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MIXED WEAK TYPE INEQUALITIES FOR ONE-SIDED OPERATORS AND ERGODIC THEOREMS

F.J. MARTÍN-REYES

This paper is essentially the talk I addressed in the CIMPA-UNESCO Argentina School 2008. It is about mixed weak type inequalities and it is based on a joint paper with S. Ombrosi [6]. The paper is organized in the following way: first, we introduce what mixed inequalities are for general operators and we state known results. Then we state the main question for us, the mixed inequalities for one-sided Hardy-Littlewood maximal operators, and we establish a conjecture. The next section is devoted to stating and commenting the results in [6] for the Hardy operator. Finally, we see the connection of this topic with ergodic theorems.

1. Mixed weak type inequalities: known results

In order to introduce what Andersen and Muckenhoupt [1] called mixed weak type inequalities, we consider sublinear operators T acting on spaces of measurable functions on $\mathbb{R}^n$ with the basic properties of subadditivity and homogeneity:

$$|T(f+g)| \leq |Tf| + |Tg|, \quad |T(\lambda f)| = |\lambda||Tf|,$$

where f and g are measurable functions and $\lambda \in \mathbb{R}$. We shall use in this paper the following definition: Given weights u and v, that is, nonnegative measurable functions on $\mathbb{R}^n$, it is said that T satisfies the weighted mixed weak type inequality if there exists C such that

$$\int_{\{x:|Tf(x)|>v(x)\}} u(x)v(x)\,dx \leq C \int |f(x)|u(x)\,dx$$

for all functions $f \in L^1(u(x)\,dx)$. Notice that the constant is independent of f but it may depend on u and v. Observe the following particular case: if $v \equiv 1$ and $f = g/\lambda$, $\lambda > 0$, then the last inequality becomes

$$\int_{\{x:|Tg(x)|>\lambda\}} u(x)\,dx \leq \frac{C}{\lambda} \int |g(x)|u(x)\,dx,$$

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or, in other words, this inequality means exactly that T is of weak type $(1,1)$ with respect to the measure with density u. Once we have the kind of weighted inequality we are interested in, we introduce the concrete operators we are going to consider.

First, the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q:x\in Q} \frac{1}{|Q|}\int_Q |f|,$$

where the supremum is taken over all the cubes with sides parallel to the axis and such that x belongs to Q. We remind that the weighted weak type $(1,1)$ inequality

$$\int_{\{x:Mf(x)>\lambda\}} u(x)\,dx \le \frac{C}{\lambda}\int |f(x)|u(x)\,dx$$

holds if and only if u satisfies the A_1 Muckenhoupt condition, that is, there exists $C>0$ such that

$$Mu(x) \le Cu(x) \quad \text{a.e.}$$

In 1982, Andersen and Muckenhoupt [1] proved a mixed weak type inequality with power weights in $\mathbb{R}$ $(n=1)$.

Theorem 1.1. *If $n=1$, $d\neq 1$ and $u\in A_1$ then there exists $C>0$ such that*

$$\int_{\{x:Mf(x)>|x|^{-d}\}} |x|^{-d}u(x)\,dx \le C\int |f(x)|u(x)\,dx,$$

for all measurable functions $f\in L^1(u(x)\,dx)$.

They proved also that the same result holds for the Hilbert transform:

$$Hf(x) = P.V.\int_{\mathbb{R}} \frac{f(x-y}{y}\,dy.$$

This result was generalized for singular integrals in $\mathbb{R}^n$ in 1990 [7].

In 1985, Eric Sawyer [10] proved a mixed weak type inequality for M in dimension 1, assuming that the weights are in the Muckenhoupt class A_1.

Theorem 1.2. *If $n=1$, $u\in A_1$ and $v\in A_1$ then there exists $C>0$ such that*

$$\int_{\{x:Mf(x)>v(x)\}} v(x)u(x)\,dx \le C\int |f(x)|u(x)\,dx,$$

for all measurable functions $f\in L^1(u(x)\,dx)$.

In the same paper, it was conjectured the corresponding result for the Hilbert transform. This conjecture was proved in 2005 by Cruz-Uribe, Martell and Pérez [2]. Furthermore, they also proved the result for Calderón-Zygmund singular integrals in dimension greater than one (and obviously for the Hardy-Littlewood maximal operator in dimension n greater than 1).

2. One-Sided Maximal Operators

The one-sided Hardy-Littlewood maximal operators M^- and M^+ are defined in the real line by

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)|\, dt \quad \text{and} \quad M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)|\, dt.$$

As for the usual Hardy-Littlewood maximal operator, we remind [11] (see also [8]) that the weighted weak type $(1,1)$ inequality (non mixed)

$$\int_{\{x: M^- f(x) > \lambda\}} u(x)\, dx \leq \frac{C}{\lambda} \int |f(x)| u(x)\, dx$$

holds if and only if u is in A_1^-, which means that

$$M^+ u(x) \leq C u(x) \quad \text{a.e.}$$

There is an analogue result for M^+: there exists $C > 0$ such that

$$\int_{\{x: M^+ f(x) > \lambda\}} u(x)\, dx \leq \frac{C}{\lambda} \int |f(x)| u(x)\, dx$$

for all measurable functions f if and only if $u \in A_1^+$, that is,

$$M^- u(x) \leq C u(x) \quad \text{a.e.}$$

(as usual, throughout the paper, the constant C will not be the same at each occurrence). Sawyer's result suggests immediately the natural conjecture for the one-sided Hardy-Littlewood maximal operators.

Conjecture 2.1. *If $u \in A_1^-$ and $v \in A_1^+$ then there exists $C > 0$ such that*

$$\int_{\{x: M^- f(x) > v(x)\}} u(x) v(x)\, dx \leq C \int |f(x)| u(x)\, dx$$

for all measurable functions f.

Let me point out another way of stating the same conjecture:

Conjecture 2.2. (The same conjecture: different statement) *If M^- applies $L^1(u)$ into weak-$L^1(u)$ and M^+ applies $L^1(v)$ into weak-$L^1(v)$ then the weighted mixed inequality holds for M^-, that is, there exists $C > 0$ such that*

$$\int_{\{x: M^- f(x) > v(x)\}} u(x) v(x)\, dx \leq C \int |f(x)| u(x)\, dx$$

for all measurable functions f.

Unfortunately, we have not been able to prove that conjecture. We asked ourselves if we could answer an apparently easier problem. More precisely, if we consider operators smaller than M^- the inequality turns out to be apparently easier. These smaller operators will be the Hardy operators that we study in the next section.

3. Mixed weak type inequalities for Hardy operators

We consider for all $c \in \mathbb{R}$ the Hardy Operator (the averaging Hardy operator) T_c defined by
$$T_c f(x) = \begin{cases} \frac{1}{x-c}\int_c^x f(y)\,dy, & \text{if } x > c; \\ 0, & \text{if } x \le c. \end{cases}$$
Usually $c = 0$ and the operator is only defined for $x > 0$ but it is better for our purposes to work in the whole line.

We have these two obvious relations with M^-: Hardy operators are smaller than M^- and the supremum over all c gives the maximal operator M^-, that is,

- $T_c|f| \le M^- f$.
- $M^- f = \sup_{c\in\mathbb{R}} T_c|f|$.

An interesting observation is that the weak type $(1,1)$ inequality of M^- is equivalent to weak type $(1,1)$ of the Hardy operators with an uniform constant and the good weights are characterized by condition A_1^-. More precisely, the following statements are equivalent:

(a) There exists a constant $C > 0$ such that
$$\int_{\{x:M^- f(x)>\lambda\}} u(x)\,dx \le \frac{C}{\lambda}\int |f(x)|u(x)\,dx$$
for all $\lambda > 0$ and all functions $f \in L^1(u(x)\,dx)$.

(b) There exists a constant $C > 0$ such that
$$\sup_{c\in\mathbb{R}} \int_{\{x:|T_c f(x)|>\lambda\}} u(x)\,dx \le \frac{C}{\lambda}\int |f(x)|u(x)\,dx$$
for all $\lambda > 0$ and all functions $f \in L^1(u(x)\,dx)$.

(c) $u \in A_1^-$.

Taking into account this equivalence we can formulate another conjecture, a weaker conjecture.

Conjecture 3.1 (A weaker conjecture). *If $u \in A_1^-$ and $v \in A_1^+$ then there exists C such that*
$$\sup_{c\in\mathbb{R}} \int_{\{x:|T_c f(x)|>v(x)\}} uv \le C\int_{\mathbb{R}} |f|u$$
for all functions $f \in L^1(u(x)\,dx)$.

It is clear that this new conjecture is weaker since the Hardy operators are smaller than the one-sided Hardy-Littlewood maximal operator. We have been able to prove this weak conjecture. In fact, we have proved a more general theorem which we shall state below. To establish the theorem, we have first characterized the mixed weak type inequality for T_c (this characterization has already appeared in [7]).

Theorem 3.2. [7, 6] *Let u and v weights in $\mathbb{R}$. Let $c \in \mathbb{R}$. The following are equivalent*

(a) *There exists* $C > 0$ *such that*

$$\int_{\{x:|T_cf(x)|>v(x)\}} uv \le C \int_{\mathbb{R}} |f|u$$

for all all functions $f \in L^1(u(x)\,dx)$.

(b) *There exists* $\widetilde{C} > 0$ *such that*

$$\sup_{a>c} \sup_{\lambda>0} \lambda \int_{\{x>a:\frac{1}{x-c}>\lambda v(x)\}} uv \le \widetilde{C}u(x) \quad a.e.\ x \in (c,a).$$

Further, if C *and* $\widetilde{C}$ *are the best constants in those inequalities then* $\widetilde{C} \le C \le 4\widetilde{C}$.

As we said at the beginning of the paper, if we take $v \equiv 1$ in the last theorem then we obtain the characterization of the weak type $(1,1)$ inequality.

Corollary 3.3. *Let* u *and* v *be weights in* $\mathbb{R}$. *Let* $c \in \mathbb{R}$. *The following are equivalent*

(a) T_c *is of weak type* $(1,1)$ *with respect to* $u(x)\,dx$, *that is, there exists a constant* $C > 0$ *such that*

$$\int_{\{x:|T_cf(x)|>\lambda\}} u \le \frac{C}{\lambda} \int_{\mathbb{R}} |f|u$$

for all $\lambda > 0$ *and all functions* $f \in L^1(u(x)\,dx)$.

(b) *The function* u *satisfies* $A_1(T_c)$ $(u \in A_1(T_c))$, *that is, for all* $a > c$

$$\sup_{y>a} \frac{1}{y-c} \int_a^y u \le \widetilde{C}u(x) \quad a.e.\ x \in (c,a).,$$

Further, if C *and* $\widetilde{C}$ *are the best constants in those inequalities then* $\widetilde{C} \le C \le 4\widetilde{C}$.

We notice that this inequality was already characterized by Andersen and Muckenhoupt [1] with a different condition but one can see easily that Andersen-Muckenhoupt condition is equivalent to the one in the corollary.

The formal adjoint operator of the Hardy operator is given by

$$T_c^*f(x) = \begin{cases} \int_x^\infty \frac{f(t)}{t-c}\,dt, & \text{if } x > c; \\ 0, & \text{if } x \le c, \end{cases}$$

Arguing as before, the weak type $(1,1)$ inequality for this adjoint operator is characterized by the condition $A_1(T_c^*)$.

Theorem 3.4. *Let* u *and* v *be weights in* $\mathbb{R}$. *Let* $c \in \mathbb{R}$. T_c^* *is of weak type* $(1,1)$ *with respect to* $v(x)dx$ *if and only if* $v \in A_1(T_c^*)$, *that is, there exist* $C > 0$ *such that*

$$\frac{1}{x-c} \int_c^x v \le Cv(x) \quad \text{for almost every } x > c.$$

Now we are prepared to state our main theorem.

Theorem 3.5. [6] *Let u and v be weights in $\mathbb{R}$. Let $c \in \mathbb{R}$. Assume that there exists $\varepsilon > 0$ such that $u^{1+\varepsilon} \in A_1(T_c)$ and $v^{1+\varepsilon} \in A_1(T_c^*)$, that is, there exists $C > 0$ such that the following two conditions hold:*

(1) $\sup_{y>a} \frac{1}{y-c} \int_a^y u^{1+\varepsilon} \leq C u^{1+\varepsilon}(x)$ *for $a > c$ and for a.e. $x \in (c, a)$.*

(2) $\frac{1}{x-c} \int_c^x v^{1+\varepsilon} \leq C v^{1+\varepsilon}(x)$ *for a.e. $x > c$.*

Then there exists C such that

$$\int_{\{x:|T_c f(x)|>v(x)\}} uv \leq C \int_{\mathbb{R}} |f|u$$

all functions $f \in L^1(u(x)\,dx)$.

Notice that in the theorem we are assuming that $u^{1+\varepsilon}$ is a good weight for the weak type inequality for T_c while $v^{1+\varepsilon}$ is a good weight for the weak type inequality of the adjoint T_c^*. Then the theorem says that the mixed weak type inequality holds with a constant which depends only on the constants in the conditions $A_1(T_c)$ and $A_1(T_c^*)$.

The proof is reduced to show, that under the assumptions, the characterization of the previous theorem hold. In order to do that, we use Kolmogorov's inequality and Hölder's inequality. Once we have this theorem, the weaker conjecture can be proved and we state it as a theorem.

Theorem 3.6. [6] *If $u \in A_1^-$ and $v \in A_1^+$ then there exists C such that*

$$\sup_{c\in\mathbb{R}} \int_{\{x:|T_c f(x)|>v(x)\}} uv \leq C \int_{\mathbb{R}} |f|u$$

all functions $f \in L^1(u(x)\,dx)$.

The proof follows from the fact that u in A_1^- and v in A_1^+ imply that, for some $\varepsilon > 0$, $u^{1+\varepsilon}$ and $v^{1+\varepsilon}$ belong to A_1^- and A_1^+ respectively [11, 8] and, therefore, $u^{1+\varepsilon}$ is in $A_1(T_c)$ and $v^{1+\varepsilon}$ is in $A_1(T_c^*)$ with a uniform constant. Now the theorem follows applying Theorem 3.5.

We finish this section with some remarks and open questions.

Remark 3.7.

(a) *An analogous conjecture for M^+ can be stated assuming that $u \in A_1^+$ and $v \in A_1^-$. The problem for M^- and M^+, that is, Conjecture 2.1 remains open.*

(b) *We have proved the weaker conjecture. However, is Theorem 3.5 true assuming only that u is in $A_1(T_c)$ and v is in $A_1(T_c^*)$? We point out that u in $A_1(T_c)$ does not imply that $u^{1+\varepsilon}$ belong to $A_1(T_c)$ [6]. So, we do not know if Theorem 3.5 for T_c is true assuming only that u is in $A_1(T_c)$ and v*

is in $A_1(T_c^*)$. A partial answer is that the result is valid if u is decreasing and v is in $A_1(T_c^*)$.

(c) *The conjecture for M^- is true if u is decreasing.*

4. Ergodic Theorems

As we have said in the previous section, we have not been able to prove Conjecture 2.1. In this section we wish to show the interest of this conjecture in the setting of Ergodic Theory. In particular, we shall see that the conjecture implies an ergodic theorem.

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\Gamma = \{T^t : t \in \mathbb{R}\}$ be a strongly continuous group of positive ($f \geq 0 \Rightarrow T^t f \geq 0$) linear operators in some $L^p(\nu)$, $1 \leq p < \infty$, where ν is a σ-finite measure equivalent to μ ($\nu(E) = 0 \Leftrightarrow \mu(E) = 0$). Consider the ergodic averages

$$\mathcal{A}_\varepsilon^+ f(x) = \frac{1}{\varepsilon}\int_0^\varepsilon T_t f(x)dt.$$

One of the main problems in Ergodic Theory is to study the almost everywhere convergence of the averages. A usual approach to this problem is to study the boundedness of the ergodic maximal operator

$$\mathcal{M}^+ f(x) = \sup_{\varepsilon>0} |\mathcal{A}_\varepsilon^+ f(x)|.$$

This is the point we are going to check more carefully in this section.

For p greater than 1, it can be proved (see [9] for the discrete case) that if the averages are uniformly bounded in $L^p(\mu)$, that is,

$$\sup_{\varepsilon>0} ||\mathcal{A}_\varepsilon^+ f||_p \leq C||f||_p$$

with a constant independent of f, then the maximal operator is bounded in the same $L^p(\mu)$: there exists $C > 0$ such that

$$||\mathcal{M}^+ f||_p \leq C||f||_p$$

for all $f \in L^p(\mu)$. In what follows we shall give an idea of the proof. The main points in the proof are the following:

- The assumptions on the group imply [5, 4] that for all t, there exist a positive measurable function v_t and a positive multiplicative linear operator Φ^t such that Φ^t is a group and $T^t f(x) = v_t(x)\Phi^t f(x)$.
- For all t, there exists a positive measurable function $H_t(x)$, s.t.

$$\int_X |T^t f(x)|^p H_t(x)d\mu(x) = \int_X |f(x)|^p d\mu(x).$$

- The uniform boundedness of the averages, that is, $\sup_{\varepsilon>0} ||\mathcal{A}_\varepsilon^+ f||_p \leq C||f||_p$ implies that for almost every x the functions $w_x(t) = H_t(x)$ belong to the one-sided A_p^+ class (see [11, 8]), with a uniform A_p^+ constant (a constant independent of x) [9, 4]. Remind [11, 8] that if w is a function on the real line and $w \in A_p^+$ then $\int_{\mathbb{R}} |M^+ f|^p(x) w(x)\,dx \le C \int_{\mathbb{R}} |f|^p(x) w(x)\,dx$.

These properties allow to use transference arguments to establish the boundedness of the ergodic maximal operator. We shall sketch the proof of this transference argument. In order to do that, we work with nonnegative functions f and we consider the truncated maximal operator

$$\mathcal{M}_\eta^+ f(x) = \sup_{0<\varepsilon\le\eta} \mathcal{A}_\varepsilon^+ f(x).$$

It is clear that it will suffice to obtain the inequality for the truncated maximal operator with a constant independent of η. Using the properties of H_t, we introduce this function in the integral.

$$\int_X (\mathcal{M}_\eta^+ f(x))^p d\mu(x) = \frac{1}{R}\int_0^R \int_X |T^t \mathcal{M}_\eta^+ f(x)|^p H_t(x) d\mu(x) dt.$$

If f^x is the function defined on $\mathbb{R}$ by $f^x(t) = T^t f(x)$, applying Fubini's theorem and the positivity of T^t we are able to dominate by the one-sided Hardy-Littlewood maximal operator of the function f^x truncated in an interval and we have

$$\begin{aligned} &\frac{1}{R}\int_0^R \int_X |T^t \mathcal{M}_\eta^+ f(x)|^p H_t(x) d\mu(x) dt \\ &\qquad \le \int_X \frac{1}{R}\int_0^\infty |M^+(f^x \chi_{[0,R+\eta]})(t)|^p H_t(x)\,dt\,d\mu(x) \end{aligned}$$

Then we are faced with a weighted inequality for M^+. Since the weights $w_x(t) = H_t(x)$ are good weights for the strong type (p,p) of M^+, that is, $w_x \in A_p^+$ with a uniform A_p^+ constant, we can follow and estimate by the L^p-norm of f^x truncated in the interval. In this way, we obtain

$$\begin{aligned} &\int_X \frac{1}{R}\int_0^\infty |M^+(f^x \chi_{[0,R+\eta]})(t)|^p H_t(x)\,dt\,d\mu(x) \\ &\qquad \le C \int_X \frac{1}{R}\int_0^{R+\eta} |f^x(t)|^p H_t(x)\,dt\,d\mu(x) \\ &\qquad = C\frac{1}{R}\int_0^{R+\eta} \int_X |T^t f(x)|^p H_t(x)\,d\mu(x)\,dt \end{aligned}$$

Applying again the property of H_t and putting together all the inequalities, we have

$$\begin{aligned} \int_X (\mathcal{M}_\eta^+ f(x))^p d\mu(x) &\le C\frac{1}{R}\int_0^{R+\eta} \int_X |T^t f(x)|^p H_t(x)\,d\mu(x)\,dt \\ &= C\frac{1}{R}\int_0^{R+\eta} \int_X |f(x)|^p\,d\mu(x)\,dt \\ &= C\frac{R+\eta}{R}\int_X |f(x)|^p\,d\mu(x). \end{aligned}$$

Letting, first R, and then η, go to infinity we obtain the inequality for the ergodic maximal operator.

$$\int_X (\mathcal{M}^+ f(x))^p d\mu(x) \leq C \int_X |f(x)|^p d\mu(x),$$

What can be said for $p = 1$? Usually, we prove the weak type $(1,1)$ inequality for $\mathcal{M}$

$$\mu(\{x : \mathcal{M}^+ f(x) > \lambda\}) \leq C \int_X f \, d\mu$$

If we start by using transference arguments, using the same properties we obtain

$$\mu(\{x : \mathcal{M}^+_\eta f(x) > \lambda\})$$
$$\leq \int_X \frac{1}{R} \int_{\{x \in (0,\infty) : |M^+(f^x \chi_{[0,R+\eta]})(t)| > \lambda v_t(x)\}} v_t(x) H_t(x) \, dt \, d\mu(x)$$

Observe that now we are faced to a mixed weighted inequality. If the conjecture (for M^-) were true, and the weights belong to the corresponding one-sided A_1 classes (with an uniform constant), that is, $t \to v_t(x) \in A_1^-$ and $t \to H_t(x) \in A_1^+$, then

$$\begin{aligned}
\mu(\{x : \mathcal{M}^+_\eta f(x) > \lambda\}) &\leq \frac{C}{\lambda} \int_X \frac{1}{R} \int_0^{R+\eta} |f^x(t)| H_t(x) \, dt \, d\mu(x) \\
&= C \frac{1}{R} \int_0^{R+\eta} \int_X |T^t f(x)| H_t(x) \, d\mu(x) \, dt \\
&= C \frac{1}{R} \int_0^{R+\eta} \int_X |f(x)| \, d\mu(x) \, dt \\
&= C \frac{R+\eta}{R} \int_X |f(x)| \, d\mu(x).
\end{aligned}$$

Letting R and η go to infinity we would be done, assuming that the conjecture (for M^-) is true, and $t \to v_t(x) \in A_1^-$ and $t \to H_t(x) \in A_1^+$.

We point out that the assumptions on the weights can be translated to properties of the group. More precisely, the first one, $t \to v_t(x) \in A_1^-$, holds if and only the averages are uniformly bounded in $L^\infty(\mu)$: there exists $C > 0$ such that

$$\sup_{\varepsilon > 0} ||\mathcal{A}^+_\varepsilon f||_\infty \leq C ||f||_\infty,$$

for all functions $f \in L^\infty(\mu)$, while the second one, $t \to H_t(x) \in A_1^+$, means that the averages are uniformly bounded in $L^1(\mu)$: there exists $C > 0$ such that

$$\sup_{\varepsilon > 0} ||\mathcal{A}^+_\varepsilon f||_1 \leq C ||f||_1$$

for all functions $f \in L^1(\mu)$. Therefore, we have that if the conjecture for the one-sided Hardy-Litlewood maximal operator is proved then we would also have that if the averages are uniformly bounded in $L^1(\mu)$ and in $L^\infty(\mu)$ then the ergodic maximal operator is of weak type $(1,1)$. But, up till now, it is only a conjecture that we state explicitly.

Conjecture 4.1 (The conjecture in Ergodic Theory)**.** *If there exists $C > 0$ such that*

(a) $\sup_{\varepsilon>0} ||\mathcal{A}_\varepsilon^+ f||_\infty \le C||f||_\infty$ *for all* $f \in L^\infty(\mu)$ *and*
(b) $\sup_{\varepsilon>0} ||\mathcal{A}_\varepsilon^+ f||_1 \le C||f||_1$ *for all* $f \in L^1(\mu)$,

then

$$\mu(\{x : \mathcal{M}^+ f(x) > \lambda\}) \le \frac{C}{\lambda} \int_X |f| \, d\mu$$

for all $\lambda > 0$ and all functions in $L^1(\mu)$.

Remind that Dunford-Schwartz ergodic theorem [3] establishes for more general semigroups that if the operators of the semigroup are contractions in L^1 and in L^∞ then the ergodic maximal operator is of weak type $(1,1)$. In this way, the conjecture in ergodic theory would be a kind of generalization of Dunford-Schwartz ergodic theorem in our setting.

As a resume, let me say first that we have a conjecture for the one-sided Hardy-Littlewood maximal operator. Second, if this conjecture is proved to be true then we would have an ergodic theorem, that would be in some sense a generalization of Dunford-Schwartz ergodic theorem. However, at this moment, we only have a weaker result for the Hardy-operators.

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F. J. Martín–Reyes
Departamento de Análisis Matemático,
Facultad de Ciencias, Universidad de Málaga,
29071 Málaga, Spain
martin_reyes@uma.es

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