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HAAR SHIFTS, COMMUTATORS, AND HANKEL OPERATORS

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ABSTRACT. Hankel operators lie at the junction of analytic and real-variables. We will explore this junction, from the point of view of Haar shifts and commutators.

1. Haar Functions

We consider operators which satisfy invariance properties with respect to two well-known groups. The first group we take to the *translation* operators

$$\operatorname{Tr}_y f(x) := f(x-y)\,, \qquad y \in \mathbb{R}\,. \tag{1.1}$$

Note that formally, the adjoint operator is $(\operatorname{Tr}_y)^* = \operatorname{Tr}_{-y}$. The collection of operators $\{\operatorname{Tr}_y \,:\, y \in \mathbb{R}\}$ is a representation of the additive group $(\mathbb{R}, +)$.

It is an important, and very general principle that a linear operator L acting on some vector space of functions, which is assumed to commute with all translation operators, is in fact given as convolution, in general with respect to a measure or distribution, thus,

$$\mathrm{L}\, f(x) = \int f(x-y)\; \mu(dy)\,.$$

For instance, with the identity operator, μ would be the Dirac pointmass at the origin.

The second group is the set of *dilations on* L^p, given by

$$\operatorname{Dil}^{(p)}_\lambda f(x) := \lambda^{-1/p} f(x/\lambda)\,, \qquad 0 < \lambda, p < \infty\,. \tag{1.2}$$

Here, we make the definition so that $\|f\|_p = \|\operatorname{Dil}^{(p)}_\lambda f\|_p$. The *scale* of the dilation $\operatorname{Dil}^{(p)}_\lambda$ is said to be λ, and these operators are a representation of the multiplicative group $(\mathbb{R}_+, *)$. The Haar measure of of this group is dy/y.

Underlying this subject are the delicate interplay between local averages and differences. Some of this interplay can be encoded into the combinatorics of *grids*, especially the *dyadic grid,* defined to be $\mathcal{D} := \{2^k(j, j+1) \,:\, j, k \in \mathbb{Z}\}$.

The Haar functions are a remarkable class of functions indexed by the dyadic grid $\mathcal{D}$. Set

$$h(x) = -\mathbf{1}_{(-1/2,0)} + \mathbf{1}_{(0,1/2)}\,,$$

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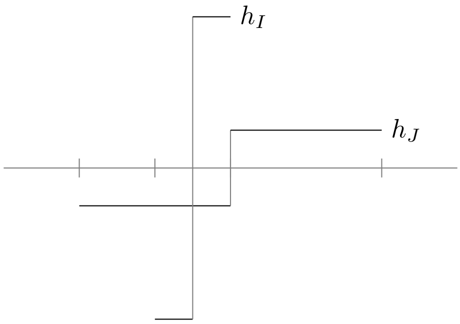

FIGURE 1. Two Haar functions.

a mean zero function supported on the interval $(-1/2, 1/2)$, taking two values, with L^2 norm equal to one. Define the *Haar function* (associated to interval I) to be

$$h_I := \mathrm{Dil}_I^2 \, h_I \tag{1.3}$$

$$\mathrm{Dil}_I^{(p)} := \mathrm{Tr}_{c(I)} \, \mathrm{Dil}_{|I|}^{(p)} , \qquad c(I) = \text{center of } I. \tag{1.4}$$

Here, we introduce the notion for the *Dilation associated with interval I.*

The Haar functions have profound properties, due to their connection to both analytical and probabilistic properties. An elemental property is that they form a basis for $L^2(\mathbb{R})$.

1.5. **Theorem.** *The set of functions $\{\mathbf{1}_{[0,1]}\} \cup \{h_I \,:\, I \in \mathcal{D},\ I \subset [0,1]\}$ form an orthonormal basis for $L^2([0,1])$. The set of functions $\{h_I \,:\, I \in \mathcal{D}\}$ form an orthonormal basis for $L^2(\mathbb{R})$.*

2. Paraproducts

Products, and certain kind of renormalized products are common objects. Let us explain the renormalized products in a very simple situation. We begin with the definition of a *paraproduct*, as a bilinear operator. Define

$$h_I^0 = h_I \,, \qquad h_I^1 = |h_I^0| = \mathrm{Dil}_I^2 \, \mathbf{1}_{[-1/2,1/2]} \,. \tag{2.1}$$

The superscript ^{0} indicates a mean-zero function, while the superscript ^{1} indicates a non-zero integral. Now define

$$\mathrm{P}^{\epsilon_1,\epsilon_2,\epsilon_3}(f_1, f_2) := \sum_{I \in \mathcal{D}} \frac{\langle f_1, h_I^{\epsilon_1} \rangle}{\sqrt{|I|}} \langle f_2, h_2^{\epsilon_2} \rangle h_I^{\epsilon_3} \,, \qquad \epsilon_j \in \{0,1\}. \tag{2.2}$$

For the most part, we consider cases where there is one choice of ϵ_j which is equal to one, but in considering fractional integrals, one considers examples where all ϵ_j are equal to one. The triple $(\epsilon_1, \epsilon_2, \epsilon_3)$ is the *signature* of the Paraproduct.

We have chosen this definition for specificity, but at the same time, it must be stressed that there is no canonical definition, and the presentation of a paraproduct can differ in a number of ways. Whatever the presentation, its single most important attribute is its signature. Indeed, in Proposition 5.3, we will see that a paraproduct arises from a computation that, while not of the form above, is clearly an operator of signature $(0,0,0)$. All the important prior work on commutators, see [1, 2, 6, 7, 9] can be interpreted in this notation. (The Lectures of M. Christ [5] are recommended as a guide to this literature.) For instance, in the notation of Coifman and Meyer [6, 7], a P_t denotes a ^{1}, while a Q_t denotes a ^{0}.

Why the name paraproduct? This is probably best explained by the identity

$$f_1 \cdot f_2 = \mathrm{P}^{1,0,0}(f_1, f_2) + \mathrm{P}^{0,0,1}(f_1, f_2) + \mathrm{P}^{0,1,0}(f_1, f_2)\,. \tag{2.3}$$

Thus, a product of two functions is a sum of three paraproducts. The three individual paraproducts in many respects behave like products, for instance we will see that there is a Hölder Inequality. And, very importantly, in certain instances they are *better* than a product.

To verify (2.3), let us first make the self-evident observation that

$$\frac{1}{|J|} \int_J g(y)\, dy = \frac{\langle g, h_I^1 \rangle}{\sqrt{|I|}} = \sum_{J \,:\, J \supsetneq I} \langle g, h_J \rangle h_J(I)\,, \tag{2.4}$$

where $h_J(I)$ is the (unique) value h_J takes on I. In (2.3), expand both f_1 and f_2 in the Haar basis,

$$f_1 \cdot f_2 = \Big\{ \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle h_I \Big\} \cdot \Big\{ \sum_{J \in \mathcal{D}} \langle f_2, h_J \rangle h_J \Big\}\,.$$

Split the resulting product into three sums, (1) $I = J$, (2) $I \subsetneq J$ (3) $J \subsetneq I$. In the first case,

$$\sum_{I,J \,:\, I=J} \langle f_1, h_I \rangle \langle f_2, h_J \rangle (h_I)^2 = \mathrm{P}^{0,0,1}(f_1, f_2)\,.$$

In the second case, use (2.4).

$$\begin{aligned} \sum_{I,J \,:\, I \subsetneq J} \langle f_1, h_I \rangle \langle f_2, h_J \rangle h_I \cdot \tfrac{1}{|I|} \int_I h_J(y)\, dy &= \sum_I \langle f_1, h_I \rangle \frac{\langle f_2, h_I^1 \rangle}{\sqrt{|I|}} h_I \\ &= \mathrm{P}^{0,1,0}(f_1, f_2)\,. \end{aligned}$$

And the third case is as in the second case, with the role of f_1 and f_2 switched.

A rudimentary property is that Paraproducts should respect Hölder's inequality, a matter that we turn to next. This Theorem is due to Coifman and Meyer [6, 7]. Also see [14, 17, 18].

2.5. **Theorem.** *Suppose at most one of $\epsilon_1, \epsilon_2, \epsilon_3$ are equal to one. We have the inequalities*

$$\|\mathrm{P}^{\epsilon_1,\epsilon_2,\epsilon_3}(f_1, f_2)\|_q \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}\,, \qquad 1 < p_1, p_2 < \infty\,,\ 1/q = 1/p_1 + 1/p_2\,. \tag{2.6}$$

3. Paraproducts and Carleson Embedding

We have indicated that Paraproducts are better than products in one way. These fundamental inequalities are the subject of this section. Let us define the notion of *(dyadic) Bounded Mean Oscillation*, BMO for short, by

$$\|f\|_{\mathrm{BMO}} = \sup_{J\in\mathcal{D}} \Bigl[|J|^{-1} \sum_{I\subset J} \langle f, h_R\rangle^2 \Bigr]^{1/2} . \tag{3.1}$$

3.2. **Theorem.** *Suppose that at exactly one of ϵ_2 and ϵ_3 are equal to* 1.

$$\bigl\| \mathrm{P}^{0,\epsilon_2,\epsilon_3}(f_1,\cdot) \bigr\|_{p\to p} \simeq \|f_1\|_{\mathrm{BMO}} , \qquad 1<p<\infty . \tag{3.3}$$

Indeed, we have

$$\bigl\| \mathrm{P}^{0,1,0}(f_1,\cdot) \bigr\|_{p\to p} \simeq \sup_J \bigl\| \mathrm{P}^{0,1,0}(f_1, |J|^{-1/p}\mathbf{1}_J) \bigr\|_p \simeq \|f_1\|_{\mathrm{BMO}} . \tag{3.4}$$

Here, we are treating the paraproduct as a linear operator on f_2, and showing that the operator norm is characterized by $\|f_1\|_{\mathrm{BMO}}$. Obviously, $\|f\|_{\mathrm{BMO}} \le 2\|f\|_\infty$, and again this a crucial point, there are unbounded functions with bounded mean oscillation, with the canonical example being $\ln x$. Thus, these paraproducts are, in a specific sense, better than pointwise products of functions.

Proof. The case $p=2$ is essential, and the only case considered in these notes. This particular case is frequently referred to as *Carleson Embedding*, a term that arises from the original application of the principal in the Corona Theorem.

Let us discuss the case of $\mathrm{P}^{0,1,0}$ in detail. Note that the dual of the operator

$$f_2 \longrightarrow \mathrm{P}^{0,1,0}(f_1,f_2) ,$$

that is we keep f_1 fixed, is the operator $\mathrm{P}^{0,0,1}(f_1,\cdot)$, so it is enough to consider $\mathrm{P}^{0,1,0}$ in the L^2 case.

One direction of the inequalities is as follows.

$$\begin{aligned} \|\mathrm{P}^{0,\epsilon_2,\epsilon_3}(f_1,\cdot)\|_{2\to2} &\ge \sup_J \|\mathrm{P}^{0,\epsilon_2,\epsilon_3}(f_1,h_J^1)\|_p \\ &\ge \|f_1\|_{\mathrm{BMO}} \end{aligned}$$

as is easy to see from inspection. Thus, the BMO lower bound on the operator norm arises solely from testing against normalized indicator sets.

For the reverse inequality, we compare to the Maximal Function. Fix f_1, f_2, and let

$$\mathcal{D}_k = \{ I\in\mathcal{D} \,:\, \frac{|\langle f_2, h_I\rangle|}{\sqrt{|I|}} \simeq 2^k \}$$

Let $\mathcal{D}_k^*$ be the maximal intervals in $\mathcal{D}_k$. The L^2-bound for the Maximal Function gives us

$$\sum_k 2^{2k} \sum_{I^*\in\mathcal{D}_k^*} |I^*| \lesssim \|\mathrm{M}\, f_2\|_2^2 \lesssim \|f\|_2^2 . \tag{3.5}$$

Then, for $I^* \in \mathcal{D}_k^*$ we have

$$\Bigl\| \sum_{\substack{I\in\mathcal{D}_k\\ I\subset I^*}} \langle f_1, h_I\rangle 2^k h_I \Bigr\|_2^2 = 2^{2k} \sum_{\substack{I\in\mathcal{D}_k\\ I\subset I^*}} \langle f_1, h_I\rangle^2 \le 2^{2k} \|f_1\|_{\mathrm{BMO}}^2 |I^*|$$

And so we are done by (3.5).

□

4. Hilbert Transform

It is a useful Theorem, one that we shall return to later, that the set of operators L that are bounded from $L^2(\mathbb{R})$ to itself, and commute with both translations and dilations have a special form. They are linear combinations of the Identity operator, and the *Hilbert* transform. The latter operator, fundamental to this study, is given by

$$\mathrm{H}\, f(x) := \text{p.v.} \int f(x-y)\, \frac{dy}{y}\,. \tag{4.1}$$

Here, we take the integral in the *principal value* sense, as the kernel $1/y$ is not integrable. Taking advantage of the fact that the kernel is odd, one can see that the limit below

$$\lim_{\epsilon\to 0} \int_{\epsilon<|y|<1/\epsilon} f(x-y)\, \frac{dy}{y} \tag{4.2}$$

exists for all x, provided f is a Schwartz function, say. Thus, H has an unambiguous definition on a dense class of functions, in all L^p. We shall take (4.2) as our general definition of principal value. The Hilbert transform is the canonical example of a *singular integral*, that is one that has to be defined in some principal value sense.

Observe that H, being convolution commutes with all translations. That is also commutes with all dilation operators follows from the observation that $1/y$ is a multiple of the multiplicative Haar measure. It can also be recovered in a remarkably transparent way from a simple to define operator based upon the Haar functions. Let us define

$$g = -\mathbf{1}_{(-1/4,-1/4)} + \mathbf{1}_{(-1/4,1/4)} - \mathbf{1}_{(1/4,1/2)} \tag{4.3}$$

$$= 2^{-1/2}\{h_{(-1/2,0)} + h_{(0,1/2)}\} \tag{4.4}$$

$$\mathfrak{H} f = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle g_I\,, \tag{4.5}$$

where as before, $g_I = \mathrm{Dil}_I^{(2)}\, g$. It is clear that $\mathfrak{H}$ is a bounded operator on L^2. What is surprising is that that it can be used to recover the Hilbert transform exactly. The succinct motivation for this definition is that $\mathrm{H}(\sin) = \cos$, so that if h_I is a local sine, then g_I is a local cosine.

4.6. **Theorem** (S. Petermichl [20]). *There is a non-zero constant c so that*

$$H = c \lim_{Y\to\infty} \int_0^Y \int_1^2 \mathrm{Tr}_y\, \mathrm{Dil}_\lambda^{(2)}\, \mathfrak{H}\, \mathrm{Dil}_{1/\lambda}^{(2)}\, \mathrm{Tr}_{-y}\, \frac{d\lambda}{\lambda}\, \frac{dy}{Y}\,. \tag{4.7}$$

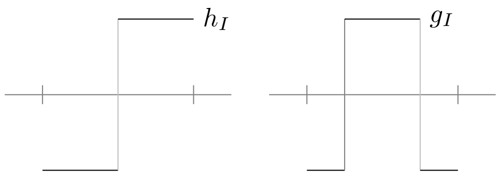

FIGURE 2. A Haar function h_I and its dual g_I.

As a Corollary, we have the estimate $\|H\|_2 \lesssim 1$, as $\mathfrak{H}$ is clearly bounded on L^2.

The operator $\mathfrak{h}$ is referred to as a *Haar shift* or as a *dyadic shift* ([22]). Certain canonical singular integrals, like the Hilbert, Riesz and Beurling transform admit remarkably simple Haar shift variants, which fact can be used to prove a range of deep results. See for instance [8, 21, 23, 24]. For applications of this notion to more general singular integrals, see [13, Section 4].

Proof. Consider the limit on the right in (4.7). This is seen to exist for each $x \in \mathbb{R}$ for Schwartz functions f. While this is elementary, it might be useful for us to define the auxiliary operators

$$\mathrm{T}_j\, f := \sum_{\substack{I \in \mathcal{D} \\ |I| \le 2^j}} \langle f, h_j \rangle g_j \,.$$

The individual terms of this series are rapidly convergent. As $|I|$ becomes small, one uses the smoothness of the function f. As $|I|$ becomes large, one uses the fact that f is integrable, and decays rapidly. Call the limit $\widetilde{\mathrm{H}} f$.

Let us also note that the operator T_j is invariant under translations by an integer multiple of 2^j. Thus, the auxiliary operator

$$2^{-j} \int_0^{2^j} \mathrm{Tr}_{-t}\, f \,\mathrm{Tr}_t \; dt$$

will be translation invariant. Thus $\widetilde{\mathrm{H}}$ is convolution with respect to a linear functional on Schwartz functions, namely a distribution.

Concerning dilations, T is invariant under dilations by a power of 2. Now, dilations form a group under multiplication on $\mathbb{R}_+$, and this group has Haar measure $d\delta/\delta$ so that the operator below will commute with all dilations.

$$\int_0^1 \mathrm{Dil}^2_{1/\delta}\, \mathrm{T}\, \mathrm{Dil}^2_\delta \,\frac{d\delta}{\delta}$$

Thus, $\widetilde{\mathrm{H}}$ commutes with all dilations.

Therefore, $\widetilde{\mathrm{H}}$ must be a linear combination of a Dirac delta function and convolution with $1/y$. (The function $1/|y|$ is also invariant under dilations, but the inner product with this function is not a linear functional on distributions.) Applying $\widetilde{\mathrm{H}}$ to a non negative Schwartz function yields a function with zero mean. Thus, $\widetilde{\mathrm{H}}$

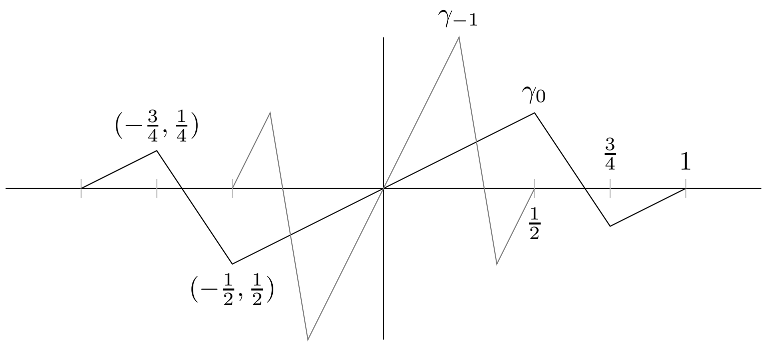

Figure 3. The graph of γ_0 and γ_{-1}.

must be a multiple of convolution with $1/y$, and we only need to see that it is non zero multiple.

Let us set G_j to be the operator

$$\mathrm{G}_j\, f := \int_0^{2^j} \mathrm{Tran}_t \sum_{\substack{I\in\mathcal{D} \\ |I|=2^j}} \langle \mathrm{Tran}_{-t}\, f, h_I\rangle h_I\, \frac{dt}{2^j}\,.$$

This operator translates with translation and hence is convolution. We can write $\mathrm{G}_j\, f = \gamma_j * f$. By the dilation invariance of the Haar functions, we will have $\gamma_j = \mathrm{Dil}^1_{2^j}\, \gamma_0$. A short calculation shows that

$$\gamma_0(y) = \int_0^1 h_I(y+t)h_I(y)\; dt$$

This function is depicted in Figure 3. Certainly the operator $\sum_j G_j$ is convolution with $\sum_j \gamma_j(x)$. This kernel is odd and is strictly positive on $[0,\infty)$. This finishes our proof.

□

5. Commutator Bound

We would like to explain a classical result on commutators.

5.1. **Theorem.** *For a function b, and $1 < p < \infty$ we have the equivalence*

$$\|[b,\mathrm{H}]\|_{p\to p} \simeq \|b\|_{\mathrm{BMO}}\,,$$

where this is the non-dyadic BMO *given by*

$$\sup_{I \text{ interval}} \left[|I|^{-1} \int_I \left| f - |I|^{-1} \int_I f(y)\; dy \right| \; dx \right]^{1/2}.$$

We refer to this as a classical result, as it can be derived from the Nehari theorem, as we will explain below. The lower bound on the operator norm is found by applying the commutator to normalized indicators of integrals, and we suppress the proof.

Both bounds are very easy, if one appeals to the Nehari Theorem. See our comments on Nehari's Theorem below. But, in many circumstances, different proofs admit different modifications, and so we present a 'real-variable' proof, deriving the upper bound from the Haar shift, and the Paraproduct bound in a transparent way.

Replacing the Hilbert transform by the Haar Shift, we prove

$$\|[b, \mathfrak{H}]\|_{p \to p} \lesssim \|b\|_{\mathrm{BMO}} \tag{5.2}$$

The last norm is dyadic-BMO, which is strictly smaller than non-dyadic BMO. But Theorem 4.6 requires that we use all translates and dilates to recover the Hilbert transform, and so the non-dyadic BMO norm will be invariant under these translations and dilations.

The Proposition is that $[b, \mathfrak{H}]$ can be explicitly computed as a sum of Paraproducts which are bounded.

5.3. **Proposition.** *We have*

$$[b, \mathfrak{H}]f = \mathrm{P}^{0,1,0}(b, \mathfrak{H}f) - \mathfrak{H} \circ \mathrm{P}^{0,1,0}(b, f) \tag{5.4}$$

$$+ \mathrm{P}^{0,0,1}(b, \mathfrak{H}f) - \mathfrak{H} \circ \mathrm{P}^{0,0,1}(b, f) \tag{5.5}$$

$$+ \widetilde{\mathrm{P}}^{0,0,0}(b, f)\,. \tag{5.6}$$

In the last line, $\widetilde{\mathrm{P}}^{0,0,0}(b, f)$ *is defined to be*

$$\widetilde{\mathrm{P}}^{0,0,0}(b, f) = \sum_{I \in \mathcal{D}} \frac{\langle b, h_I^0 \rangle}{\sqrt{I}} \langle f, h_I^0 \rangle (h^0_{I_{\mathrm{left}}} + h^0_{I_{\mathrm{right}}})\,.$$

Each of the five terms on the right are L^p-bounded operators on f, provided $b \in \mathrm{BMO}$, so that the upper bound on the commutator norm in Theorem 5.1 follows as an easy corollary. The paraproduct in (5.6) does not hew to our narrow definition of a Paraproduct, but it is degenerate in that it is of signature $(0, 0, 0)$, and thus even easier to control than the other terms.

Proof. Now, $[b, \mathfrak{H}]f = b\mathfrak{H}f - \mathfrak{H}(b \cdot f)$. Apply (2.3) to both of these products. We see that

$$[b, \mathfrak{H}]f = \sum_{\vec{\epsilon} = (1,0,0),(0,1,0),(0,0,1)} \mathrm{P}^{\vec{\epsilon}}(b, \mathfrak{H}f) - \mathfrak{H}\,\mathrm{P}^{\vec{\epsilon}}(b, f)\,.$$

The choices of $\vec{\epsilon} = (0, 1, 0), (0, 0, 1)$ lead to the first four terms on the right in (5.4).

The terms that require more care are the difference of the two terms in which a 1 falls on a b. In fact, we will have

$$\mathrm{P}^{\vec{\epsilon}}(b, \mathfrak{H}f) - \mathfrak{H}\,\mathrm{P}^{\vec{\epsilon}}(b, f) = \widetilde{\mathrm{P}}^{0,0,0}(b, f)\,.$$

To analyze this difference quickly, let us write

$$\langle \mathfrak{H} f, h_I \rangle = \operatorname{sgn}(I) \langle f, h_{\operatorname{Par}(I)} \rangle$$

where $\operatorname{Par}(I)$ is the 'parent' of I, and $\operatorname{sgn}(I) = 1$ if I is the left-half of $\operatorname{Par}(I)$, and is otherwise -1. This definition follows immediately from the definition of g_I in (4.3). Now observe that

$$\begin{aligned} \langle \mathrm{P}^{\vec{\epsilon}}(b, \mathfrak{H} f), h_I^0 \rangle &= \langle \mathfrak{H} f, \mathrm{P}^{\vec{\epsilon}}(b, h_I^0) \rangle \\ &= \frac{\langle b, h_I^1 \rangle}{\sqrt{|I|}} \cdot \langle \mathfrak{H} f, h_I^0 \rangle \\ &= \langle f, h^0_{\operatorname{Par}(I),} \rangle \operatorname{sgn}(I) \frac{\langle b, h_I^1 \rangle}{\sqrt{|I|}} \end{aligned}$$

And on the other hand, we have

$$\langle \mathfrak{H} P^{1,0,0}(b, f), h_I \rangle = \frac{\langle b, h^1_{\operatorname{Par}(I)} \rangle}{\sqrt{|\operatorname{Par}(I)|}} \operatorname{sgn}(I) \langle f, h^0_{\operatorname{Par}(I)} \rangle$$

Comparing these two terms, we see that we should examine the term that falls on b. But a calculation shows that

$$\sqrt{2} h_I^1 - h^1_{\operatorname{Par}(I)} = -\operatorname{sgn}(I) h^0_{\operatorname{Par}(I)}.$$

Thus, we see that this difference is of the claimed form.

□

6. The Nehari Theorem

We define Hankel operators on the real line. On $L^2(\mathbb{R})$, we have the Fourier transform

$$\widehat{f}(\xi) = \int f(x) \, \mathrm{e}^{-i\xi x} \; dx \, .$$

Define the orthogonal projections onto positive and negative frequencies

$$\mathrm{P}_{\pm} f(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}_\pm} \widehat{f}(\xi) \, \mathrm{e}^{i\xi x} \; dx \, .$$

Define Hardy spaces $H^2(\mathbb{R}) \stackrel{\text{def}}{=} \mathrm{P}_+ L^2(\mathbb{R})$. Functions $f \in H^2(\mathbb{R})$ admit an analytic extension to the upper half plane $\mathbb{C}_+$. As in the case of the disk, it is convenient to refer to functions in $H^2(\mathbb{R})$ as *analytic*.

A *Hankel operator with symbol* b is then a linear operator from $H^2(\mathbb{C}_+)$ to $H^2_+(\mathbb{C}_+)$ given by $\mathrm{H}_b \, \varphi \stackrel{\text{def}}{=} \mathrm{P}_+ \mathrm{M}_b \overline{\varphi}$. This only depends on the analytic part of b. It is typical to include the notation $\mathbb{C}_+$ to emphasize the connection with analytic function theory, and the relevant domain upon which one is working. Below, we will suppress this notation.

The result that we are interested in is:

6.1. **Nehari's Theorem** ([19]). *The Hankel operator* H_b *is bounded from* H^2 *to* H^2 *iff there is a bounded function* β *with* $P_+b = P_+\beta$*. Moreover,*

$$\|\mathrm{H}_b\| = \inf_{\beta\,:\,\mathrm{P}_+\,\beta=\mathrm{P}_+\,b} \|\beta\|_\infty \tag{6.2}$$

Less exactly, we have $\|\mathrm{H}_b\| \simeq \|\mathrm{P}_+\,b\|_{\mathrm{BMO}}$*, where we can take the last norm to be non-dyadic* BMO*.*

This Theorem was proved in 1954, appealing to the following classical fact.

6.3. **Proposition.** *Each function* $f \in H^1$ *is a product of functions* $f_1, f_2 \in H^2$*. In particular,* f_1 *and* f_2 *can be chosen so that*

$$\|f\|_{H^1} = \|f_1\|_{H^2}\|f_2\|_{H^2}$$

Given a bounded Hankel operator H_b, we want to show that we can construct a bounded function β so that the analytic part of b and β agree.

This proof is the one found by Nehari [19]. We begin with a basic computation of the norm of the Hankel operator H_b:

$$\begin{aligned}
\|\mathrm{H}_b\| &= \sup_{\|\varphi\|_{H^2}=1}\ \sup_{\|\psi\|_{H^2}=1} \int \mathrm{H}_b\,\psi\cdot\overline{\varphi}\;dx \\
&= \sup_{\|\varphi\|_{H^2}=1}\ \sup_{\|\psi\|_{H^2}=1} \int \mathrm{P}_+\,\mathrm{M}_b\,\overline{\psi}\cdot\overline{\varphi}\;dx \\
&= \sup_{\|\varphi\|_{H^2}=1}\ \sup_{\|\psi\|_{H^2}=1} \int (\mathrm{P}_+\,b)\overline{\psi\cdot\varphi}\;dx \\
&= \sup_{\|\varphi\|_{H^2}=1}\ \sup_{\|\psi\|_{H^2}=1} \langle(\mathrm{P}_+\,b), \psi\cdot\varphi\rangle
\end{aligned} \tag{6.4}$$

But, the $H^1 = H^2 \cdot H^2$, as we recalled in Proposition 6.3. We read from the equality above that the analytic part of b defines a bounded linear functional on H^1 a subspace of L^1.

The Hahn Banach Theorem applies, giving us an extension of this linear functional to all of L^1, with the same norm. But a linear function on L^1 is a bounded function, hence we have constructed a bounded function β with the same analytic part as b.

The calculation (6.4) is more general than what we have indicated here, a point that we return to below.

Let us remark that the H^p variant of Nehari's Theorem holds. On the one hand, one has $H^p \cdot H^{p'} \subset H^1$, so that the upper bound on the norm $\|\mathrm{H}_b\|_{H^p\to H^p}$ follows. On the other, Proposition 6.3 extends to the H^p-$H^{p'}$ factorization, whence the same argument for the lower bound can be used.

There is a close connection between commutators $[b, \mathrm{H}]$ and Hankel operators. Indeed, we have

$$[b, \mathrm{H}] = [b, \mathrm{H}] = 2\,\mathrm{P}_-\,b\,\mathrm{P}_+ - 2\,\mathrm{P}_+\,b\,\mathrm{P}_-\ . \tag{6.5}$$

The two terms on the right can be recognized as two Hankel operators with orthogonal domains and ranges. Indeed, keep in mind the elementary identities $\mathrm{P}_+^2 = \mathrm{P}_+$, $\mathrm{P}_+ \mathrm{P}_- = 0$, $\mathrm{H} = \mathrm{I} - 2\,\mathrm{P}_-$, and $[b, \mathrm{I}] = 0$. Then, observe

$$\begin{aligned}\mathrm{P}_+[b,\mathrm{H}]\,\mathrm{P}_- &= -2\,\mathrm{P}_+[b,\mathrm{P}_-]\,\mathrm{P}_- \\ &= -\,\mathrm{P}_+\,b\,\mathrm{P}_-^2 + \mathrm{P}_+\,\mathrm{P}_-\,b\,\mathrm{P}_- = -\,\mathrm{P}_+\,b\,\mathrm{P}_- \\ \mathrm{P}_-[b,\mathrm{H}]\,\mathrm{P}_- &= \mathrm{P}_-[b,\mathrm{P}_+]\,\mathrm{P}_- = 0\end{aligned}$$

There are two additional calculations, which are dual to these and we omit them.

7. Further Applications

The author came to the Haar shift approach to the commutator from studies of Multi-Parameter Nehari Theorem [10, 16]. The paper [15] surveys these two papers. This subject requires an understanding of the structure of product BMO that goes beyond the foundational papers of S.-Y. Chang and R. Fefferman [3, 4] on the subject.

In particular, as in Nehari's Theorem, the upper bound on the Hankel operator is trivial, as one direction of the factorization result is trivial: $H^2 \cdot H^2 \subset H^1$. The lower bound is however very far from trivial, as factorization is known to fail in product Hardy spaces. Indeed, Nehari's theorem is equivalent to so-called *weak* factorization, one of the points of interest in the Theorem. See [10, 15, 16] for a discussion of this important obstruction to the proof, and relevant references.

There are different critical ingredients needed for the proof of the lower bound. One of them is a very precise quantitative understanding of the proof of the upper bound. It is at this point that the techniques indicated in this paper are essential. The fundamentals of the multi-parameter Paraproduct theory were developed by Journé [11, 12]. The subject has been revisited recently to develop novel Leibnitz rules by Muscalu, Pipher, Tao and Thiele [17, 18]. Also see [14].

An influential extension of the classical Nehari Theorem to a real-variable setting was found by Coifman, Rochberg and Weiss [9]: Real-valued BMO on $\mathbb{R}^n$ can be characterized in terms of commutators with Riesz Transforms. The real-variable setting implies a complete loss of analyticity, making neither bound easy. Recently, the author, with Pipher, Petermichl and Wick, have proved the multi-parameter extension of the this result [13]. This paper includes in it a quantification of the Proposition 5.3 to the higher dimensional setting, for (smooth) Calderón Zygmund operators T: $[b, T]$ is a sum of bounded paraproducts, a crucial Lemma in that paper. See [13, Proposition 5.11]. Such an observation is not new, as it can be found in e. g. [1] for instance. Still the presentation of Proposition 5.3 in this paper is as simple as any the author is aware of in the literature.

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