_E)$, where ∇_E is defined as in (ii).

2. The variety $\mathcal{C}$

In this section we will study the subvariety $\mathcal{C}$ generated by chains within $\mathcal{QH}$. Observe that if $(H; \nabla) \in \mathcal{C}$, then $H \in \mathcal{H}_L$, that is, $\mathcal{C} \subseteq \mathcal{Q}H_L \subseteq \mathcal{QH}$. Consequently, $\mathcal{C} \models (x \to y) \vee (y \to x) \approx 1$.

Recall that in the variety of Heyting algebras, congruences are determined by filters. Precisely, if $H \in \mathcal{H}$ and F is a filter of H, then $\theta_F = \{(a, b) \in H \times H : (a \to b) \wedge (b \to a) \in F\}$ is a congruence on H, and the correspondence $F \mapsto \theta_F$ establishes an isomorphism from the lattice of filters of H on $Con_{\mathcal{H}}\ H$, the lattice of congruences of H. If F is generated by an element a, $F = [a)$, we write $\theta_a = \theta_{[a)}$.

Observe that if C is a Heyting chain and F is a filter of C, $(a, b) \in \theta_F$ if and only if $a = b$ or $a, b \in F$. Then, $Con_{\mathcal{QH}}\ (C; \nabla) = Con_{\mathcal{H}}\ C$. As a consequence of this, we have that if C is a chain, $(C; \nabla)$ is a subdirectly irreducible algebra in $\mathcal{QH}$ if and only if C is a subdirectly irreducible algebra in $\mathcal{H}$, that is, C has a unique dual atom.

A quantifier ∇ on an algebra $H \in \mathcal{QH}$ is said to be *multiplicative* if $\nabla(a \wedge b) = \nabla a \wedge \nabla b$, for every $a, b \in H$.

Let $\mathcal{M}$ be the subvariety of $\mathcal{QH}$ characterized by the equation $\nabla(x \wedge y) \approx \nabla x \wedge \nabla y$.

Lemma 2.1. *If $(H; \nabla) \in \mathcal{M}$, $Con_{\mathcal{QH}}\ (H; \nabla) = Con_{\mathcal{H}}\ H$.*

Proof Let $\theta_F \in Con_{\mathcal{H}}\ H$ and $(a, b) \in \theta_F$, i.e., $(a \to b) \wedge (b \to a) \in F$. As $a \wedge (a \to b) = a \wedge b$, then $\nabla a \wedge \nabla(a \to b) \leq \nabla b$, that is, $\nabla(a \to b) \leq \nabla a \to \nabla b$. So $(a \to b) \wedge (b \to a) \leq \nabla((a \to b) \wedge (b \to a)) = \nabla(a \to b) \wedge \nabla(b \to a) \leq$

$(\nabla a \to \nabla b) \wedge (\nabla b \to \nabla a)$. Thus $(\nabla a \to \nabla b) \wedge (\nabla b \to \nabla a) \in F$, so $(\nabla a, \nabla b) \in \theta_F$. Therefore, $\theta_F \in Con_{\mathcal{QH}}\ (H; \nabla)$. ■

Observe that $\mathcal{C} \models \{\nabla(x \wedge y) \approx \nabla x \wedge \nabla y,\ (x \to y) \vee (y \to x) \approx 1\}$, that is, $\mathcal{C} \subseteq \mathcal{M} \cap \mathcal{Q}H_L$. Let us see that $\mathcal{C} = \mathcal{M} \cap \mathcal{Q}H_L$.

Lemma 2.2. *Let $(H; \nabla)$ be a subdirectly irreducible algebra in $\mathcal{M} \cap \mathcal{Q}H_L$. Then H is a chain.*

Proof Let $(H; \nabla) \in \mathcal{M} \cap \mathcal{Q}H_L$ be a subdirectly irreducible algebra. Then $Con_{\mathcal{QH}}\ (H; \nabla) = Con_{\mathcal{H}}\ H$ and hence H is subdirectly irreducible in $\mathcal{H}$, that is, H has a unique dual atom. Since for every $a, b \in H$, $(a \to b) \vee (b \to a) = 1$, then $a \to b = 1$ or $b \to a = 1$, that is, $a \leq b$ or $b \leq a$. So H is a chain. ■

Corollary 2.3. $\mathcal{C} = \mathcal{M} \cap \mathcal{Q}H_L$.

As a consequence of this corollary we have that $\mathcal{C}$ is characterized within $\mathcal{QH}$ by the identities $\nabla(x \wedge y) \approx \nabla x \wedge \nabla y$ and $(x \to y) \vee (y \to x) \approx 1$.

The following theorem characterizes the dual space of an algebra in $\mathcal{M}$.

Theorem 2.4. *Let $(H; \nabla)$ be a Q-Heyting algebra, let $(X(H); E)$ be the associated Q-Heyting space and $\{E_i\}_{i \in I}$ the partition of $X(H)$ determined by E. Then, $(H; \nabla) \in \mathcal{M}$ if and only if each E_i has exactly one maximal element.*

Proof Suppose that $(H; \nabla) \in \mathcal{M}$ and there exists $i_0 \in I$ such that E_{i_0} has two maximal elements M, N, $M \neq N$. Let $a \in H$ be such that $M \in \sigma(a)$ and $N \notin \sigma(a)$. For each $P \in \sigma(a) \cap E_{i_0}$, we have that $N \not\subseteq P$. Thus there exists $b_P \in H$ such that $N \in \sigma(b_P)$ and $J \in X \setminus \sigma(b_P)$. Consequently

$$\sigma(a) \cap E_{i_0} \subseteq \bigcup_{P \in \sigma(a) \cap E_{i_0}} X \setminus \sigma(b_P).$$

As $\sigma(a) \cap E_{i_0}$ is closed, by a compacteness argument

$$\sigma(a) \cap E_{i_0} \subseteq \bigcup_{i=1}^{n} X \setminus \sigma(b_i) = X \setminus \sigma(\bigwedge_{i=1}^{n} b_i) = X \setminus \sigma(b)$$

and $N \in \sigma(b)$. So $M \in \sigma(a) \cap E_{i_0}$ and $N \in \sigma(b) \cap E_{i_0}$. This implies that $E_{i_0} \subseteq \sigma(\nabla a \wedge \nabla b)$ and consequently $\sigma(\nabla a \wedge \nabla b) \cap E_{i_0} = E_{i_0}$. On the other hand, $\nabla_E\, \sigma(a \wedge b) \cap E_{i_0} = \nabla_E\, (\sigma(a) \cap \sigma(b)) \cap E_{i_0} = \emptyset$, which contradicts that $(H; \nabla) \in \mathcal{M}$.

Conversely, we know that $D(X(H)) \in \mathcal{QH}$. Let us see that $\nabla_E\, \sigma(a) \cap \nabla_E\, \sigma(b) = \nabla_E\, \sigma(a \wedge b)$. Since ∇_E is a quantifier, $\nabla_E\, \sigma(a \wedge b) \subseteq \nabla_E\, \sigma(a) \cap \nabla_E\, \sigma(b)$. Let us prove the other inclusion. Let $P \in \nabla_E\, \sigma(a) \cap \nabla_E\, \sigma(b)$ and $i_0 \in I$ such that $P \in E_{i_0}$. Since $\sigma(a) \cap E_{i_0} \neq \emptyset$ and $\sigma(b) \cap E_{i_0} \neq \emptyset$, if $\{M_{i_0}\} = max\, E_{i_0}$, then $M_{i_0} \in \sigma(a) \cap \sigma(b) = \sigma(a \wedge b)$. Therefore $E_{i_0} \subseteq \nabla_E\, \sigma(a \wedge b)$ and so $P \in \nabla_E\, \sigma(a \wedge b)$. ■

Lemma 2.5. *$\mathcal{M}$ is the greatest subvariety of $\mathcal{QH}$ such that every filter determines a congruence.*

Proof Let $(H;\nabla) \in \mathcal{QH}$ such that $(H;\nabla) \notin \mathcal{M}$. We are going to construct a filter in H which does not determine a congruence. From $(H;\nabla) \notin \mathcal{M}$, there exist $a, b \in H$ such that $\nabla a \wedge \nabla b \not\leq \nabla(a \wedge b)$. Then there exists a prime ideal M such that $\nabla(a \wedge b) \in M$ and $\nabla a \wedge \nabla b \notin M$. Since M is an ideal we have that $\nabla a \notin M$ and $\nabla b \notin M$. Consider the filter $F = [a)$. Then $(a \wedge b, b) \in \theta_F$, being that $((a \wedge b) \to b) \wedge (b \to (a \wedge b)) = b \to a \geq a$. Let us see that $(\nabla(a \wedge b), \nabla b) \notin \theta_F$. Suppose on the contrary that $(\nabla(a \wedge b), \nabla b) \in \theta_F$. Thus $\nabla b \to \nabla(a \wedge b) \geq a$, which implies that $\nabla b \to \nabla(a \wedge b) \geq \nabla a$ (*) since the image of ∇ is closed under implication. On the other hand, $\nabla b \wedge (\nabla b \to \nabla(a \wedge b)) \leq \nabla(a \wedge b)$, so $\nabla b \wedge (\nabla b \to \nabla(a \wedge b)) \in M$. Since M is a prime ideal and $\nabla b \notin M$ we have that $\nabla b \to \nabla(a \wedge b) \in M$. This, together with (*), implies that $\nabla a \in M$, which is a contradiction. ■

3. Free algebras

In this section we characterize the free algebra in $\mathcal{C}$ with n generators. Following a path analogous to that of M. Abad and L. Monteiro in [1], we will provide a method to construct the order set $\Pi(n)$ of all join-irreducible elements of the free algebra, and as a consequence, we will obtain a formula to compute $|\Pi(n)|$.

It is clear that for any subset X of a chain $(C;\nabla)$, the subalgebra of $(C;\nabla)$ generated by X is $S(X) = X \cup \nabla(X) \cup \{0, 1\}$. Thus, every n-generated subalgebra of a chain of $\mathcal{C}$ has at most $2n + 2$ elements, that is, the class of all chains in $\mathcal{C}$ is uniformly locally finite. So $\mathcal{C}$ is generated by a uniformly locally finite class, and consequently, $\mathcal{C}$ is a variety locally finite [6, Theorem 3.7].

If $(H;\nabla) \in \mathcal{C}$ is a finite algebra, the Q-Heyting space $(X(H);\leq,\tau,E)$ has the discrete topology and $(X(H);\leq)$ is anti-isomorphic to the ordered set $(\Pi(H);\leq)$ of join-irreducible elements of H. In this section we will use the set $\Pi(H)$ instead of $X(H)$ and we will consider the relation E defined on $\Pi(H)$, that is we consider $(\Pi(H);E)$. If $\{E_i\}_{i\in I}$ is the partition determined by E in $\Pi(H)$, we say that $E_i \leq E_j$ if and only if $min\, E_i \leq min\, E_j$. This is an order relation.

Theorem 3.1. [10] *A Heyting algebra is linear if and only if the family of prime filters which contain a prime filter is a chain.*

Definition 3.2. *Let $(H;\nabla) \in \mathcal{C}$ be a finite algebra. Let $p \in \Pi(H)$ and let $E_1 \leq \cdots \leq E_{r+1}$, such that $(p] \cap E_j \neq \emptyset$, $1 \leq j \leq r+1$, where $(p] = \{q \in \Pi(H) : q \leq p\}$. We say that p has coordinates $(m, m_1, \ldots, m_{r+1})$, if the chain $(p]$ is of length $m+1$ and if $m_j = |(p] \cap E_j|$, $1 \leq j \leq r+1$.*

Notice that the set $(p]$ of the previous definition is considered within $\Pi(H)$.

Let m be a non negative integer. Let $C_m = \{0, a_1, \ldots, a_m, 1\}$ be the chain with $m+2$ elements. Let $\nabla(C_m) = \{b_0 = 0, b_1, \ldots, b_r, b_{r+1} = 1\}$, $r \leq m$, with $b_i < b_j$ for $i < j$. Let $(b_i, b_j]$ be the interval in C_m consisting of the elements $a \in C_m$ such that $b_i < a \leq b_j$. We denote $C_{m,m_1,\ldots,m_{r+1}}$ the algebra $(C_m;\nabla)$, where $m_i = |(b_{i-1}, b_i]|$, $i = 1, \ldots, r+1$.

Observe that if $p \in \Pi(H)$, H/θ_p is a chain. More precisely, $(p]$ is of length $m+1$ if and only if H/θ_p is a chain with $m+2$ elements [1, p. 7]. If $\pi : H \to H/\theta_p$ is the natural homomorphism and $a_1, \ldots, a_m \in H$ are such that $\pi(0) < \pi(a_1) < \cdots < \pi(a_m) < \pi(1)$, then there exist join-irreducible elements $q_1, \ldots, q_m$ in H such that $q_1 < \cdots < q_m < p$ and $\pi(q_i) = \pi(a_i)$, $1 \leq i \leq m$. Moreover, taking into account that $q_i E q_j$ en $\Pi(H)$ if and only if $\nabla q_i = \nabla q_j$ en H, it follows that $p \in \Pi(H)$ has coordinates $(m, m_1,, \ldots, m_{r+1})$ if and only if $H/\theta_p = C_{m,m_1,\ldots,m_{r+1}}$. Since $\cap_{p\in\Pi(H)}\theta_p$ is the trivial relation, we have that $(H; \nabla)$ is a subdirect product of the chains $\{H/\theta_p\}_{p\in\Pi(H)}$.

Let $L(n)$ be the free $\mathcal{C}$-algebra with a finite set of generators of cardinal $n > 0$. For the sake of simplicity we will write $\Pi(n)$ instead of $\Pi(L(n))$.

We know that every n-generated subalgebra of a chain of $\mathcal{C}$ has at most $2n+2$ elements. Since $L(n)/\theta_p$ is a chain generated by at most n elements, we have the following

Lemma 3.3. *If $p \in \Pi(n)$, then $|L(n)/\theta_p| \leq 2\,n + 2$.*

If $p \in \Pi(n)$ then from Lemma 3.3, p has coordinates $(m, m_1, \ldots, m_{r+1})$, for some m, $0 \leq m \leq 2n$ and $m_1, \ldots, m_{r+1} \in \mathbb{N}$ such that $\sum_{j=1}^{r+1} m_j = m+1$.

Consider the following sets:

$$M_1 = \{b_j : |(b_{j-1}, b_j]| = 1, 1 \leq j \leq r\}$$

and

$$N = C_{m,m_1,\ldots,m_{r+1}} \setminus \nabla(C_{m,m_1,\ldots,m_{r+1}}).$$

For a subset T of $C_{m,m_1,\ldots,m_{r+1}}$ to generate the algebra $C_{m,m_1,\ldots,m_{r+1}}$, every non constant element must be contained in T, that is, $N \subseteq T$. Besides, every constant can be obtained from N, except the constants of M_1. So we have that $T \supseteq N \cup M_1$ and consequently, $|T| \geq |N| + |M_1| = m - r + |M_1| = m - (r - |M_1|)$.

For every m, $0 \leq m \leq 2n$, consider the sets

$$N_m(n) = \{(m_1, \ldots, m_{r+1}) : m_j \in \mathbb{N}, 1 \leq j \leq r+1, \sum_{j=1}^{r+1} m_j = m+1, m-r+|M_1| \leq n\}.$$

We will denote N_m instead of $N_m(n)$. Observe that $N_0 = \{(1)\}$, $N_1 = \{(2), (1, 1)\}$ and $N_{2n} = \{(2, 2, \ldots, 2, 1)\}$ for every $n \in \mathbb{N}$, that is, N_{2n} consists of one $(n+1)$-tuple whose n first coordinates are equal to 2. Moreover, if $n \geq 2$, we have that $N_2 = \{(3), (2, 1), (1, 2), (1, 1, 1)\}$, but if $n = 1$, $N_2 = \{(2, 1)\}$.

Let $\Pi_{m,m_1,\ldots,m_{r+1}}(n) = \{p \in \Pi(n) : L(n)/\theta_p = C_{m,m_1,\ldots,m_{r+1}}\}$. It is clear that

$$\Pi(n) = \bigcup_{m=0}^{2n} \bigcup_{(m_1,\ldots,m_{r+1})\in N_m} \Pi_{m,m_1,\ldots,m_{r+1}}(n)$$

and that $\Pi_{m,m_1,\ldots,m_l}(n) \cap \Pi_{m',m'_1,\ldots,m'_{l'}}(n) = \emptyset$ for $(m_1, \ldots, m_l) \neq (m'_1, \ldots, m'_{l'})$.

Let $\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$ be the set of all functions f from the set G of free generators of $L(n)$ into $C_{m,m_1,\ldots,m_{r+1}}$ such that $S(f(G)) = C_{m,m_1,\ldots,m_{r+1}}$. Observe that every $\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$ is nonempty, as $f \in \mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$ if and only if $M_1 \cup N \subseteq f(G)$, that is $m - r + |M_1| \leq n$.

Recall that a filter P in a finite Heyting algebra is prime if and only if $P = [p)$, where p is join-irreducible element.

If $f \in \mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$, f can be extended to a unique homomorphism $\overline{f}$ from $L(n)$ onto $C_{m,m_1,\ldots,m_{r+1}}$. If $N(\overline{f}) = \{a \in L(n) : \overline{f}(a) = 1\}$ is the kernel of $\overline{f}$, it is well known that $N(\overline{f})$ is a prime filter in $L(n)$, so $N(\overline{f}) = [p_f)$, with $p_f \in \Pi_{m,m_1,\ldots,m_{r+1}}$. Thus, for each $(m_1, \ldots, m_{r+1}) \in N_m,\ 0 \leq m \leq 2n$, we have a function

$$\psi_{m,m_1,\ldots,m_{r+1}} : \mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n) \rightarrow \Pi_{m,m_1,\ldots,m_{r+1}}(n)$$

defined by $\psi_{m,m_1,\ldots,m_{r+1}}(f) = p_f$.

Lemma 3.4. *The following holds* $|\Pi_{m,m_1,\ldots,m_{r+1}}(n)| = |\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)|,\ 0 \leq m \leq 2n,\ (m_1, \ldots, m_{r+1}) \in N_m$.

Proof Let us see that $\psi_{m,m_1,\ldots,m_{r+1}}$ is onto. For $p \in \Pi_{m,m_1,\ldots,m_{r+1}}(n)$, consider h the natural homomorphism from $L(n)$ onto $L(n)/\theta_p = C_{m,m_1,\ldots,m_{r+1}}$, and $f = h\restriction_G$ the restriction of h to G. Then $S(f(G)) = S(h(G)) = h(L(n)) = C_{m,m_1,\ldots,m_{r+1}}$ and therefore $f \in \mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$. Let $\overline{f}$ be the extension of f. Since $\overline{f}\restriction_G = f = h\restriction_G$, then $\overline{f} = h$ and therefore $\psi_{m,m_1,\ldots,m_{r+1}}(f) = p_f = p$.

Let us prove that the function $\psi_{m,m_1,\ldots,m_{r+1}}$, is one-to-one. If the functions f_1, $f_2 \in \mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$ satisfy $N(\overline{f_1}) = N(\overline{f_2})$ then there is an automorphism α of $C_{m,m_1,\ldots,m_{r+1}}$ such that $\alpha \circ \overline{f_1} = \overline{f_2}$. But the only automorphism of $C_{m,m_1,\ldots,m_{r+1}}$ is the identity, then $\overline{f_1} = \overline{f_2}$ and then $f_1 = f_2$. ■

Lemma 3.5.

$$\begin{aligned} |\Pi(n)| &= \sum_{m=0}^{2n} \sum_{(m_1,\ldots,m_{r+1}) \in N_m} |\Pi_{m,m_1,\ldots,m_{r+1}}(n)| \\ &= \sum_{m=0}^{2n} \sum_{(m_1,\ldots,m_{r+1}) \in N_m} |\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)|. \end{aligned}$$

If $NS(a, b)$ is the number of functions from a set with a elements onto a set with b elements, then:

$$NS(a, b) = \begin{cases} \sum_{i=0}^{b-1} (-1)^i \binom{b}{i} (b-i)^a & \text{if } a \geq b \\ 0 & \text{if } a < b \end{cases}.$$

Let $l = r - |M_1|$. Then, for each $(m_1, \ldots, m_{r+1}) \in N_m,\ 0 \leq m \leq 2n$,

$$|\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)| = \sum_{k=0}^{l+2} \binom{l+2}{k} NS(n, m - l + k).$$

In particular, for $\mathbf{F}_{0,1}(n)$, $m-l=0$ and then $|\mathbf{F}_{0,1}(n)| = NS(n,0)+2NS(n,1)+NS(n,2) = 2^n$. For $\mathbf{F}_{1,2}(n)$, $m=1$, $r=|M_1|=0$ and for $\mathbf{F}_{1,1,1}(n)$, $m=r=|M_1|=1$. So in both cases, $m-l=1$, then $|\mathbf{F}_{1,2}(n)| = |\mathbf{F}_{1,1,1}(n)| = NS(n,1)+2NS(n,2)+NS(n,3) = 3^n-2^n$. And for $\mathbf{F}_{2n,2,2,\ldots,2,1}(n)$, $m=2n$, $r=n$, $|M_1|=0$, then $m-l=n$ and $|\mathbf{F}_{2n,2,2,\ldots,2,1}(n)| = n!$.

Consequently,

$$|\Pi(1)| = |\mathbf{F}_{0,1}(1)| + |\mathbf{F}_{1,2}(1)| + |\mathbf{F}_{1,1,1}(1)| + |\mathbf{F}_{2,2,1}(1)| = 2+1+1+1 = 5.$$

Consider the set

$$\mathbf{F}(n) = \bigcup_{m=0}^{2n} \bigcup_{(m_1,\ldots,m_{r+1})\in N_m} \mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n).$$

If $f\in\mathbf{F}(n)$, there is a unique m and a unique $(m_1,\ldots,m_{r+1})\in N_m$ such that $f\in\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$. If we put $\psi(f)=\psi_{m,m_1,\ldots,m_{r+1}}(f)$ we have a one-to-one mapping from $\mathbf{F}(n)$ onto $\Pi(n)$.

The following lemma is immediate (recall that $(p_f] = \{q\in\Pi(n) : q\le p_f\}$).

Lemma 3.6. *$p_f\in\Pi(n)$ has coordinates $(0,1)$ if and only if $f(g)\in\{0,1\}$ for all $g\in G$.*

As a consequence, the set $\Pi(n)$ has 2^n minimal elements.

Lemma 3.7. *For $1\le m\le 2n$, and $(m_1,\ldots,m_{r+1})\in N_m$, $p_f\in\Pi(n)$ has coordinates $(m,m_1,\ldots,m_{r+1})$ if and only if $N\cup M_1\subseteq f(G)\subseteq C_{m,m_1,\ldots,m_{r+1}}$.*

Proof From the proof of Lemma 3.4, p_f has coordinates $(m,m_1,\ldots,m_{r+1})$ if and only if $f\in\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$, and from the comment preceding that lemma, this is equivalent to $M_1\cup N\subseteq f(G)\subseteq C_{m,m_1,\ldots,m_{r+1}}$. ■

Remark 3.8. We know that if $f\in\mathbf{F}_{m,m_1,\ldots,m_{r+1}}(n)$, the extension homomorphism $\overline{f}$ and the natural homomorphism h from $L(n)$ into $L(n)/N(\overline{f})$ satisfy $h=\overline{f}$. Then if in $\Pi(n)$, $(p_f] = \{p_1,\ldots,p_m,p_{m+1}=p_f\}$, we have

$$\overline{f}(x) = \begin{cases} 1 & \text{if } x\in N(\overline{f}) \\ a_i & \text{if } x\in[p_i)\setminus[p_{i+1}),\ 1\le i\le m \\ 0 & \text{if } x\notin[p_1) \end{cases}.$$

The proof of the following lemma will be omitted since it is an adaptation of that of [1, Lemma 3.13].

We say q covers p if $p<q$ and $p\le r<q$ implies $r=p$.

Lemma 3.9. *If p, $q\in\Pi(n)$, q covers p if and only if the following conditions hold:*

(i) $p<q$,
(ii) $p\in\Pi_{m,m_1,\ldots,m_{r+1}}(n)$, $0\le m\le 2n-1$, $(m_1,\ldots,m_{r+1})\in N_m$.
(iii) $q\in\Pi_{m+1,m_1,\ldots,m_{r+1}+1}(n)$ *or* $q\in\Pi_{m+1,m_1,\ldots,m_{r+1},1}(n)$, $0\le m\le 2n-1$, $(m_1,\ldots,m_{r+1}+1)\in N_{m+1}$ *and* $(m_1,\ldots,m_{r+1},1)\in N_{m+1}$.

In the following theorem we denote $a_0 = 0$.

Theorem 3.10. *Let $f,\, h \in \mathbf{F}(n)$. Then $\psi(h) = p_h$ covers $\psi(f) = p_f$ if and only if $f \in \mathbf{F}_{m,m_1,\dots,m_{r+1}}(n)$, $h \in \mathbf{F}_{m+1,m_1,\dots,m_{r+1}+1}(n)$ or $h \in \mathbf{F}_{m+1,m_1,\dots,m_{r+1},1}(n)$, $0 \le m \le 2n-1$, $(m_1,\dots,m_{r+1}) \in N_m$, and for $g \in G$ the following conditions hold:*

(*I*) *$f(g) = a_i$ if and only if $h(g) = a_i$, $0 \le i \le m$.*
(*II*) *$f(g) = 1$ if and only if $h(g) = 1$ or $h(g) = a_{m+1}$.*

Proof Suppose that p_h covers p_f. The first part of the theorem is an immediate consequence of Lemma 3.9.

Since in $\Pi(n)$, $p_1 < \cdots < p_m < p_f < p_h$, we have

$$\overline{f}(x) = \overline{h}(x) = 0 \text{ if and only if } x \notin [p_1),$$
$$\overline{f}(x) = \overline{h}(x) = a_i,\ 1 \le i \le m \text{ if and only if } x \in [p_i) \setminus [p_{i+1}),\ 1 \le i \le m,$$
$$\overline{f}(x) = \overline{h}(x) = 1 \text{ if and only if } x \in [p_h),$$
$$\overline{f}(x) = 1 \text{ and } \overline{h}(x) = a_{m+1} \text{ if and only if } x \in [p_f) \setminus [p_h).$$

In particular, we have the conditions (I) and (II).

Conversely, let $f,\, h \in \mathbf{F}(n)$ be such that

$$f \in \mathbf{F}_{m,m_1,\dots,m_{r+1}}(n), \quad h \in \mathbf{F}_{m+1,m_1,\dots,m_{r+1}+1}(n) \cup \mathbf{F}_{m+1,m_1,\dots,m_{r+1},1}(n),$$

and satisfying (I) and (II). Then,

$$p_f \in \Pi_{m,m_1,\dots,m_{r+1}}(n) \text{ and } p_h \in \Pi_{m+1,m_1,\dots,m_{r+1}+1}(n) \cup \Pi_{m+1,m_1,\dots,m_{r+1},1}(n).$$

From Lemma 3.9, we must prove that $p_f < p_h$.

Consider in $\Pi(n)$

$$p_1 < \cdots < p_m < p_{m+1} = p_f$$

and

$$q_1 < \cdots < q_m < q_{m+1} < q_{m+2} = p_h$$

the chains $(p_f]$ and $(p_h]$ respectively and consider the following sets:

$$\begin{aligned} C_{m+2} &= [p_h) \cap [p_f), \\ C_{m+1} &= ([q_{m+1}) \setminus [q_{m+2})) \cap [p_{m+1}), \\ C_i &= ([q_i) \setminus [q_{i+1})) \cap ([p_i) \setminus [p_{i+1})),\ 1 \le i \le m, \\ C_0 &= (L(n) \setminus [q_1)) \cap (L(n) \setminus [p_1)) = L(n) \setminus ([q_1) \cup [p_1)). \end{aligned}$$

Then

$$z \in C_{m+2} \text{ if and only if } \overline{h}(z) = 1 \text{ and } \overline{f}(z) = 1,$$
$$z \in C_{m+1} \text{ if and only if } \overline{h}(z) = a_{m+1} \text{ and } \overline{f}(z) = 1,$$
$$z \in C_i \text{ if and only if } \overline{h}(z) = a_i \text{ and } \overline{f}(z) = a_i, 0 \le i \le m.$$

We have that C_{m+2} is a filter, C_0 is an ideal and C_i, $0 \le i \le m$, are nonempty sets, being that $a_i \in \overline{h}(L(n))$, $a_i \in \overline{f}(L(n))$, $0 \le i \le m$. C_{m+1} is also nonempty. Indeed, if $h \in \mathbf{F}_{m+1,m_1,\dots,m_{r+1}+1}(n)$, since $f \in \mathbf{F}_{m,m_1,\dots,m_{r+1}}(n)$, there is $g \in G$

such that $f(g) \neq h(g)$, then from (I) and (II), $\overline{h}(g) = a_{m+1}$ and $\overline{f}(g) = 1$, that is $g \in C_{m+1}$. If $h \in \mathbf{F}_{m+1,m_1,\ldots,m_{r+1},1}(n)$, there is $g \in G$ such that $\nabla f(g) \neq \nabla h(g)$, then from (I) and (II), $\overline{h}(\nabla g) = a_{m+1}$ and $\overline{f}(\nabla g) = 1$, that is $\nabla g \in C_{m+1}$.

It is clear that the sets C_i, $0 \leq i \leq m+2$, are pairwise disjoint. Observe that $C_{m+2} \cup C_{m+1} = [q_{m+1}) \cap [p_{m+1})$, and so it is a filter. Using these remarks it is a routine matter to show that the set $S = \cup_{i=0}^{m+2} C_i$ is a subalgebra of $L(n)$.

Let us see that $G \subseteq S$. If $g \in G$, $h(g) \in \{0, a_1, \ldots, a_m, a_{m+1}, 1\}$.

If $h(g) = 1$, $g \in [q_{m+2})$ and from (II), $f(g) = 1$, that is, $g \in [p_{m+1})$. Then $g \in C_{m+2} \subseteq S$.

If $h(g) = a_{m+1}$, $g \in [q_{m+1}) \setminus [q_{m+2})$ and from (II), $f(g) = 1$, that is $g \in [p_{m+1})$. So $g \in C_{m+1} \subseteq S$.

If $h(g) = a_i$, $0 \leq i \leq m$, then $g \in [q_i) \setminus [q_{i+1})$ and from (I), $f(g) = a_i$, that is, $g \in [p_i) \setminus [p_{i+1})$. Then $g \in C_i \subseteq S$.

Therefore, $G \subseteq S$ and consequently $S = L(n)$.

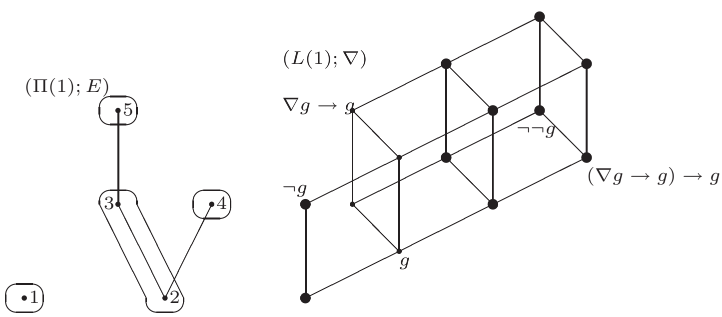
Then we can write, $[p_h) = [q_{m+2}) \cap L(n) = [q_{m+2}) \cap (\cup_{i=0}^{m+2} C_i) = \cup_{i=0}^{m+2}([q_{m+2}) \cap C_i) = [q_{m+2}) \cap C_{m+2} = [q_{m+2}) \cap [p_{m+1}) = [p_h) \cap [p_f)$.

Since $p_h \neq p_f$, we have $p_f < p_h$. ■

The previous theorem allows us to construct the ordered set of join-irreducible elements of the free algebra $L(n)$. By virtue of Lemma 3.6, there exists a one-to-one correspondence between the set of minimal elements of $\Pi(n)$ and the set of functions f from G into $\{0,1\}$. Since $(p]$ is a chain, for every $p \in \Pi(n)$, then $(f]$ is also a chain for $f \in \mathbf{F}(n)$. So, the ordered-connected components of $\mathbf{F}(n)$ are $[f)$, where f is minimal, that is, the order-connected components of $\mathbf{F}(n)$ are the sets $[f)$, where $f : G \to \{0,1\}$.

We have constructed the Q-Heyting space $(\Pi(n); E)$. The free algebra $L(n) \in \mathcal{C}$ with a finite set of generators of cardinality $n > 0$, is the algebra obtained from $(\Pi(n); E)$ considering the decreasing subsets of $\Pi(n)$ with the quantifier given by $\nabla_E U = \{q \in \Pi(n) : qEp \text{ for same } p \in U\}$, for each U decreasing set of $\Pi(n)$.

Example 3.11. In the next figure we give the free algebra $(L(1); \nabla)$ generated by an element g, and the ordered set $\Pi(1)$ of its join-irreducible elements, with the equivalence relation which determines the quantifier.



Where $1 = (0) \in \mathbf{F}_{0,1}(1)$, $2 = (1) \in \mathbf{F}_{0,1}(1)$, $3 = (a_1) \in \mathbf{F}_{1,2}(1)$, $4 = (a_1) \in \mathbf{F}_{1,1,1}(1)$ and $5 = (a_1) \in \mathbf{F}_{1,2,1}(1)$. We denote $\neg g = g \to 0$.

In the rest of this section we investigate the poset $\Pi(n)$ in order to obtain a recursive formula for the number of elements of $\Pi(n)$.

Let $K_j(n)$ be the family of order-connected components $[f)$, with f minimal, such that $|f^{-1}(1)| = j$, $0 \le j \le n$. It is clear that $|K_0(n)| = 1$, and if $K_0(n) = \{K\}$, then $|K| = 1$. In general, $|K_j(n)| = \binom{n}{j}$.

For a given j, all the order-connected components in $K_j(n)$ have the same number of elements. So if $K_j(n) = \left\{K_1, K_2, \dots, K_{\binom{n}{j}}\right\}$ and $N(n,j) = |K|$ for $K \in K_j(n)$, then

$$\left|\bigcup_{i=1}^{\binom{n}{j}} K_i\right| = \binom{n}{j} N(n,j).$$

We are going to determine $N(n,j)$.

Consider $K = [f) \in K_j(n)$. From Theorem 3.10, we know that $h \in [f)$ covers f if and only if $h \in \mathbf{F}_{1,2}(n)$ or $h \in \mathbf{F}_{1,1,1}(n)$ and

(I) $h(g) = 0$ if and only if $f(g) = 0$.

(II) $h(g) \in \{a_1, 1\}$ if and only if $f(g) = 1$.

In particular, there are $\binom{j}{1} + \cdots + \binom{j}{j} = 2^j - 1$ funtions h in $\mathbf{F}_{1,2}(n)$ covering f, and similarly there are $2^j - 1$ funtions h in $\mathbf{F}_{1,1,1}(n)$ covering f. So, there are $2(2^j - 1)$ functions h covering f, $2\binom{j}{t}$ of which satisfy $|h^{-1}(a_1)| = t$, $1 \le t \le j$.

In these conditions we have the following result.

Proposition 3.12. (1) *If $h \in \mathbf{F}_{1,1,1}(n)$, there exists $f_1 \in \mathbf{F}_{0,1}(n)$ with $[f_1) \in K_{j-t}(n)$ such that $[h)$ and $[f_1)$ are order-isomorphic.*

(2) *If $h \in \mathbf{F}_{1,2}(n)$ and h_1 covers h, with $h_1 \in \mathbf{F}_{2,2,1}(n)$ and $h_1(g) = h(g)$, for every $g \in G$, then there exists $f_1 \in \mathbf{F}_{0,1}(n)$, $[f_1) \in K_{j-t}(n)$ such that $[h) \setminus [h_1)$, $[h_1)$ and $[f_1)$ are order-isomorphic.*

Proof

(1) If $f_1 : G \to \{0,1\}$ is the function defined by:

$$(*)\ f_1(g) = \begin{cases} 1 & \text{if} \quad h(g) = 1 \\ 0 & \text{if} \quad h(g) = a_1 \quad \text{or} \quad h(g) = 0 \end{cases},$$

f_1 is clearly a minimal element of $\mathbf{F}(n)$, $f_1 \in \mathbf{F}_{0,1}(n)$ and $[f_1) \in K_{j-t}(n)$.

Let us see that $[h)$ and $[f_1)$ are order-isomorphic. Observe that if $u \in [h)$, then $u \in \mathbf{F}_{1+i,1,m_2,\dots,m_{r+1}}(n)$, where $0 \le i \le 2(j-t)$ and $1 \le r \le j-t+1$. We define $\alpha : [h) \to [f_1)$ by means of $\alpha(u) = v$, where

$$(**)\ v(g) = \begin{cases} 0 & \text{if} \quad u(g) = 0 \\ 1 & \text{if} \quad u(g) = 1 \\ a_{k-1} & \text{if} \quad u(g) = a_k \quad 1 \le k \le 1+i \end{cases},\ v \in \mathbf{F}_{i,m_2,\dots,m_{r+1}}(n).$$

Clearly α is an isomorphism.

(2) Observe that, if $u \in [h) \setminus [h_1)$, $u \neq h$, then $u \in \mathbf{F}_{1+i,m_1,\ldots,m_{r+1}}(n)$, $1 \leq i \leq 2(j-t)$, $m_1 \geq 2$ and $0 \leq r \leq j-t$. If f_1 is the function defined by $(*)$, then $[h) \setminus [h_1)$ and $[f_1)$ are order-isomorphic. Indeed, if we define $\alpha : [h) \setminus [h_1) \to [f_1)$ by means of $\alpha(u) = v$, where $u \in [h) \setminus [h_1)$ and v defined as in $(**)$, $v \in \mathbf{F}_{i,m_1-1,\ldots,m_{r+1}}(n)$ and it can be proved that α is an isomorphism.

Finally, consider $\beta : [h_1) \to [f_1)$ defined by $\beta(u) = v$, where $u \in [h_1)$, $u \in \mathbf{F}_{2+i,2,m_2,\ldots,m_{r+1}}(n)$, $0 \leq i \leq 2(j-t)$, $1 \leq r \leq j-t+1$ and

$$v(g) = \begin{cases} 0 & \text{if } \ u(g) = 0 \text{ or } u(g) = a_1 \\ 1 & \text{if } \ u(g) = 1 \\ a_{k-2} & \text{if } \ u(g) = a_k, \quad 2 \leq k \leq i+2 \end{cases}, \; v \in \mathbf{F}_{i,m_2,\ldots,m_{r+1}}(n).$$

Clearly β is an isomorphism.

■

From the previous proposition,

$$N(n,j) = \sum_{t=1}^{j} \left[3 \binom{j}{t} N(n, j-t) \right] + 1.$$

Therefore

$$\left| \bigcup_{i=1}^{\binom{n}{j}} K_i \right| = \binom{n}{j} \left[\sum_{t=1}^{j} \left[3 \binom{j}{t} N(n, j-t) \right] + 1 \right],$$

and then

$$|\mathbf{F}(n)| = |\Pi(n)| = 1 + \left[\sum_{j=1}^{n} \binom{n}{j} \left[\sum_{t=1}^{j} \left[3 \binom{j}{t} N(n, j-t) \right] + 1 \right] \right].$$

Acknowledgment: I gratefully acknowledge helpful comments of the referees. In particular, one of them suggested the algebraic proof of Lemma 2.5.

References

[1] M. Abad and L. Monteiro, *On free L-algebras*, Notas de Lógica Matemática 35(1987), 1-20.

[2] M. Abad and J. P. Díaz Varela, *Free Q-distributive lattices from meet semilattices*, Discrete Math. 224 (2000), 1-14.

[3] M. Abad, J. P. Díaz Varela, L. Rueda and A. Suardíaz, *Varieties of three-valued Heyting algebras with a quantifier*, Studia Logica 65 (2000), 181-198.

[4] M. E. Adams and W. Dziobiak, *Quasivarieties of distributive lattices with a quantifier*, Discrete Math. 135(1994), 12-28.

[5] R. Balbes and Ph. Dwinger, Distributive Lattices, University of Missouri Press, Columbia, Missouri, 1974.

[6] G. Bezhanishvili, *Locally finite varieties*, Algebra Universalis 46(2001), No.4, 531-548.

[7] R. Cignoli, *Quantifiers on distributive lattices*, Discrete Math. 96(1991), 183-197.

[8] R. Cignoli, *Free Q-distributive lattices*, Studia Logica 56(1996), 23-29.

[9] P. R. Halmos, *Algebraic Logic I. Monadic Boolean algebras*, Compositio Math. 12 (1955), 217-249.

[10] A. Monteiro, *Sur les Algèbres de Heyting Symétriques*, Portugaliae Mathematica, 39 (1980), 1-237.

[11] A. Petrovich, *Equations in the theory of Q-distributive lattices*, Discrete Math. 17(1997), 211-219.

[12] H. A. Priestley, *Ordered sets and duality for distributive lattices*, Ann. Discrete Math. 23(1984), 39-60.

[13] H. A. Priestley, *Natural dualities for varieties of distributive lattices with a quantifier*, Banach Center Publications, Volume 28, Warszawa 1993, 291-310.

[14] L. Rueda, *Algebras de Heyting lineales con un cuantificador* Ph. D. Thesis, Universidad Nacional del Sur, Bahía Blanca, 2000.

[15] L. Rueda, *Linear Heyting algebras with a quantifier*, Annals of Pure and Applied Logic 108 (2001), 327-343.

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Recibido: 12 de noviembre de 2007
Aceptado: 2 de junio de 2009