

## NOTES ON THE SPACES OF BILINEAR MULTIPLIERS

OSCAR BLASCO

ABSTRACT. A locally integrable function  $m(\xi, \eta)$  defined on  $\mathbb{R}^n \times \mathbb{R}^n$  is said to be a bilinear multiplier on  $\mathbb{R}^n$  of type  $(p_1, p_2, p_3)$  if

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta$$

defines a bounded bilinear operator from  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$  to  $L^{p_3}(\mathbb{R}^n)$ . The study of the basic properties of such spaces is investigated and several methods of constructing examples of bilinear multipliers are provided. The special case where  $m(\xi, \eta) = M(\xi - \eta)$  for a given  $M$  defined on  $\mathbb{R}^n$  is also addressed.

### 1. INTRODUCTION.

Throughout the paper  $C_{00}(\mathbb{R}^n)$  denotes the space of continuous functions defined in  $\mathbb{R}^n$  with compact support,  $\mathcal{S}(\mathbb{R}^n)$  denotes the Schwartz class on  $\mathbb{R}^n$ , i.e.  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $f \in C^\infty(\mathbb{R}^n)$  and  $x^\alpha \frac{\partial^{|\beta|} f(x)}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}$  is bounded for any  $\beta = (\beta_1, \dots, \beta_n)$  and  $\alpha = (\alpha_1, \dots, \alpha_n)$  where  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$  and  $|\beta| = \beta_1 + \dots + \beta_n$  and  $\mathcal{P}(\mathbb{R}^n)$  stands for the set of functions in  $\mathcal{S}(\mathbb{R}^n)$  such that  $\hat{f} \in C_{00}(\mathbb{R}^n)$  where  $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$ .

We shall use the notation  $\mathcal{M}_{p,q}(\mathbb{R}^n)$  (respect.  $\tilde{\mathcal{M}}_{p,q}(\mathbb{R}^n)$ ), for  $1 \leq p, q \leq \infty$ , for the space of distributions  $u \in \mathcal{S}'(\mathbb{R}^n)$  such that  $u * \phi \in L^q(\mathbb{R}^n)$  for all  $\phi \in L^p(\mathbb{R}^n)$  (respect. for the space of bounded functions  $m$  such that  $T_m$  defines a bounded operator from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$  where  $\widehat{T_m(\phi)}(\xi) = m(\xi) \hat{f}(\xi)$ .) We endow the space  $\tilde{\mathcal{M}}_{p,q}(\mathbb{R}^n)$  with the “norm” of the operator  $T_m$ , that is  $\|m\|_{p,q} = \|T_m\|$ .

Let us start off by mentioning some well known properties of the space of linear multipliers (see [1, 14]):  $\mathcal{M}_{p,q}(\mathbb{R}^n) = \{0\}$  whenever  $q < p$ ,  $\mathcal{M}_{p,q}(\mathbb{R}^n) = \mathcal{M}_{q',p'}(\mathbb{R}^n)$  for  $1 < p \leq q < \infty$  and for  $1 \leq p \leq 2$ ,  $\mathcal{M}_{1,1}(\mathbb{R}^n) \subset \mathcal{M}_{p,p}(\mathbb{R}^n) \subset \mathcal{M}_{2,2}(\mathbb{R}^n)$ . We also have the identifications

$$\tilde{\mathcal{M}}_{2,2}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n),$$

$$\mathcal{M}_{1,q}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : u \in L^q(\mathbb{R}^n)\}, 1 < q < \infty,$$

$$\mathcal{M}_{1,1}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : u = \mu \in M(\mathbb{R}^n)\}.$$

In this paper we shall be dealing with their bilinear analogues.

---

*Key words and phrases.* spaces of bilinear multipliers, bilinear Hilbert transform, bilinear fractional transform.

Partially supported by Proyecto MTM2008-04594/MTM.

**Definition 1.1.** Let  $1 \leq p_1, p_2 \leq \infty$  and  $0 < p_3 \leq \infty$  and let  $m(\xi, \eta)$  be a locally integrable function on  $\mathbb{R}^n \times \mathbb{R}^n$ . Define

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta$$

for  $f, g \in \mathcal{P}(\mathbb{R}^n)$ .

$m$  is said to be a bilinear multiplier on  $\mathbb{R}^n$  of type  $(p_1, p_2, p_3)$  if there exists  $C > 0$  such that

$$\|B_m(f, g)\|_{p_3} \leq C \|f\|_{p_1} \|g\|_{p_2}$$

for any  $f, g \in \mathcal{P}(\mathbb{R}^n)$ , i.e.  $B_m$  extends to a bounded bilinear operator from  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$  to  $L^{p_3}(\mathbb{R}^n)$  (where we replace  $L^\infty(\mathbb{R}^n)$  for  $C_0(\mathbb{R}^n)$  in the case  $p_i = \infty$  for  $i = 1, 2$ ).

We write  $\mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R}^n)$  for the space of bilinear multipliers of type  $(p_1, p_2, p_3)$  and  $\|m\|_{p_1, p_2, p_3} = \|B_m\|$ .

The study of bilinear multipliers for smooth symbols (where  $m(\xi, \eta)$  is a “nice” regular function) goes back to the work by R.R. Coifman and Y. Meyer in [6].

Particularly simple examples are the following bilinear convolution-type operators: For a given  $K \in L^1_{loc}(\mathbb{R}^n)$  we define

$$C_K(f, g)(x) = \int_{\mathbb{R}^n} f(x - y)g(x + y)K(y)dy \tag{1}$$

for  $f$  and  $g$  belonging to  $C_{00}(\mathbb{R}^n)$ .

If  $K \in L^1(\mathbb{R}^n)$  then  $m(\xi, \eta) = \hat{K}(\xi - \eta)$  defines a multiplier in  $\mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R}^n)$  for  $1/p_1 + 1/p_2 = 1/p_3$  if  $p_3 \geq 1$  and  $\|m\|_{p_1, p_2, p_3} \leq \|K\|_1$ .

Indeed, for  $f$  and  $g \in \mathcal{S}(\mathbb{R}^n)$ , one has  $f(x - y) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \langle x - y, \xi \rangle} d\xi$  and  $g(x + y) = \int_{\mathbb{R}^n} \hat{g}(\eta) e^{2\pi i \langle x + y, \eta \rangle} d\eta$ . Hence we have

$$\begin{aligned} C_K(f, g)(x) &= \int_{\mathbb{R}^n} f(x - y)g(x + y)K(y)dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) K(y) e^{2\pi i \langle x - y, \xi \rangle} e^{2\pi i \langle x + y, \eta \rangle} d\xi d\eta dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \left( \int_{\mathbb{R}^n} K(y) e^{-2\pi i \langle \xi - \eta, y \rangle} dy \right) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{g}(\eta) \hat{f}(\xi) \hat{K}(\xi - \eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta. \end{aligned}$$

This motivates the introduction of the following class of multipliers.

**Definition 1.2.** Let  $1 \leq p_1, p_2 \leq \infty$  and  $0 < p_3 \leq \infty$ . We denote by  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R}^n)$  the space of measurable functions  $M : \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $m(\xi, \eta) = M(\xi - \eta) \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R}^n)$ , that is to say

$$B_M(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) M(\xi - \eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta$$

extends to a bounded bilinear map from  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$  into  $L^{p_3}(\mathbb{R}^n)$ . We keep the notation  $\|M\|_{p_1, p_2, p_3} = \|B_M\|$ .

It was only in the last decade that the cases  $M_0(x) = \frac{1}{|x|^{1-\alpha}}$  were shown to define bilinear multipliers of type  $(p_1, p_2, p_3)$  for  $1/p_3 = 1/p_1 + 1/p_2 - \alpha$  for  $1 < p_1, p_2 < \infty$  and  $0 < \alpha < 1/p_1 + 1/p_2$  (see (3) in Theorem 1.3) and, in the case  $n = 1$ ,  $M_1(x) = -isign(x)$  was shown to define a bilinear multiplier of type  $(p_1, p_2, p_3)$  for  $1/p_3 = 1/p_1 + 1/p_2$  for  $1 < p_1, p_2 < \infty$  and  $p_3 > 2/3$  (see (2) in Theorem 1.3). These two main examples correspond to the following bilinear operators: the *bilinear fractional integral* defined by

$$I_\alpha(f, g)(x) = \int_{\mathbb{R}} \frac{f(x-y)g(x+y)}{|y|^{1-\alpha}} dy, \quad 0 < \alpha < 1$$

and the *bilinear Hilbert transform* defined by

$$H(f, g)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(x-y)g(x+y)}{y} dy$$

respectively.

Let us collect the results about their boundedness which are known nowadays.

**Theorem 1.3.** *Let  $1 < p_1, p_2 < \infty$ ,  $0 < \alpha < 1/p_1 + 1/p_2$ ,  $1/q = 1/p_1 + 1/p_2 - \alpha$ ,  $1/p_3 = 1/p_1 + 1/p_2$  and  $2/3 < p_3 < \infty$ . Then there exist constants  $A$  and  $B$  such that*

$$\|H(f, g)\|_{p_3} \leq A \|f\|_{p_1} \|g\|_{p_2} \quad (\text{Lacey-Thiele, [12, 13]}), \quad (2)$$

$$\|I_\alpha(f, g)\|_q \leq B \|f\|_{p_1} \|g\|_{p_2}. \quad (\text{Kenig-Stein [11], Grafakos-Kalton [10]}). \quad (3)$$

Our objective is to study the basic properties of the classes  $\mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\mathcal{M}_{p_1, p_2, p_3}(\mathbb{R})$ , to find examples of bilinear multipliers in these classes and to get methods to produce new ones.

As usual, if  $f \in L^1(\mathbb{R}^n)$  we denote by  $\tau_x$ ,  $M_x$  and  $D_t^p$  the translation  $\tau_x f(y) = f(y - x)$  for  $x \in \mathbb{R}^n$ , the modulation  $M_x f(y) = e^{2\pi i \langle x, y \rangle} f(y)$  and the dilation  $D_t^p f(x) = t^{-n/p} f(\frac{x}{t})$  for  $0 < p, t < \infty$ .

With this notation out of the way one has, for  $1 \leq p \leq \infty$  and  $1/p + 1/p' = 1$ ,

$$(\widehat{\tau_x f})(\xi) = M_{-x} \hat{f}(\xi), \quad (\widehat{M_x f})(\xi) = \tau_x \hat{f}(\xi), \quad (\widehat{D_t^p f})(\xi) = D_{t^{-1}}^{p'} \hat{f}(\xi). \quad (4)$$

Clearly  $\tau_x, M_x$  and  $D_t^p$  are isometries on  $L^p(\mathbb{R}^n)$  for any  $0 < p \leq \infty$ .

Although most of the results presented in what follows have a formulation in  $n \geq 1$  we shall restrict ourselves to the case  $n = 1$  for simplicity. The reader is referred to [2, 3, 4, 5, 7] for several similar results on other groups, and to find same methods of transference.

## 2. BILINEAR MULTIPLIERS: THE BASICS

Let us start by pointing out a characterization, for  $p_3 \geq 1$ , in terms of the duality, whose elementary proof is left to the reader.

**Proposition 2.1.** *Let  $1 \leq p_3 \leq \infty$ . Then  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  if and only if there exists  $C > 0$  such that*

$$\left| \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq C \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3'}$$

for all  $f, g, h \in \mathcal{P}(\mathbb{R})$ .

We now present a basic example of a bilinear multiplier. For a Borel regular measure in  $\mathbb{R}$   $\mu$  we denote  $\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} d\mu(x)$  its Fourier transform.

**Proposition 2.2.** *Let  $p_3 \geq 1$  and  $1/p_1 + 1/p_2 = 1/p_3$  and let  $m(\xi, \eta) = \hat{\mu}(\alpha\xi + \beta\eta)$  where  $\mu$  is a Borel regular measure in  $\mathbb{R}$  and  $(\alpha, \beta) \in \mathbb{R}^2$ . Then  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|m\|_{p_1, p_2, p_3} \leq \|\mu\|_1$ .*

*Proof.* Let us first rewrite the value  $B_m(f, g)$  as follows:

$$\begin{aligned} B_m(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{\mu}(\alpha\xi + \beta\eta)e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\left(\int_{\mathbb{R}} e^{-2\pi i(\alpha\xi+\beta\eta)t} d\mu(t)\right)e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(x-\alpha t)\xi} e^{2\pi i(x-\beta t)\eta} d\xi d\eta\right) d\mu(t) \\ &= \int_{\mathbb{R}} f(x - \alpha t)g(x - \beta t) d\mu(t). \end{aligned}$$

Hence, using Minkowski's inequality, one has

$$\begin{aligned} \|B_m(f, g)\|_{p_3} &\leq \int_{\mathbb{R}} \|f(\cdot - \alpha t)g(\cdot - \beta t)\|_{p_3} d|\mu|(t) \\ &\leq \int_{\mathbb{R}} \|f(\cdot - \alpha t)\|_{p_1} \|g(\cdot - \beta t)\|_{p_2} d|\mu|(t) \\ &= \|f\|_{p_1} \|g\|_{p_2} \int_{\mathbb{R}} d|\mu|(t) = \|\mu\|_1 \|f\|_{p_1} \|g\|_{p_2}. \end{aligned}$$

■

Let us start with some elementary properties of the bilinear multipliers when composing with translations, modulations and dilations.

**Proposition 2.3.** *Let  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ .*

- (a) *If  $m_1 \in \tilde{\mathcal{M}}_{s_1, p_1}(\mathbb{R})$  and  $m_2 \in \tilde{\mathcal{M}}_{s_2, p_2}(\mathbb{R})$  then  $m_1(\xi)m(\xi, \eta)m_2(\eta) \in \mathcal{BM}_{(s_1, s_2, p_3)}(\mathbb{R})$ . Moreover*

$$\|m_1 m m_2\|_{s_1, s_2, p_3} \leq \|m_1\|_{s_1, p_1} \|m\|_{p_1, p_2, p_3} \|m_2\|_{s_2, p_2}$$

- (b)  *$\tau_{(\xi_0, \eta_0)} m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  for each  $(\xi_0, \eta_0) \in \mathbb{R}^2$  and*

$$\|\tau_{(\xi_0, \eta_0)} m\|_{p_1, p_2, p_3} = \|m\|_{p_1, p_2, p_3}.$$

- (c)  *$M_{(\xi_0, \eta_0)} m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  for each  $(\xi_0, \eta_0) \in \mathbb{R}^2$  and*

$$\|M_{(\xi_0, \eta_0)} m\|_{p_1, p_2, p_3} = \|m\|_{p_1, p_2, p_3}$$

- (d) *If  $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$  and  $0 < t < \infty$  then  $D_t^q m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  and*

$$\|D_t^q m\|_{p_1, p_2, p_3} = \|m\|_{p_1, p_2, p_3}.$$

*Proof.* Use (4) to deduce the following formulas

$$B_{m_1 m m_2}(f, g) = B_m(T_{m_1} f, T_{m_2} g). \tag{5}$$

$$B_{\tau_{(\xi_0, \eta_0)} m}(f, g) = M_{\xi_0 + \eta_0} B_m(M_{-\xi_0} f, M_{-\eta_0} g). \tag{6}$$

$$B_{M_{(\xi_0, \eta_0)} m}(f, g) = B_m(\tau_{-\xi_0} f, \tau_{-\eta_0} g). \tag{7}$$

$$B_m(D_t^{p_1} f, D_t^{p_2} g) = D_t^{p_3} B_{D_t^q m}(f, g). \tag{8}$$

Let us check only the validity of last one. The other ones follow easily from the previous facts.

$$\begin{aligned} B_m(D_t^{p_1} f, D_t^{p_2} g)(x) &= \int_{\mathbb{R}^2} t^{\frac{1}{p_1}} \hat{f}(t\xi) t^{\frac{1}{p_2}} \hat{g}(t\eta) m(\xi, \eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}^2} t^{\frac{1}{p_1}} \hat{f}(\xi) t^{\frac{1}{p_2}} \hat{g}(\eta) m\left(\frac{\xi}{t}, \frac{\eta}{t}\right) e^{2\pi i(\xi+\eta)\frac{x}{t}} t^{-2} d\xi d\eta \\ &= t^{-\frac{1}{p_3}} \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{g}(\eta) t^{-\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_3}} m\left(\frac{\xi}{t}, \frac{\eta}{t}\right) e^{2\pi i(\xi+\eta)\frac{x}{t}} d\xi d\eta \\ &= D_t^{p_3} B_{D_t^q m}(f, g)(x). \end{aligned}$$

■

From (8) we can see that the condition  $1/p_1 + 1/p_2 = 1/p_3$  is also connected to the homogeneity of the symbol.

**Proposition 2.4.** *Let  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  such that  $m(t\xi, t\eta) = m(\xi, \eta)$  for any  $t > 0$ . Then  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ .*

*Proof.* From assumption  $D_t^\infty m = m$ . Using (8) we have

$$B_m(D_t^{p_1} f, D_t^{p_2} g) = t^{1/p_3 - (1/p_1 + 1/p_2)} D_t^{p_3} B_m(f, g)$$

and therefore

$$\begin{aligned} \|B_m(f, g)\|_{p_3} &= \|D_t^{p_3} B_m(f, g)\|_{p_3} \\ &= t^{-1/p_3 + (1/p_1 + 1/p_2)} \|B_m(D^{p_1} f, D_t^{p_2} g)\|_{p_3} \\ &\leq t^{-1/p_3 + (1/p_1 + 1/p_2)} \|B_m\| \|f\|_{p_1} \|g\|_{p_2}. \end{aligned}$$

For this to hold for any  $0 < t < \infty$  one needs  $1/p_1 + 1/p_2 = 1/p_3$ .

■

Let us combine the previous results to get new bilinear multipliers from a given one.

**Proposition 2.5.** *Let  $p_3 \geq 1$  and  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ .*

- (a) *If  $Q = [a, b] \times [c, d]$  and  $1 < p_1, p_2 < \infty$  then  $m\chi_Q \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|m\chi_Q\|_{p_1, p_2, p_3} \leq C \|m\|_{p_1, p_2, p_3}$ .*
- (b) *If  $\Phi \in L^1(\mathbb{R}^2)$  then  $\Phi * m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|\Phi * m\|_{p_1, p_2, p_3} \leq \|\Phi\|_1 \|m\|_{p_1, p_2, p_3}$ .*
- (c) *If  $\Phi \in L^1(\mathbb{R}^2)$  then  $\hat{\Phi} m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|\hat{\Phi} m\|_{p_1, p_2, p_3} \leq \|\Phi\|_1 \|m\|_{p_1, p_2, p_3}$ .*
- (d) *If  $\psi \in L^1(\mathbb{R}^+, t^{\frac{1}{p_3} - (\frac{1}{p_1} + \frac{1}{p_2})})$  then  $m_\psi(\xi, \eta) = \int_0^\infty m(t\xi, t\eta) \psi(t) dt \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ . Moreover  $\|m_\psi\|_{p_1, p_2, p_3} \leq \|\psi\|_1 \|m\|_{p_1, p_2, p_3}$ .*

*Proof.* (a) Use that  $\chi_{[a,b]} \in \tilde{\mathcal{M}}_{p_1,p_1}$  for  $1 < p_1 < \infty$  and  $\chi_{[c,d]} \in \tilde{\mathcal{M}}_{p_2,p_2}$  for  $1 < p_2 < \infty$  together with Proposition 2.3 part (a).

(b) Note that

$$\begin{aligned} B_{\Phi * m}(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta) \left( \int_{\mathbb{R}^2} m(\xi - u, \eta - v)\Phi(u, v)dudv \right) e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)m(\xi - u, \eta - v)e^{2\pi i(\xi+\eta)x} d\xi d\eta \right) \Phi(u, v)dudv \\ &= \int_{\mathbb{R}^2} B_{\tau_{(u,v)}m}(f, g)(x)\Phi(u, v)dudv. \end{aligned}$$

From the vector-valued Minkowski inequality and Proposition 2.3 part (b), we have

$$\begin{aligned} \|B_{\Phi * m}(f, g)\|_{p_3} &\leq \int_{\mathbb{R}^2} \|B_{\tau_{(u,v)}m}(f, g)\|_{p_3} |\Phi(u, v)| dudv \\ &\leq \|m\|_{p_1,p_2,p_3} \|f\|_{p_1} \|g\|_{p_2} \|\Phi\|_1. \end{aligned}$$

(c) Observe that

$$\begin{aligned} B_{\hat{\Phi}m}(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta) \left( \int_{\mathbb{R}^2} M_{(-u,-v)}m(\xi, \eta)\Phi(u, v)dudv \right) e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}^2} B_{M_{(-u,-v)}m}(f, g)(x)\Phi(u, v)dudv. \end{aligned}$$

Argue as above, using now Proposition 2.3 part (c), to conclude the result.

(d) Use now Proposition 2.3 part (d), for  $\frac{1}{p_3} - (\frac{1}{p_1} + \frac{1}{p_2}) = -\frac{2}{q}$ ,

$$\begin{aligned} B_{m_\psi}(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta) \left( \int_0^\infty D_{t^{-1}}^q m(\xi, \eta)t^{-2/q}\psi(t)dt \right) e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_0^\infty B_{D_{t^{-1}}^q m}(f, g)(x)t^{-2/q}\psi(t)dt. \end{aligned}$$

■

With all these procedures we have several useful methods to produce multipliers in  $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ . Let us mention one application of each of them.

**Example 2.6.** (1) If  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ ,  $m_1 \in \tilde{\mathcal{M}}_{(p_1,p_1)}$  and  $m_2 \in \tilde{\mathcal{M}}_{(p_2,p_2)}$  then  $m(\xi, \eta) = m_1(\xi)m_2(\eta) \in \mathcal{BM}_{p_1,p_2,p_3}$ .  
 (2) If  $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ ,  $p_3 \geq 1$  and  $Q_1, Q_2$  are bounded measurable sets in  $\mathbb{R}$  then

$$\frac{1}{|Q_1||Q_2|} \int_{Q_1 \times Q_2} m(\xi + u, \eta + v)dudv \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}).$$

(3) If  $\Phi \in L^1(\mathbb{R}^2)$  then  $\hat{\Phi} \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$  for  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ ,  $p_3 \geq 1$ .

(4) If  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ ,  $|\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}| < 1$  then

$$m_1(\xi, \eta) = \int_0^\infty m(t\xi, t\eta) \frac{dt}{1+t^2} \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R}).$$

A combination of the previous results gives the following examples of bilinear multipliers in  $\mathcal{BM}_{(1,1,p_3)}(\mathbb{R})$  whose proof is left to the reader.

**Corollary 2.7.** Let  $\Phi \in L^1(\mathbb{R}^2)$ ,  $\psi_1 \in L^{p_1}(\mathbb{R})$  and  $\psi_2 \in L^{p_2}(\mathbb{R})$  and  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \leq 1$  then

$$m(\xi, \eta) = \hat{\psi}_1(\xi)\hat{\Phi}(\xi, \eta)\hat{\psi}_2(\eta) \in \mathcal{BM}_{(1,1,p_3)}(\mathbb{R}).$$

Let us use Proposition 2.1 and interpolation to get a sufficient integrability condition to guarantee that  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ .

**Theorem 2.8.** Let  $1 \leq p_1, p_2 \leq p \leq 2$  and  $p_3 \geq p'$  such that  $\frac{1}{p_1} + \frac{1}{p_2} - \frac{2}{p} = \frac{1}{p_3}$ . If  $m \in L^p(\mathbb{R}^2)$  then  $m \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ .

*Proof.* Let us show first that  $m \in \mathcal{BM}_{(p,p,\infty)}(\mathbb{R})$ . Let  $f, g \in L^p(\mathbb{R})$  and  $h \in L^1(\mathbb{R})$ . Using Hölder and Hausdorff-Young's inequalities one gets

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi + \eta)m(\xi, \eta)d\xi d\eta \right| &\leq \|m\|_{L^p(\mathbb{R}^2)}\|\hat{h}\|_\infty\|\hat{f}\|_{p'}\|\hat{g}\|_{p'} \\ &\leq \|m\|_{L^p(\mathbb{R}^2)}\|h\|_1\|f\|_p\|g\|_p. \end{aligned}$$

Similarly, changing the variables  $\xi + \eta = u$ ,  $\xi = -v$ , one has

$$\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi + \eta)m(\xi, \eta)d\xi d\eta = \int_{\mathbb{R}^2} \hat{f}(-v)\hat{g}(u+v)\hat{h}(u)m(-v, u+v)dvdu.$$

An argument as above gives also the estimate

$$\left| \int_{\mathbb{R}^2} \hat{f}(-v)\hat{g}(u+v)\hat{h}(u)m(-v, u+v)dvdu \right| \leq \|m\|_{L^p(\mathbb{R}^2)}\|g\|_1\|f\|_p\|h\|_p.$$

This shows that  $m \in \mathcal{BM}_{(p,1,p')}(\mathbb{R})$ . A similar argument shows also that  $m \in \mathcal{BM}_{(1,p,p')}(\mathbb{R})$ .

Given  $1 \leq \tilde{p}_1 \leq p$  and  $p' \leq \tilde{p}_3 \leq \infty$  with  $\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_3} = \frac{1}{p}$  we have  $0 \leq \theta \leq 1$  such that  $\frac{1}{\tilde{p}_1} = \frac{1-\theta}{p} + \frac{\theta}{\tilde{p}_3}$  and  $\frac{1}{\tilde{p}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{p'}$ . Hence, by interpolation,  $m \in \mathcal{BM}_{(\tilde{p}_1, p, \tilde{p}_3)}(\mathbb{R})$ .

Similarly  $m \in \mathcal{BM}_{(p, \tilde{p}_2, \tilde{q}_3)}(\mathbb{R})$  whenever  $1 \leq \tilde{p}_2 \leq p$  and  $p' \leq \tilde{q}_3 \leq \infty$  with  $\frac{1}{\tilde{p}_2} - \frac{1}{\tilde{q}_3} = \frac{1}{p}$ .

To finish the proof we observe that if  $1 < p_1 < p$  and  $1 < p_2 < p$  then for each  $0 < \theta < 1$  there exist  $1 \leq \tilde{p}_1 \leq p_1 < p$  and  $1 \leq \tilde{p}_2 \leq p_2 < p$  such that

$$\frac{1}{p_1} - \frac{1}{p} = (1 - \theta)\left(\frac{1}{\tilde{p}_1} - \frac{1}{p}\right), \quad \frac{1}{p_2} - \frac{1}{p} = \theta\left(\frac{1}{\tilde{p}_2} - \frac{1}{p}\right).$$

Denoting  $\tilde{p}_3, \tilde{q}_3$  the values such that  $\frac{1}{\tilde{p}_2} - \frac{1}{p} = \frac{1}{\tilde{p}_3}$  and  $\frac{1}{p_2} - \frac{1}{p} = \frac{1}{\tilde{q}_3}$  one obtains that

$$\frac{1}{p_1} = \frac{(1 - \theta)}{\tilde{p}_1} + \frac{\theta}{p}, \quad \frac{1}{p_2} = \frac{(1 - \theta)}{p} + \frac{\theta}{\tilde{p}_1}, \quad \frac{1}{p_3} = \frac{(1 - \theta)}{\tilde{p}_3} + \frac{\theta}{\tilde{q}_3}.$$

Hence the result follows again from interpolation between the last ones. ■

3. BILINEAR MULTIPLIERS DEFINED BY FUNCTIONS IN ONE VARIABLE

Let us restrict ourselves to a smaller family of multipliers where  $m(\xi, \eta) = M(\xi - \eta)$  for some  $M$  defined in  $\mathbb{R}$ . These multipliers satisfy

$$B_m(M_x f, M_x g) = M_{2x} B_m(f, g). \tag{9}$$

As in the introduction we use the notation  $\tilde{\mathcal{M}}_{p_1, p_2, p_3}(\mathbb{R})$  for the space of functions  $M : \mathbb{R} \rightarrow \mathbb{C}$  such that  $m(\xi, \eta) = M(\xi - \eta) \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ , that is to say

$$B_M(f, g)(x) = \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{g}(\eta) M(\xi - \eta) e^{2\pi i(\xi + \eta)x} d\xi d\eta,$$

defined for  $\hat{f}$  and  $\hat{g}$  compactly supported, extends to a bounded bilinear map from  $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$  into  $L^{p_3}(\mathbb{R})$ . We keep the notation  $\|M\|_{p_1, p_2, p_3} = \|B_M\|$ .

The reader should be aware that the starting assumption on the function  $M$  is only relevant for the definition of the bilinear mapping to make sense when acting on certain classes of “nice” functions. Then a density argument allows to extend functions belonging to Lebesgue spaces. We would like to point out the following observation.

**Remark 3.1.** *If  $M_n \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  are functions such that  $M_n(x) \rightarrow M(x)$  a.e and  $\sup_n \|M_n\| < \infty$  then  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|M\|_{p_1, p_2, p_3} \leq \sup_n \|M_n\|_{p_1, p_2, p_3}$ . Indeed, this fact follows from Fatou’s lemma, since*

$$\|B_M(f, g)\|_{p_3} \leq \liminf \|B_{M_n}(f, g)\|_{p_3} \leq \sup_n \|M_n\|_{p_1, p_2, p_3} \|f\|_{p_1} \|g\|_{p_2}.$$

**Remark 3.2.** *The case  $M(x) = \frac{1}{|x|^{1-\alpha}}$  (and even the  $n$ -dimensional case) corresponds to the bilinear fractional integral. This was first shown by C. Kenig and E. Stein in [11] to belong to  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  for any  $1 < p_1, p_2 < \infty$ ,  $0 < \alpha < 1/p_1 + 1/p_2$  and  $1/p_1 + 1/p_2 = 1/p_3 - \alpha$ . Another very important and non trivial example is the bilinear Hilbert transform, given by  $M(x) = -isign(x)$ , which was shown by M. Lacey and C.Thiele in [12, 13] to belong to  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  for any  $1 < p_1, p_2 < \infty$ ,  $1/p_1 + 1/p_2 = 1/p_3$  and  $p_3 > 2/3$ . These results were extended to other cases in [10] and [8, 9] respectively.*

We start reformulating the definition of this class of bilinear multipliers.

**Proposition 3.3.** *Let  $M \in L^1_{loc}(\mathbb{R})$ ,  $f, g \in \mathcal{P}(\mathbb{R})$ . Then*

$$B_M(f, g)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \hat{f}\left(\frac{u+v}{2}\right) \hat{g}\left(\frac{u-v}{2}\right) M(v) e^{2\pi i u x} du dv \tag{10}$$

$$B_M(f, g)(-x) = \int_{\mathbb{R}} (\widehat{\tau_x g * M})(\xi) \widehat{\tau_x f}(\xi) d\xi. \tag{11}$$

$$\widehat{B_M(f, g)}(x) = \frac{1}{2} C_M(\widehat{D^1_{1/2} f}, \widehat{D^1_{1/2} g})(x). \tag{12}$$



*Proof.* (10) follows changing variables.

To show (11) observe that

$$\begin{aligned} B_M(f, g)(-x) &= \int_{\mathbb{R}^2} \widehat{\tau_x f}(\xi) \widehat{\tau_x g}(\eta) M(\xi - \eta) d\xi d\eta \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \widehat{\tau_x g}(\eta) M(\xi - \eta) d\eta \right) \widehat{\tau_x f}(\xi) d\xi \\ &= \int_{\mathbb{R}} (\widehat{\tau_x g} * M)(\xi) \widehat{\tau_x f}(\xi) d\xi \end{aligned}$$

Finally, using (10), we have

$$\begin{aligned} B_M(f, g)(x) &= \frac{1}{2} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \hat{f}\left(\frac{u+v}{2}\right) \hat{g}\left(\frac{u-v}{2}\right) M(v) dv \right) e^{2\pi i u x} du \\ &= \frac{1}{2} \int_{\mathbb{R}} C_M(D_{1/2}^\infty \hat{f}, D_{1/2}^\infty \hat{g})(u) e^{2\pi i u x} du. \end{aligned}$$

This implies (12). ■

For symbols  $M$  which are integrable we can write  $B_M$  in terms  $C_K$  for some kernel  $K$ .

**Proposition 3.4.** *Let  $M \in L^1(\mathbb{R})$  and set  $K(t) = \hat{M}(-t)$ . Then  $B_M = C_K$ , i.e*

$$B_M(f, g) = \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt$$

*Proof.*

$$\begin{aligned} C_K(f, g)(x) &= \int_{\mathbb{R}} f(x-t)g(x+t)\hat{M}(-t)dt \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(x-t)\xi}e^{2\pi i(x+t)\eta}d\xi d\eta \right) \hat{M}(-t)dt \\ &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta) \left( \int_{\mathbb{R}} \hat{M}(t)e^{2\pi i(\xi-\eta)t}dt \right) e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= B_M(f, g)(x). \end{aligned}$$

■

This class does have much richer properties than  $\mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$ . As above use the notation  $f_t(x) = D_t^1 f(x) = \frac{1}{t} f\left(\frac{x}{t}\right)$  for a function  $f$  defined in  $\mathbb{R}$ . The following facts are immediate.

$$\tau_y B_M(f, g) = B_M(\tau_y f, \tau_y g), y \in \mathbb{R}. \tag{13}$$

$$M_{2y} B_M(f, g) = B_M(M_y f, M_y g), y \in \mathbb{R}. \tag{14}$$

$$(B_M(f, g))_t = B_{D_{t^{-1}}^1} M(f_t, g_t), t > 0. \tag{15}$$

When specializing the properties obtained for  $m(\xi, \eta)$  to the case  $M(\xi - \eta)$  we get the following facts:

$$B_M(\tau_{-y}f, \tau_y g) = B_{M_y M}(f, g), y \in \mathbb{R}. \tag{16}$$

$$B_M(M_y f, M_{-y}g) = B_{\tau_{2y}M}(f, g), y \in \mathbb{R}. \tag{17}$$

For  $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$  we have

$$B_M(D_t^{p_1} f, D_t^{p_2} g) = D_t^{p_3} B_{D_t^q M}(f, g), t > 0. \tag{18}$$

As in the previous section we can generate new multipliers in  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ .

**Proposition 3.5.** *Let  $p_3 \geq 1$ ,  $\phi \in L^1(\mathbb{R})$  and  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ . Then*

- (a)  $\phi * M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|\phi * M\|_{p_1, p_2, p_3} \leq \|\phi\|_1 \|M\|_{p_1, p_2, p_3}$ .
- (b)  $\hat{\phi}M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|\hat{\phi}M\|_{p_1, p_2, p_3} \leq \|\phi\|_1 \|M\|_{p_1, p_2, p_3}$ .
- (c) If  $\psi \in L^1(\mathbb{R}^+, t^{\frac{1}{p_3} - (\frac{1}{p_1} + \frac{1}{p_2})})$  then  $M_\psi(\xi) = \int_0^\infty M(t\xi)\psi(t)dt \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ .  
Moreover  $\|M_\psi\|_{p_1, p_2, p_3} \leq \|\psi\|_1 \|M\|_{p_1, p_2, p_3}$ .

*Proof.* (a) Apply Minkowski's inequality to the following fact:

$$\begin{aligned} B_{\phi * M}(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta) \left( \int_{\mathbb{R}} M(\xi - \eta - u)\phi(u)du \right) e^{2\pi i(\xi + \eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} \widehat{M_{-u}f}(\xi)\hat{g}(\eta)M(\xi - \eta) e^{2\pi i(\xi + \eta)x} d\xi d\eta \right) e^{2\pi iux} \phi(u)du \\ &= \int_{\mathbb{R}} M_u B_M(M_{-u}f, g)(x)\phi(u)du. \end{aligned}$$

(b) Observe that

$$\begin{aligned} B_{\hat{\phi}M}(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta) \left( \int_{\mathbb{R}} (M_{-u}m)(\xi - \eta)\phi(u)du \right) e^{2\pi i(\xi + \eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}^2} B_{M_u M}(f, g)(x)\phi(u)du. \end{aligned}$$

Use now Minkowski's again and (17).

(c) Write  $\frac{1}{p_3} - (\frac{1}{p_1} + \frac{1}{p_2}) = -\frac{1}{q}$ ,

$$\begin{aligned} B_{M_\psi}(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta) \left( \int_0^\infty D_{t^{-1}}^q M(\xi - \eta)t^{-1/q}\psi(t)dt \right) e^{2\pi i(\xi + \eta)x} d\xi d\eta \\ &= \int_0^\infty B_{D_{t^{-1}}^q M}(f, g)(x)t^{-1/q}\psi(t)dt. \end{aligned}$$

The result follows from (18) and Minkowski's again. ■

$$M(\xi - \eta)$$

**Proposition 3.6.** *Let  $p_3 \geq 1$ ,  $\phi \in L^1(\mathbb{R})$  and  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ . Then  $m(\xi, \eta) = M(\xi - \eta)\hat{\phi}(\xi + \eta) \in \mathcal{BM}_{(p_1, p_2, p_3)}(\mathbb{R})$  and  $\|m\|_{p_1, p_2, p_3} \leq \|\phi\|_1 \|M\|_{p_1, p_2, p_3}$ .*

*Proof.* Apply Young's inequality to the following fact:

$$\begin{aligned} B_m(f, g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi - \eta)\left(\int_{\mathbb{R}} \phi(y)e^{-2\pi i(\xi+\eta)y} dy\right)e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi - \eta)e^{2\pi i(\xi+\eta)(x-y)} d\xi d\eta\right)\phi(y)dy \\ &= \phi * B_M(f, g)(x). \end{aligned}$$

■

Let us show that the classes  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  are reduced to  $\{0\}$  for some values of the parameters.

**Theorem 3.7.** *Let  $p_3 \geq 1$  such that  $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{p_3}$ . Then  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R}) = \{0\}$ .*

*Proof.* Let  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ . Using Proposition 3.5 we have that  $\phi * M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  for any  $\phi$  continuous with compact support. Hence we may assume that  $M \in L^1(\mathbb{R})$ . Using Proposition 3.4 one has that

$$B_M(f, g)(x) = \int_{(x+B_R)\cap(-x+B_R)} f(x-t)g(x+t)\hat{M}(-t)dt$$

for any  $f$  and  $g$  continuous functions supported in a ball  $B_R = \{|x| \leq R\}$ . Therefore one concludes that  $supp(B_M(f, g)) \subset B_{2R}$  in such a case. On the other hand for any compactly supported function  $h$ ,  $0 < p < \infty$  and  $y$  big enough one can say that  $\|h \pm \tau_y f\|_p = 2^{1/p}\|f\|_p$ .

Consider  $\{r_k\}$  the Rademacher system in  $[0, 1]$  and observe that, for each  $N \in \mathbb{N}$  and  $y \in \mathbb{R}$ , the orthonormality of the system gives

$$\int_0^1 B_M\left(\sum_{k=0}^N r_k(t)\tau_{ky}f, \sum_{k=0}^N r_k(t)\tau_{ky}f\right)dt = \sum_{k=0}^N B_M(\tau_{ky}f, \tau_{ky}g)$$

Therefore, since  $\sum_{k=0}^N B_M(\tau_{ky}f, \tau_{ky}g) = \sum_{k=0}^N \tau_{ky}B_M(f, g)$ , we conclude that for  $y$  big enough

$$\left\|\sum_{k=0}^N \tau_{ky}B_M(f, g)\right\|_{p_3}^{p_3} = (N + 1)\|B_M(f, g)\|_{p_3}^{p_3}.$$

On the other hand, for  $p_3 \geq 1$ ,

$$\begin{aligned} & \left\| \int_0^1 B_M \left( \sum_{k=0}^N r_k(t) \tau_{ky} f, \sum_{k=0}^N r_k(t) \tau_{ky} g \right) dt \right\|_{p_3} \\ & \leq \int_0^1 \left\| B_M \left( \sum_{k=0}^N r_k(t) \tau_{ky} f, \sum_{k=0}^N r_k(t) \tau_{ky} g \right) \right\|_{p_3} dt \\ & \leq \int \left\| B_M \right\| \left\| \sum_{k=0}^N r_k(t) \tau_{ky} f \right\|_{p_1} \left\| \sum_{k=0}^N r_k(t) \tau_{ky} g \right\|_{p_2} dt \\ & \leq \|B_M\| \sup_{0 < t < 1} \left\| \sum_{k=0}^N r_k(t) \tau_{ky} f \right\|_{p_1} \sup_{0 < t < 1} \left\| \sum_{k=0}^N r_k(t) \tau_{ky} g \right\|_{p_2} \\ & \leq \|B_M\| (N + 1)^{1/p_1} \|f\|_{p_1} (N + 1)^{1/p_2} \|g\|_{p_2}. \end{aligned}$$

This implies that  $(N + 1)^{1/p_3} \|B_M(f, g)\|_{p_3} \leq C(N + 1)^{1/p_1 + 1/p_2} \|f\|_{p_1} \|g\|_{p_2}$ . Hence  $1/p_1 + 1/p_2 \geq 1/p_3$ . ■

The following elementary lemma is quite useful to get necessary conditions on multipliers.

**Lemma 3.8.** *Let  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ . If  $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$  then there exists  $C > 0$  such that*

$$\left| \int_{\mathbb{R}} e^{-\lambda^2 \xi^2} M(\xi) d\xi \right| \leq C \|M\|_{p_1, p_2, p_3} \lambda^{\frac{1}{q} - 1}$$

for any  $\lambda > 0$ .

*Proof.* Let  $\lambda > 0$  and denote  $G_\lambda$  such that  $\hat{G}_\lambda(\xi) = e^{-2\lambda^2 \xi^2}$ . Using (10) one concludes that

$$\begin{aligned} B_M(G_\lambda, G_\lambda)(x) &= \frac{1}{2} \int_{\mathbb{R}^2} e^{-\lambda^2 v^2} e^{-\lambda^2 u^2} M(v) e^{2\pi i u x} dudv \\ &= \frac{1}{2} \left( \int_{\mathbb{R}} e^{-\lambda^2 v^2} M(v) dv \right) \left( \frac{1}{\lambda} \int_{\mathbb{R}} e^{-u^2} e^{2\pi i u \frac{x}{\lambda}} du \right) \\ &= C \frac{1}{\lambda} e^{-\pi^2 \frac{x^2}{\lambda^2}} \left( \int_{\mathbb{R}} e^{-\lambda^2 v^2} M(v) dv \right). \end{aligned}$$

Since  $\|G_\lambda\|_p = C_p \lambda^{\frac{1}{p} - 1}$  and  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  one gets that

$$\|B_M(G_\lambda, G_\lambda)\|_{p_3} = C \left| \int_{\mathbb{R}} e^{-\lambda^2 v^2} M(v) dv \right| \lambda^{\frac{1}{p_3} - 1} \leq C \|M\|_{p_1, p_2, p_3} \lambda^{\frac{1}{p_1} - 1} \lambda^{\frac{1}{p_2} - 1}.$$

Therefore  $\left| \int_{\mathbb{R}} e^{-\lambda^2 \xi^2} M(\xi) d\xi \right| \leq C \|M\|_{p_1, p_2, p_3} \lambda^{\frac{1}{q} - 1}$ . ■

**Theorem 3.9.** *If there exists a non-zero continuous and integrable function  $M$  belonging to  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  then*

$$\frac{1}{p_3} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p_3} + 1.$$

*Proof.* Assume first that  $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{p_3}$ . Use Lemma 3.8 applied to  $\tau_{-2y}M$  for any  $y \in \mathbb{R}$  together with (17) to obtain

$$\left| \lambda \int_{\mathbb{R}} e^{-\lambda^2 \xi^2} M(\xi + 2y) d\xi \right| \leq C \|M\|_{p_1, p_2, p_3} \lambda^{\frac{1}{q}}.$$

Therefore, using the continuity of  $M$  and  $q < 0$  one gets

$$\lim_{\lambda \rightarrow \infty} \left| \lambda \int_{\mathbb{R}} e^{-\lambda^2 \xi^2} M(\xi + 2y) d\xi \right| = |M(2y)| = 0.$$

Hence  $M = 0$ .

Assume now that  $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} > 1$ . Using again Lemma 3.8, applied to  $M_y M$ , together with (17) we obtain

$$\left| \int_{\mathbb{R}} e^{-\lambda^2 \xi^2} M(\xi) e^{2\pi i y \xi} d\xi \right| \leq C \|M\|_{p_1, p_2, p_3} \lambda^{\frac{1}{q} - 1}.$$

Therefore, taking limits again as  $\lambda \rightarrow 0$ , since  $1/q - 1 > 0$  we get  $|\hat{M}(y)| = 0$ . Hence  $M = 0$ . ■

**Corollary 3.10.** (see [16, Prop 3.1]) *Let  $p_3 \geq 1$  such that  $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{p_3}$  or  $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{p_3} + 1$ . Then  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R}) = \{0\}$ .*

*Proof.* Let  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ . From Proposition 3.5 we have that  $\phi * M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  for any  $\phi$  compactly supported and continuous. Now use Theorem 3.9 to conclude that  $\phi * M = 0$  for any compactly supported and continuous  $\phi$ . This implies that  $M = 0$ . ■

Let us now use some interpolation methods to get more examples of multipliers in  $\tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$ . First note that, selecting  $\alpha = 1$  and  $\beta = -1$  in Proposition 2.2 we obtain the following simple example.

**Proposition 3.11.** *If  $\mu \in M(\mathbb{R})$  then  $M = \hat{\mu} \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  for  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \leq 1$  and  $\|M\| \leq \|\mu\|_1$ .*

**Theorem 3.12.** *Let  $\frac{1}{p_3} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \min\{2, \frac{1}{p_3} + 1\}$ . If  $M \in L^1(\mathbb{R})$  and  $M = \hat{K}$  for some  $K \in L^q(\mathbb{R})$  where  $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = 1 - \frac{1}{q}$  then  $M \in \tilde{\mathcal{M}}_{(p_1, p_2, p_3)}(\mathbb{R})$  with  $\|M\|_{p_1, p_2, p_3} \leq C \|K\|_q$ .*

*Proof.* Consider the trilinear form

$$T(K, f, g) = \int_{\mathbb{R}} f(x - t)g(x + t)K(t)dt.$$

From Proposition 3.4 we have  $B_M(f, g) = T(K, f, g)$  for  $M = \hat{K}$ . Now use Proposition 3.11 to conclude that  $T$  is bounded in  $L^1(\mathbb{R}) \times L^{q_1}(\mathbb{R}) \times L^{q_2}(\mathbb{R}) \rightarrow L^{s_1}(\mathbb{R})$  where  $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{s_1} \leq 1$  and it has norm bounded by 1.

Assume first that  $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ . Hence  $T$  is bounded in  $L^1(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  for  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ .

On the other hand, using Hölder's inequality

$$\sup_x \left| \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt \right| \leq \|f\|_{p_1} \|g\|_{p_2} \|K\|_{p'}.$$

This shows that  $T$  is also bounded in  $L^{p'}(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ . Therefore, by interpolation, selecting  $0 < \theta < 1$  such that  $\frac{1}{p_3} = \frac{1-\theta}{p}$ , one obtains that  $T$  is bounded in  $L^q(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^{p_3}(\mathbb{R})$  for  $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = 1 - \frac{1}{q}$ .

Assume now that  $1 < \frac{1}{p_1} + \frac{1}{p_2} \leq 2$ .

Using that  $\int_{\mathbb{R}} f(x-t)g(x+t)dt = f * g(2x)$ , Young's inequality implies that

$$\left\| \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt \right\|_{r_3} \leq \|K\|_\infty \|D_{1/2}^\infty(|f| * |g|)\|_{r_3} \leq C \|f\|_{r_1} \|g\|_{r_2} \|K\|_\infty$$

whenever  $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$  and  $\frac{1}{r_1} + \frac{1}{r_2} - 1 = \frac{1}{r_3}$ .

Hence  $T$  is bounded in  $L^\infty(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  where  $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p} \leq 1$ .

Using duality,  $\langle T(K, f, g), h \rangle = \langle T(h, \bar{f}, g), K \rangle$ , where  $\bar{f}(x) = f(-x)$ , that is

$$\int_{\mathbb{R}^2} f(x-t)g(x+t)K(t)h(x)dt dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \bar{f}(t-x)g(x+t)h(x)dx \right) K(t)dt.$$

Therefore  $T$  is also bounded in  $L^{p'}(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ .

Select  $0 \leq \theta \leq 1$  such that  $\frac{1}{p_3} = \frac{1}{p} + \frac{\theta}{p'}$ . Now using interpolation  $T$  will be bounded in  $L^q(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^{p_3}(\mathbb{R})$  for  $\frac{1}{q} = \frac{\theta}{p'} = \frac{1}{p_3} - \frac{1}{p} = \frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2} + 1$ . ■

## REFERENCES

- [1] Bergh, J., Löfström, J., *Interpolation spaces*. Springer Verlag [1970].
- [2] Blasco, O., *Bilinear multipliers and transference*. Int. J. Math. Math. Sci. **2005**(4) [2005], 545-554.
- [3] Berkson, E., Blasco, O., Carro, M. and Gillespie, A., *Discretization versus transference for bilinear operators*. Banach Spaces their Applications in Analysis, 11-30, De Gruyter [2007].
- [4] Blasco, O., Carro, M., and Gillespie, A., *Bilinear Hilbert transform on measure spaces*. J. Fourier Anal. and Appl. **11** [2005], 459-470.
- [5] Blasco, O., Villarroja, F., *Transference of bilinear multipliers on Lorentz spaces*. Illinois J. Math. **47**(4), [2005], 1327-1343
- [6] Coifman R.R., Meyer, Y., *Fourier Analysis of multilinear convolution, Calderón theorem and analysis of Lipschitz curves*. Euclidean Harmonic Analysis (Proc. Sem. Univ. Maryland, College Univ., Md) Lecture Notes in Maths. **779**, [1979], 104-122.
- [7] Fan, D., Sato, S., *Transference on certain multilinear multiplier operators*. J. Austral. Math. Soc. **70**, [2001], pp. 37-55.
- [8] Gilbert, J, Nahmod, A., *Bilinear operators with non-smooth symbols*, J. Fourier Anal. Appl., **7**[2001], pp. 435-467.
- [9] Gilbert, J, Nahmod, A., *Boundedness of bilinear operators with nonsmooth symbols*, Mat. Res. Letters, **7**[2000], pp. 767-778.
- [10] Grafakos, L., Kalton, N., *Some remarks on multilinear maps and interpolation*. Math. Annalen **319** [2001], pp. 151-180

- [11] Kenig, C. E., Stein, E.M., *Multilinear estimates and fractional integration*. Math. Res. Lett. **6** [1999], pp. 1-15.
- [12] Lacey M., Thiele C.,  *$L^p$  estimates on the bilinear Hilbert transform for  $2 < p < \infty$* . Annals Math. 146, [1997], pp. 693-724.
- [13] Lacey, M., Thiele, C., *On Calderón's conjecture*. Ann. Math. **149** 2 [1999] pp. 475-496.
- [14] Stein, E. M., Weiss, G., *Introduction to Fourier Analysis on Euclidean spaces*. Princeton Univ. Press [1971].
- [15] Villarroya, P., *La transformada de Hilbert bilineal*. Doctoral Dissertation. Univ. of Valencia [2002].
- [16] Villarroya, P., *Bilinear multipliers on Lorentz spaces*. Czechoslovak Math. J. 58(133) [2008], no. 4, 1045–1057.

*Oscar Blasco*  
Department of Mathematics,  
Universitat de Valencia,  
Burjassot 46100 (Valencia), Spain  
`oscar.blasco@uv.es`

*Recibido: 1 de agosto de 2008*  
*Aceptado: 18 de diciembre de 2008*