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TRANSFERENCE OF L^p-BOUNDEDNESS BETWEEN HARMONIC ANALYSIS OPERATORS FOR LAGUERRE AND HERMITE SETTINGS

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Abstract. In this paper we discuss a transference method of L^p-boundedness properties for harmonic analysis operators in the Hermite setting to the corresponding operators in the Laguerre context. As a byproduct of our procedure we obtain new characterizations of certain classes of Banach spaces and Köethe spaces.

1. Introduction

From May 26 to June 6, 2008, was held in La Falda (Córdoba, Argentina) the congress CIMPA School 2008. In this paper we collect the main results that appeared in the talk presented by the author there. This project research about transference of L^p boundedness properties for harmonic analysis operators in the Hermite and Laguerre setting has been developed jointly with José Luis Torrea, from the Universidad Autónoma de Madrid, Juan Carlos Fariña, Lourdes Rodríguez-Mesa and Alejandro Sanabria, from the Universidad de La Laguna (Tenerife). Most of the results with their proofs can be encountered in [2], [3] and [4].

In the monograph of Stein [28] harmonic analysis operators (maximal operators, multipliers, Riesz transforms, Littlewood-Paley g-functions,...) related to semigroup of operators are defined. Here we consider the operators associated with semigroups in the Hermite and Laguerre context.

We denote by H the second order Hermite operator defined by

$$H = -\frac{d^2}{dx^2} + x^2 = -\frac{1}{2}\left[\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right)+\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)\right], \quad x \in \mathbb{R}\,. \tag{1.1}$$

This operator is symmetric with respect to the Lebesgue measure on $\mathbb{R}$. For every $n \in \mathbb{N}$, we have that $Hh_n = (2n+1)h_n$, where $h_n(x) = (\sqrt{\pi}2^n n!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x)$, $x \in \mathbb{R}$, represents the n- th Hermite function and H_n is the n-th Hermite polynomial (see [33]).

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The heat semigroup $\{W_t\}_{t\geq 0}$ associated with the sequence $\{h_n\}_{n\in\mathbb{N}}$ of eigenfunctions for the Hermite operator H is defined by

$$W_t(f)(x) = \int_{-\infty}^{+\infty} W_t(x,y)f(y)dy, \quad x\in\mathbb{R},\;\; f\in L^2(\mathbb{R}) \;\; and \;\; t>0,$$

being

$$\begin{aligned} W_t(x,y) &= \sum_{n=0}^{\infty} e^{-(2n+1)t}h_n(x)h_n(y) \\ &= \frac{1}{\sqrt{\pi}}\left(\frac{e^{-t}}{1-e^{-4t}}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-4t}}{1-e^{-4t}}+\frac{2xye^{-2t}}{1-e^{-4t}}}, \;\; x,y\in\mathbb{R},\, t>0. \end{aligned}$$

As usual, the maximal operator W_* of the heat semigroup $\{W_t\}_{t\geq 0}$ is

$$W_*(f) = \sup_{t>0}|W_t(f)|, \;\; f\in L^2(\mathbb{R}).$$

According to the Stein's ideas ([28]) the factorization (1.1) of H suggests to define the Riesz transform in the Hermite setting by

$$\begin{aligned} R(f)(x) &= \left(\frac{d}{dx}+x\right)H^{-1/2}f(x) \\ &= \sum_{n=0}^{\infty}\left(\frac{2n}{2n+1}\right)^{1/2} h_{n-1}(x)a_n(f), \;\; f\in C_c(\mathbb{R}), \end{aligned}$$

where, for every $n\in\mathbb{N}$ and $f\in L^2(\mathbb{R})$, $a_n(f)=\int_{-\infty}^{\infty} f(x)h_n(x)dx$. $C_c(\mathbb{R}$ represents the space of $C^\infty(\mathbb{R})$ having compact support. The fractional integral $H^{-1/2}$ is defined by using the heat semigroup as follows

$$H^{-1/2}(f)(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty W_t(f)(x)t^{-1/2}dt, \;\; f\in L^2(\mathbb{R}).$$

From the results established by Muckenhoupt ([21]) we can deduce that, for every $f\in L^2(\mathbb{R})$,

$$R(f)(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} R(x,y)f(y)dy, \;\; a.e. \;\; x\in\mathbb{R},$$

being

$$R(x,y) = \frac{1}{\sqrt{\pi}}\int_0^\infty \left(\frac{d}{dx}+x\right)W_t(x,y)t^{-1/2}dt, \;\; x,y\in\mathbb{R}.$$

The subordinated Poisson semigroup associated with $\{h_n\}_{n\in\mathbb{N}}$ is given by

$$P_t(f)(x) = \int_{-\infty}^{\infty} P_t(x,y)f(y)dy, \;\; f\in L^2(\mathbb{R}),$$

being

$$P_t(x,y) = \frac{t}{2\sqrt{\pi}}\int_0^\infty u^{-3/2}e^{-t^2/(4u)}W_u(x,y)du, \;\; t\in(0,\infty), x,y\in\mathbb{R}.$$

For every $1 < q < \infty$, we denote by g_q^H the Littlewood-Paley function for the Poisson semigroup for H, that is,

$$g_q^H(f)(x) = \Big(\int_{-\infty}^{\infty} \Big| t\frac{\partial}{\partial t} P_t(f)(x)\Big|^q \frac{dt}{t}\Big)^{1/q}.$$

Similar definitions for the heat semigroup for H can be made.

Harmonic analysis associated with the Hermite polynomials was began by Muckenhoupt ([20] and [21]). Maximal operators, Riesz transforms and Littlewood-Paley g-functions in the Hermite setting have been investigated by Torrea and Stempak ([30], [31] and [32]) and Thangavelu ([34]).

For every $\alpha > -1$ we consider the Laguerre differential operator

$$L_\alpha = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right)\right), \quad x \in (0,\infty).$$

This operator can be factorized in the following way

$$L_\alpha = \frac{1}{2}\mathfrak{D}_\alpha^* \mathfrak{D}_\alpha + \alpha + 1, \tag{1.2}$$

where $\mathfrak{D}_\alpha f = \left(-\dfrac{\alpha+1/2}{x} + x + \dfrac{d}{dx}\right) f = x^{\alpha+\frac{1}{2}}\dfrac{d}{dx}(x^{-\alpha-\frac{1}{2}} f) + xf$, and $\mathfrak{D}_\alpha^*$ denotes the formal adjoint of $\mathfrak{D}_\alpha$ in $L^2((0,\infty), dx)$. For every $n \in \mathbb{N}$, we have that

$$L_\alpha \varphi_n^\alpha = (2n + \alpha + 1)\varphi_n^\alpha,$$

where the Laguerre function φ_n^α is defined by

$$\varphi_n^\alpha(x) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2}} x^{\alpha+\frac{1}{2}} L_n^\alpha(x^2), \quad x \in (0,\infty),$$

and L_n^α denotes the n-th Laguerre polynomial of type α ([33, p. 100] and [34, p. 7]).

The heat semigroup $\{W_t^\alpha\}_{t\geq 0}$ generated by the operator $-L_\alpha$ takes the form

$$W_t^\alpha(f)(x) = \int_0^\infty W_t^\alpha(x,y) f(y)dy, \quad x \in (0,\infty),\ f \in L^2((0,\infty), dx)\ ,$$

where, for every $t, x, y \in (0,\infty)$,

$$\begin{aligned} W_t^\alpha(x,y) &= \sum_{n=0}^{\infty} e^{-t(2n+\alpha+1)} \varphi_n^\alpha(x)\varphi_n^\alpha(y) \\ &= \left(\frac{2e^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}} \left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{\frac{1}{2}} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}}. \end{aligned}$$

Here I_α represents the modified Bessel function of the first kind and order α.

The maximal operator associated with the heat semigroup $\{W_t^\alpha\}_{t\geq 0}$ is defined by $W_*^\alpha(f) = \sup_{t>0} |W_t^\alpha(f)|$. The Riesz transform in the Laguerre setting is defined by (see (1.2))

$$R^\alpha(f) = D_\alpha L_\alpha^{-1/2}(f), \ \ f \in C_c(0,\infty).$$

This operator is actually a principal value integral operator given by

$$R^\alpha(f)(x) = \lim_{\varepsilon\to 0}\int_{0,\,|x-y|>\varepsilon}^{\infty} R^\alpha(x,y)f(y)dy, \quad a.e.\ x\in\mathbb{R},$$

being

$$R_\alpha(x,y) = \frac{1}{\sqrt{\pi}}\int_0^\infty D_{\alpha,x}W_t^\alpha(x,y)t^{-1/2}dt, \quad x,y\in(0,\infty).$$

The Poisson semigroup $\{P_t^\alpha\}_{t\geq 0}$ generated by $-\sqrt{L_\alpha}$ is given in terms of $\{W_t^\alpha\}_{t\geq 0}$ by the subordination formula.

The Littlewood-Paley $g_q^{L_\alpha}$-function, when $1<q<\infty$, for the Poisson semigroup $\{P_t^\alpha\}_{t\geq 0}$ is defined by

$$g_q^{L_\alpha}(f)(x) = \Big(\int_{-\infty}^\infty \Big|t\frac{\partial}{\partial t}P_t^\alpha(f)(x)\Big|^q\frac{dt}{t}\Big)^{1/q}.$$

L^p-boundedness properties for maximal operators, Riesz transforms and Littlewood-Paley functions in the Laguerre setting have been established by [8], [13], [14], [16], [20], [22], [24], [29], and [34], amongst others.

In [2], [3] and [4] a new method is employed to study L^p boundedness properties for the harmonic analysis operators in the Laguerre setting by using corresponding properties for operators in the Hermite context. Our procedure allows to get new and shorter proofs of known results and also to obtain new results.

The idea of transferring boundedness properties from Hermite to Laguerre settings was used also by Gutiérrez, Incognito, and Torrea [13] (see also [12] and [16]). They employed formulae relating Hermite polynomials in dimension n with Laguerre polynomials in dimension 1 and $\alpha=\frac{n}{2}-1$. The procedure developed in [13] only allows us to obtain boundedness properties for operators in the Laguerre setting for these special values of α. Then, it is necessary to use some kind of transplantation result to extend the result to other values of α.

Our method is esentially different to the one considered in [13]. We connect the harmonic analysis operators in the Laguerre setting for every $\alpha>-1$ with the corresponding operators in the Hermite context. More precisely, the kernel of the operator under consideration is broken in the part close to the diagonal (local part) and in the part far away from the diagonal (global part). In the local part the transference between Hermite and Laguerre contexts works. In the global part the operators can be controlled by positive operators whose L^p-boundedness properties are wellknown.

In the following sections of this paper we show how our method can be applied to analyze L^p-boundedness properties for the maximal operator associated with the heat semigroup, Riesz transform and Littlewood-Paley g functions for the Poisson semigroup in the Laguerre setting. Moreover, we can obtain new characterizations of certain geometric properties (UMD, Hardy-Littlewood and Lusin type and cotype) by using the harmonic analysis operators in the Laguerre context.

Throughout this paper by C we always denote a positive constant that can change from one line to the other line.

2. Maximal operator for the Laguerre heat semigroup.

In this section we analyze the L^p-boundedness properties for the maximal operator of the heat semigroup in the Laguerre setting. Also we establish new characterizations of certain Banach lattices with the Hardy-Littlewood property.

The key property is the following pointwise estimate involving the kernels of the heat semigroup for the Hermite and Laguerre operators.

Proposition 2.1. *let $\alpha > -1$. There exists $C > 0$ such that*
(i) $W_t^\alpha(x, y) \leq C y^{\alpha+1/2} x^{-\alpha-3/2}$, $t > 0$ *and* $0 < y < x/2$.
(ii) $W_t^\alpha(x, y) \leq C x^{\alpha+1/2} y^{-\alpha-3/2}$, $t > 0$ *and* $0 < 2x < y$.
(iii) $|W_t^\alpha(x, y) - W_{t/2}(x, y)| \leq C x^{-1}$, $t > 0$ *and* $x/2 < y < 2x$.

Note that the comparison between the kernels W_t^α and $W_{t/2}$ for the Laguerre and Hermite heat semigroups, respectively, is got in the local region (close to the diagonal), as it is shown in (iii). The estimates obtained in (i) and (ii) say that the maximal operator W_*^α restricted to the global region is controlled by Hardy type operators. L^p-boundedness properties of the Hardy type operators are wellknown ([22] and [8]). Then, L^p-boundedness of W_*^α is implied by the L^p-boundedness of W_*.

Suppose that B is a Banach space consisting of equivalence classes, modulo equality almost everywhere, of locally integrable real functions on a complete σ-finite measure space (Ω, Σ, μ). This class of Banach spaces is named Köethe function spaces ([18] and [26]) when the following two conditions are satisfied.

(a) If $|f(w)| \leq |g(w)|$, a.e. $w \in \Omega$, with f measurable and $g \in B$, then $f \in B$ and $\|f\|_B \leq \|g\|_B$.

(b) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A of A belongs to B.

Each Köethe function space is a Banach lattice with the natural order ($f \geq 0 \Leftrightarrow f(w) \geq 0$, a.e. $w \in \Omega$). This lattice is σ-order complete.

Banach lattices and Köethe function spaces with the Hardy-Littlewood property were introduced in [10]. If f is a real locally integrable B-valued function, where B is a Köethe space, and J is a finite subset of $\mathbb{Q}_+$, the set of positive rational numbers, we define

$$\mathfrak{M}_J(f)(x) = \sup_{r \in J} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^n.$$

We say that B has the Hardy-Littlewood property ([10]) when for a certain $1 < p < \infty$ there exists $C > 0$ such that

$$\|\mathfrak{M}_J f\|_{L^p_B(\mathbb{R})} \leq C \|f\|_{L^p_B(\mathbb{R})}, \quad f \in L^p_B(\mathbb{R}),$$

for every finite subset J of $\mathbb{Q}_+$. Maximal operators associated with heat semigroup for the Ornstein-Uhlenbeck operator (in the Hermite polynomial setting) was used in [15] to characterize Köethe spaces with the Hardy-Littlewood property. The corresponding properties in the Hermite and Laguerre context is included in the following proposition. In its proof the pointwise estimates established in Proposition 2.1 play a fundamental role.

Proposition 2.2. *Let B be a Köethe function space and $\alpha > -1$. The following properties are equivalent.*

(i) *B has the Hardy-Littlewood property.*
(ii) *The maximal operator W_* is bounded from $L^p_B(\mathbb{R}, w(x)dx)$ into itself, for every $1 < p < \infty$ and $w \in A_p(\mathbb{R})$.*
(iii) *The maximal operator W_* is bounded from $L^1_B(\mathbb{R}, w(x)dx)$ into $L^{1,\infty}_B(\mathbb{R}, w(x)dx)$, for every $w \in A_1(\mathbb{R})$.*
(iv) *The maximal operator W^α_* is bounded from $L^p_B((0,\infty), dx)$ into itself for some $1 < p < \infty$, when $\alpha > -1/2$, and for $2/(2\alpha+3) < p < -2/(2\alpha+1)$, when $-1 < \alpha \leq -1/2$.*
(v) *The maximal operator W^α_* is bounded from $L^p_B((0,\infty), x^\sigma dx)$ into itself for every $1 < p < \infty$ and $-1 - p(\alpha + 1/2) < \sigma < p(\alpha + 3/2) - 1$.*

3. Laguerre Riesz transforms

The UMD property for Banach spaces was introduced by Burkholder ([7]) in a probability setting. For a Banach space B the UMD property is equivalent to the fact that the Hilbert transform admits a B valued extension to $L^p_B(\mathbb{R})$ as a bounded operator of $L^p_B(\mathbb{R})$ for some (any) $1 < p < \infty$ ([6] and [7]). Recently, Abu-Falahah and Torrea ([1]) have obtained a characterization of UMD Banach spaces in terms of the Riesz transform R associated with the Hermite operator. In [2, Theorem 4.1] we obtain the corresponding result for the Riesz transform in the Laguerre setting. To prove this property are crucial the next pointwise estimates involving the kernels for the Laguerre and Hermite Riesz transforms.

Proposition 3.1. *Let $\alpha > -1$. Then*

(i) $|R_\alpha(x,y)| \leq C y^{\alpha+1/2} x^{-\alpha-3/2}$, $0 < y < x/2$.
(ii) $|R_\alpha(x,y)| \leq C x^{\alpha+3/2} y^{-\alpha-5/2}$, $2x < y$.
(iii) $\left| R_\alpha(x,y) - \int_0^\infty (\frac{d}{dx} + x) W_{t/2}(x,y) \right| \frac{dt}{\sqrt{t}} \leq \frac{C}{y}\Big(1 + \frac{(xy)^{1/4}}{|x-y|^{1/2}}\Big)$, $0 < x/2 < y < 2x$.

Note that the Laguerre and Hermite kernels differ in the local part, at most, by an absolutely integrable function on $(0,\infty)$.

By using Proposition 3.1 the following result can be proved.

Proposition 3.2. *Let $\alpha > -1$ and let B be a Banach space. Then the following statements are equivalent.*

(i) *B has the UMD property.*
(ii) *Riesz transform R_α admits a bounded extension from $L^p_B((0,\infty), dx)$ into itself, for some p such that $\max\{1, 2/(2\alpha+3)\} < p < \infty$.*
(iii) *Riesz transform R_α admits a bounded extension from $L^p_B((0,\infty), x^\sigma dx)$ into itself, for every $1 < p < \infty$ and $-p(\alpha+3/2) - 1 < \sigma < p(\alpha+3/2) - 1$.*

4. Littlewood-Paley g-functions for the Laguerre Poisson semigroup.

Xu ([35]) introduced the Lusin cotype and type properties for a Banach space B as follows. Let f be in $L^1_B(\mathbb{T})$, where $\mathbb{T}$ denotes the one dimensional torus. We consider, for every $1 < q < \infty$, the generalized Littlewood-Paley g function defined by

$$g_q(f)(z) = \left(\int_0^1 (1-r)^q \|\nabla P_r * f(z)\|_B^q \frac{dr}{1-r} \right)^{\frac{1}{q}},$$

where $P_r(\theta)$ represents the Poisson kernel on $\mathbb{T}$. We say that B has a Lusin cotype $q \geq 2$ when for some $p \in (1, \infty)$ it has

$$\|g_q(f)\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p_B(\mathbb{T})}, \;\; f \in L^p_B(T).$$

It is said that B has a Lusin type $1 \leq q \leq 2$ when for some $p \in (1, \infty)$ the following inequality holds

$$\|f\|_{L^p_B(\mathbb{T})} \leq C \left(\|\hat{f}(0)\|_B + \|g_q(f)\|_{L^p(\mathbb{T})} \right).$$

The Lusin type and Lusin cotype of B do not depend on $p \in (1, \infty)$. Moreover Xu proved in [35, Theorem 3.1] that a Banach space B has Lusin cotype q (Lusin type q) if and only if B has a martingale cotype q (martingale type q). We recall that the double inequality

$$\frac{1}{C_p}\|f\|_{L^p_B(\mathbb{T})} \leq |\hat{f}(0)| + \|g_2(f)\|_{L^p_B(\mathbb{T})} \leq C_p\|f\|_{L^p_P(\mathbb{T})}, \tag{4.1}$$

holds when $B = \mathbb{C}$, that is, when f is a scalar valued function. For a Banach space B the above double inequality holds if and only if B is isomorphic to a Hilbert space ([17]). Recently, Martínez, Torrea and Xu [19] have extended the results obtained by Xu in [35] to subordinated Poisson semigroups of general symmetric difusion Markovian semigroups. Also, Harboure, Torrea and Viviani [15] characterized the Lusin cotype of a Banach space by using Littlewood-Paley g-function in the Ornstein-Uhlenbeck setting. In [3] it is shown that our method of comparison between Laguerre and Hermite contexts allows us to describe the martingale (Lusin) cotype and type in terms of the Littlewood-Paley g functions associated with the Hermite and Laguerre Poisson semigroups.

Proposition 4.2. *Let B a Banach space, $q \geq 2$ and $\alpha > -1$. We denote $\Omega_\alpha = (1, \infty)$, when $\alpha > -\frac{1}{2}$, and $\Omega_\alpha = \left(\frac{2}{2\alpha+3}, \frac{-2}{2\alpha+1}\right)$, when $-1 < \alpha \leq -\frac{1}{2}$. The following assertions are equivalent.*

(i) B has q-martingale cotype.

(ii) For every (or, equivalently, for some) $p \in \Omega_\alpha$ there exists $C_p > 0$ such that

$$||g^\alpha_q(f)||_{L^p(0,\infty)} \leq C_p||f||_{L^p_B(0,\infty)}, \qquad f \in L^p_B(0, \infty).$$

(iii) For every (or, equivalently, for some) $p \in \Omega_\alpha$ there exists $C_p > 0$ such that

$$||g_q(f)||_{L^p(\mathbb{R})} \leq C_p||f||_{L^p_B(\mathbb{R})}, \qquad f \in L^p_B(\mathbb{R}).$$

To establish the above result is fundamental to prove a pointwise estimate involving the kernels of the Laguerre and Hermite Litlewood-Paley functions.

Proposition 4.3. *Let B a Banach space, $1 < q \leq 2$ and $\alpha > -1$. Ω_α is defined as in Theorem 4.2. Then the following assertions are equivalent.*

(i) B has q-martingale type.

(ii) For every (or, equivalently, for some) $p \in \Omega_\alpha$ there exists $C_p > 0$ such that

$$||f||_{L^p_B(0,\infty)} \leq C_p ||g^\alpha_q(f)||_{L^p(0,\infty)}.$$

(iii) For every (or, equivalently, for some) $p \in \Omega_\alpha$ there exists $C_p > 0$ such that

$$||f||_{L^p_B(\mathbb{R})} \leq C_p ||g_q(f)||_{L^p(\mathbb{R})}.$$

5. Other operators in the Laguerre setting.

Our procedure also works for other harmonic analysis operators in the Laguerre context.

The factorization (1.2) suggests to define (formally), for every $k \in \mathbb{N}$, the k-th Riesz transform $R^{(k)}_\alpha$ associated with the Laguerre operator by

$$R^{(k)}_\alpha = \mathcal{D}^k_\alpha L^{-k/2}_\alpha.$$

Here $L^{-\beta}_\alpha$, $\beta > 0$, denotes the β-th power of the operator L_α defined by

$$L^{-\beta}_\alpha(f)(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} W^\alpha_t(f)(x)dt, \;\; x \in (0,\infty).$$

In the Laguerre polynomial context, Graczyk, Loeb, López, Nowak and Urbina [12] investigated the corresponding higher Riesz transform $\mathbb{R}^{(k)}_\alpha$ when $k \in \mathbb{N}$ and $\alpha = \frac{n}{2} - 1$, $n \in \mathbb{N}$. They use the connection between n-dimensional Hermite polynomial and Laguerre polynomials of order $\alpha = \frac{n}{2} - 1$, that had been exploited by Incognito, Gutiérrez and Torrea [13]. Also, Nowak and Stempak [25] studied weighted L^p-boundedness properties for $R^{(k)}_\alpha$ by using Calderón-Zygmund theory. An application of our method, by comparing $R^{(k)}_\alpha$ with the corresponding k-th order Riesz transform in the Hermite setting, allows us to improve the results in [25] in the one dimensional case. In [4] the following result was established. The class of weights admitted in Proposition 5.1 bellow is wider than the one considered in [25] when k is odd. Also it is obtained a representation of the higher order Riesz transform $R^{(k)}_\alpha$ as principal value integral operators.

Proposition 5.1. *Let $\alpha > -1$ and $k \in \mathbb{N}$. For every $\phi \in C^\infty_c(0,\infty)$ it has that*

$$R^{(k)}_\alpha \phi(x) = w_k \phi(x) + \lim_{\varepsilon \to 0^+} \int_{0, |x-y|>\varepsilon}^\infty R^{(k)}_\alpha(x,y)\phi(y)dy, \quad x \in (0,\infty),$$

where

$$R^{(k)}_\alpha(x,y) = \frac{1}{\Gamma\left(\frac{k}{2}\right)} \int_0^\infty t^{\frac{k}{2}-1} \mathfrak{D}^k_\alpha W^\alpha_t(x,y)dt, \quad x,y \in (0,\infty),$$

and $w_k = 0$, when k is odd and $w_k = -2^{\frac{k}{2}}$, when k is even.

The operator $R_\alpha^{(k)}$ can be extended, defining it by

$$R_\alpha^{(k)} f(x) = w_k f(x) + \lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^{\infty} R_\alpha^{(k)}(x,y) f(y) dy, \qquad a.e.\ x \in (0,\infty),$$

as a bounded operator from $L^p((0,\infty), x^\delta dx)$ into itself, for $1 < p < \infty$ and

(a) $-\Big(\alpha + \frac{3}{2}\Big)p - 1 < \delta < \Big(\alpha + \frac{3}{2}\Big)p - 1$, when k is odd;

(b) $-\Big(\alpha + \frac{1}{2}\Big)p - 1 < \delta < \Big(\alpha + \frac{3}{2}\Big)p - 1$, when k is even;

and as a bounded operator from $L^1((0,\infty), x^\delta dx)$ into $L^{1,\infty}((0,\infty), x^\delta dx)$ when

(c) $-\alpha - \frac{5}{2} \le \delta \le \alpha + \frac{1}{2}$, when k is odd;

(d) $-\alpha - \frac{3}{2} \le \delta \le \alpha + \frac{1}{2}$, for $\alpha \neq -\frac{1}{2}$, and $-1 < \delta \le 1$, for $\alpha = -\frac{1}{2}$, when k is even.

We can use our method also to study L^p-boundedness properties of other Littlewood-Paley functions in the Laguerre setting. In [5] the behaviour of the area Littlewood-Paley function for the Poisson and heat semigroups for the Laguerre operator is studied on weighted Lebesgue spaces. To use our procedure it is needed in a first step to establish the corresponding results for the area Littlewood-Paley functions for the Poisson and heat semigroups for the Hermite operator.

Sasso [27] investigated L^p-boundedness properties for the spectral multipliers of Laguerre expansions when the multiplier is the Laplace transform of a bounded function. After writing the multiplier operator in terms of the Poisson of heat kernel, our method can be used to establish weighted L^p-boundedness for that operators by using the corresponding properties for the spectral multiplier associated with Hermite expansions.

Recently, Garrigós, Harboure, Signes, Torrea and Viviani [11] have studied power weighted inequalities for Mihlin multipliers associated with Laguerre expansions. They used the ideas developed in [13] and transplantation theorems. It is an interesting question to analyze the applicability of our method in the problem considered in [11] by using the Mihlin-Hormander multiplier theorem for Hermite expansions (see [34]).

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