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FORMULAS FOR THE EULER-MASCHERONI CONSTANT

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ABSTRACT. We give several integral representations for the Euler-Mascheroni constant using a combinatorial identity for $\sum_{n=1}^{N} \frac{1}{(n+x)(n+y)}$. The derivation of this combinatorial identity is done in an elemental way.

Introduction. There exist many formulas for Euler-Mascheroni constant $\gamma = lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} - \ln(n)$, see for example [7], [6]. Indeed the irrationality of γ would follow from criteria given in [3] (see also [5]).

The purpose of this note is to give integral representations for γ which seem to be new. As usual we write $(x)_n = (x+n-1)(x+n-2) \ \dots \ x$.

Theorem. *If* $f(x,n) := \frac{3x}{2n} + 2 + \frac{n+x+\frac{1}{2}}{2n-1}$, $g(y,n) := \frac{2n}{n+y} - \frac{y}{2n} + \frac{n+\frac{1}{2}}{2n-1}$ *then*

$$i) \ \gamma = \sum_{n=1}^{\infty} (-1)^n \int_0^1 \frac{(-x)_n (x)_n}{(x)_{2n+1}} f(x,n)dx.$$

$$ii) \ \gamma = \sum_{n=1}^{\infty} (-1)^n \int_0^1 \frac{(-y)_n (y)_n}{(2n)!y} g(y,n)dy$$

Remark 1. The formulae stated converge more rapidly than the usual definition. For example, notice that for $1 \le n$

$$|\int_0^1 \frac{(-x)_n (x)_n}{(x)_{2n+1}} f(x,n)dx| \le \frac{6}{n^2 \binom{2n}{n}}$$

Indeed this follows form the fact that for $0 \le x \le 1$ one has $|\frac{(-x)_n (x)_n}{(x)_{2n+1}}| \le \frac{1}{n^2\binom{2n}{n}}$ and $f(x,n) \le f(1,n) \le 6$ if $1 \le n$.

Proof. We use the following formula: if $f_1(n,x,y) := \frac{(2n+x)}{(n+y)} + \frac{1}{(2n-1)}(n+x+\frac{1}{2}) + \frac{(x-y)}{2n}$ and $f_2(n,N,x,y) := \frac{1}{2(1-2n)} + \frac{N+x}{(1-2n)} - \frac{(x-y)}{2n}$, then

$$\sum_{n=1}^{N} \frac{1}{(n+x)(n+y)} = \tag{1}$$

$$\sum_{n=1}^{N} (-1)^{n-1} \frac{(1^2-(x-y)^2)\dots((n-1)^2-(x-y)^2)}{(2n+x)(2n-1+x)...(x+1)} f_1(n,x,y)+$$

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$$+\sum_{n=1}^{N}(-1)^{n-1}\frac{(1^2-(x-y)^2)\dots((n-1)^2-(x-y)^2)}{(N+n+x)(N+n-1+x)\dots(N-n+x+1))}f_2(n,N,x,y)=:$$

$$A_N(x,y)+B_N(x,y),$$

where we set $(1^2-(x-y)^2)\dots((n-1)^2-(x-y)^2)=1$ if $n=1$.

Recall the well-known representation

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)}=-\gamma+x\sum_{n=1}^{\infty}\frac{1}{n(n+x)}. \tag{2}$$

Notice that in (1), $B_N(x,y)\to 0$ as $N\to\infty$ if x and y are bounded . We prove this in a moment.

Now i) follows from integrating (2) from 0 to 1 and using (1) with $y=0$, letting $N\to\infty$. The first formula of ii) is proved in the same way putting $x=0$ in (1).

Now we prove (1): set $C_k:=C_k(n,x,y)=\frac{b_1\dots b_k}{(n+k+x)\dots(n-k+x)}\frac{1}{(n+y)}$; $C_0:=C_0(n,x,y)=\frac{1}{(n+x)(n+y)}$ where $b_k:=b_k(x,y)=(x-y)^2-k^2$, and define $b_1\dots b_{i-1}=1$ if $i=1$.

Add from $n=1$ to N the trivial identity $C_0-C_{n-1}=(C_0-C_1)+\dots+(C_{n-2}-C_{n-1})$ to get

$$\sum_{n=1}^{N}\frac{1}{(n+x)(n+y)}-\sum_{n=1}^{N}\frac{b_1\dots b_{n-1}}{(n+y).((2n-1+x)\dots(x+1))}=$$

$$\sum_{n=1}^{N}\sum_{k=1}^{n-1}(C_{k-1}-C_k)=\sum_{n=1}^{N}\sum_{k=1}^{n-1}\frac{b_1\dots b_{k-1}}{(n+k+x)\dots(n-k+x)}(n+2x-y)= \tag{3}$$

$$=\sum_{n=1}^{N}\sum_{k=1}^{n-1}\epsilon_{n,k}(x,y)-\epsilon_{n-1,k}(x,y)=\sum_{k=1}^{N}\epsilon_{N,k}(x,y)-\sum_{k=1}^{N}\epsilon_{k,k}(x,y),$$

with $\epsilon_{n,k}(x,y):=b_1\dots b_{k-1}.\frac{\left(\frac{1}{2(1-2k)}+\frac{n+x}{(1-2k)}+(x-y)(-\frac{1}{2k})\right)}{(n+k+x)\dots(n-k+1+x)}$.

From the equality of the first expression in (3) and the last one, we obtain (1).

We now prove that if $0\le x\le 1$, $0\le y\le 1$ then $B_N(x,y)\to 0$ as $N\to\infty$ (the proof for x,y bounded is similar). Indeed in this range of x and y one has

$$\mid B_N(x,y)\mid=O(\sum_{n=1}^{N}\frac{1}{\binom{N+n}{2n}\binom{2n}{n}}\frac{N}{n^3})=O(\sum_{1\le n\le N/4}\frac{1}{\binom{N+n}{2n}\binom{2n}{n}}\frac{N}{n^3})+O(1/N)$$

But $O(\sum_{1\le n\le N/4}\frac{1}{\binom{N+n}{2n}\binom{2n}{n}}\frac{N}{n^3})=O(\sum_{1\le n\le N/4}\frac{1}{N^2}\frac{N}{n^3})=O(1/N)$, where we have used $N(N+1)/2\le\binom{N+n}{2n}$ for $1\le n\le N/4$, $N\ge 4$.

This finishes the proof of the theorem. ■

A corollary of formula (1) is the following

Corollary. *Let* $h(n,M,y) := \frac{1/2+M+n}{2n-1} - \frac{y}{2n} + \frac{M+2n}{M+n+y}$. *Set*

$$D_M := (\sum_{n=1}^{M} \frac{1}{n}) - \ln(M+1) + \int_0^1 y \sum_{1\le n\le M/2} (-1)^{n-1} \frac{(1^2-y^2)\dots((n-1)^2-y^2)}{(2n)!\binom{2n+M}{2n}} h(n,M,y)dy$$

Then

$$\gamma - D_M = O(1/8^M)$$

Proof. From (1), letting $N \to \infty$ one gets

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)(n+y)} =$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1^2-(x-y)^2)\dots((n-1)^2-(x-y)^2)}{(2n+x)(2n-1+x)\dots(x+1)} f_1(n,x,y).$$

Now substitute y by $M+y$ and x by M to get for $0 \le y \le 1$

$$\sum_{n=M+1}^{\infty} \frac{1}{n(n+y)} =$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1^2-y^2)\dots((n-1)^2-y^2)}{(2n)!\binom{2n+M}{2n}} f_1(n,M,M+y) =$$

$$\sum_{1\le n\le M/2} + \sum_{M/2<n<\infty} = \sum_{1\le n\le M/2} + O(1/8^M)$$

Now the corollary follows from this last formula inserted in (recall formula (2))

$$\gamma = \int_0^1 y\{\sum_{n=1}^{M} \frac{1}{n(n+y)} + \sum_{n=M+1}^{\infty} \frac{1}{n(n+y)}\}dy = \sum_{n=1}^{M} \frac{1}{n} - \ln(M+1) + \int_0^1 y \sum_{n=M+1}^{\infty} \frac{1}{n(n+y)} dy,$$

observing that $h(n,M,y) = f_1(n,M,M+y)$. ■

Remark 2. The corollary stated seems to give clean approximation formulas. Indeed

$$D_1 = 1 - \ln 2, D_2 = \frac{283}{144} - \ln 4, D_3 = \frac{35}{16} - \ln 5,$$
$$D_4 = \frac{169553}{67200} - \ln 7, D_5 = \frac{192809}{72576} - \ln 8$$

Numerically we have checked that D_M is always of the form $r - \ln n$, with r a rational number and $2 \le n \le 2M$, $n \epsilon Z$.

Notice that (1) or derivates of (1) give formulae for Hurwitz-Riemann's zeta function $\sum_{n=1}^{\infty} \frac{1}{(n+x)^s}$ for $s = 2, 3, 4, \dots$.

We mention without proof that formula ii) of theorem 1 is equivalent to

$$\gamma = \int_0^1 \{y \ {}_3F_2[1/2, 1-y, 1+y; 3/2, 3/2; -1/4] + \frac{y}{1+y} \ {}_3F_2[1-y, 1+y, 1+y; 3/2, 2+y; -1/4]$$

$$-\frac{y^2}{4}\ {}_4F_3[1,1,1-y,1+y;3/2,2,2;-1/4]-\frac{1}{y}\operatorname{Sinh}^2(y.\operatorname{ArcSinh}(1/2))\}dy.$$

Here ${}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;z]=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{z^k}{k!}$ is the general hypergeometric function.

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