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GELFAND PAIRS RELATED TO GROUPS OF HEISENBERG TYPE

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ABSTRACT. In this article we collect some known results concerning (generalized) Gelfand pairs (K, N), where N is a group of Heisenberg type and K is a subgroup of automorphisms of N. We also give new examples.

1. Introduction and preliminary results

Let N be a two step nilpotent Lie group and assume that K acts on N by automorphisms. We denote by $K \rhd < N$ the semidirect product of N and K.

In this note we will describe some known results on (generalized) Gelfand pairs of the form $(K \rhd < N, K)$, and will also give some new examples in the case that N is a group of Heisenberg type.

Definition 1. *Let K be a compact subgroup of the automorphism group of N. We say that $(K \rhd < N, K)$ (or (K, N)) is a Gelfand pair if the convolution algebra $L^1_K(N)$ of $K-$invariant, integrable functions on N is commutative.*

Examples.

1. Let us consider $N = \mathbb{R}^n$ and $K = SO(n)$, the orthogonal group.

$$L^1_K(\mathbb{R}^n) = \left\{ f : R^n \to C \text{ radial such that } \int_0^\infty f(r)\, r^{n-1} dr < \infty \right\}$$

2. The Heisenberg group H_n is identified with $\mathbb{C}^n \times \mathbb{R}$ with law $(z, t)(z', t') = \left(z + z', t + t' + \frac{1}{2}\mathrm{Im} z.\overline{z'}\right)$ Then the unitary group $U(n)$ acts on H_n (by automorphisms) by

$$g(z, t) = (gz, t) \tag{1.1}$$

Let T^n be a maximal torus of $U(n)$.The pairs $(U(n), H_n)$ and (T^n, H_n) are Gelfand pairs. To see this, we check a well known criterion for Gelfand pairs: for each $(z, t) \in H_n$, there exists an automorphism in T^n that sends $(z, t) \to (-z, -t)$. Indeed, let us consider the involutive automorphism $\tau : (z, t) \to (\overline{z}, -t)$ and then compose τ with some $g \in T^n$ such that $g(\overline{z}) = -z$.

The subgroups K of $U(n)$ such that (K, H_n) are Gelfand pairs were determined by Benson, Jenkins and Ratcliff in [1].

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The main ingredients for the proof are a Carcano criterion and a Kac list, which we now describe. We recall that the irreducible, unitary, representations of H_n are

* Infinite-dimensional representations, parametrized by $0 \neq \lambda \in \mathbb{R}$. The corresponding representation π_λ is realized on the Fock space F_λ of entire functions on $\mathbb{C}^n$, which are square integrable with respect to the measure $e^{-|z|^2}$. We have that the polynomial algebra $P(\mathbb{C}^n) \subset F_\lambda$.

* Unitary characters, $\chi_w(z,t) = e^{i\langle z,w\rangle}$, defined for each $w \in \mathbb{C}^n$.

Let us consider $K \subset U(n)$. For each π_λ and $k \in K$, let π_k be the representation

$$\pi_\lambda^k(n) = \pi_\lambda(kn). \tag{1.2}$$

Since K acts trivially on the center of H_n, we have $\pi_\lambda^k \simeq \pi_\lambda$. So for $k \in K$, we can choose an operator $\omega_\lambda(k)$ which intertwines π_λ and π_λ^k. By Schur Lemma, ω_λ is a projective representation of $U(n)$, called the *metaplectic representation.*

Explicity

$$\omega_\lambda(k)\,p(z) = p\left(k^{-1}z\right) \tag{1.3}$$

Up to a factor of $\det(k)^{\frac{1}{2}}$, ω_λ lifts to a representation on the double covering of $U(n)$.

Theorem 1. *(Carcano, see* [1]*)* (K, H_n) *is a Gelfand pair if and only if the action of* ω_λ *on* F_λ *is multiplicity free, that is, each irreducible (proyective) representation of* K *appears in* $(\omega_\lambda, F_\lambda)$ *at most once.*

The Kac list (see [4]) gives precisely the triplets (K_c, W, ρ) where K_c is a complex group,

$$\rho : K \to GL(W)$$

is an irreducible representation and the induced action on $P(W)$ is multiplicity free.

For $p \in P(W)$, the action is given by

$$(\rho(g)\,p)(v) = p\left(\rho\left(g^{-1}\right)v\right) \tag{1.4}$$

So

Theorem 2. *Let* K *be a connected subgroup of* $U(n)$. *Then* (K, H_n) *is a Gelfand pair if and only if* $(K_c, \mathbb{C}^n)$ *appears in table1.8, page 415, in* [1].

We now introduce the groups of Heisenberg type (see [5]).

Consider two vector spaces V and Z, endowed with inner products $\langle,\rangle_V$ and $\langle,\rangle_Z$, a nondegeneraterate skew-symmetric bilinear form $\Psi : V \times V \to Z$, and define a Lie algebra $\eta = V \oplus Z$ by $[(v,z),(v',z')] = (0, \Psi(v,v'))$.

For $V = \mathbb{R}^{2n}$ and $Z = \mathbb{R}$, there is (up to isomorphism) only one such Ψ, and the corresponding η is the Heisenberg Lie algebra.

We say that η is of Heisenberg type if $J_z : V \to V$ given by

$$\langle J_z v, w\rangle = \langle z, \Psi(v,w)\rangle \tag{1.5}$$

is an orthogonal transformation for all $z \in Z$ with $|z| = 1$.

A connected and simply connected Lie group N is of Heisenberg type if its Lie algebra is of type H. Since for $|z| = 1$, J_z is both orthogonal and skew-symmetric we have

$$J_z^2 = -Id.$$

So by linearity and polarization we have for $z, w \in Z$

$$J_z J_w + J_w J_z = -2 \langle z, w \rangle Id. \tag{1.6}$$

Let $m := \dim Z$ and let $C(m)$ be the Clifford algebra $C\left(Z, -|.|^2\right)$. Then the action J of Z on V extends to a representation of $C(m)$. It is well known that $C(m)$ is isomorphic to a matrix algebra, over the real, complex or quaternionic numbers, for $m \equiv 1, 2, 4, 5, 6, 8 \pmod 8$. So in this cases $C(m)$ has, up to equivalence, only one irreducible module. For $m \equiv 3, 7 \pmod 8$, $C(m)$ is isomorphic to a direct sum of two matrix algebras and it has two inequivalent irreducible modules, say V_+ and V_-.

We say that η is *irreducible or isotypical*, if so is V as a representation of $C(m)$.

Let $A(N)$ be the group of automorphisms of N that acts by orthogonal transformations on η. Kaplan and Ricci raised in [7] the question of when (K, N) is a Gelfand pair, for some specific subgroups K of $A(N)$.

The structure of $A(N)$ has been given by Riehm in [12] .

$$\text{Let } U_0 = \left\{g \in A(N) : g_{|Z} = Id\right\}, \text{ and}$$

$$\text{let } Pin(m) \text{ be the group generated by } \{(-\rho_z, J_z) : z \in \mathfrak{z}, |z| = 1\},$$

where $\rho_z : \mathfrak{z} \to \mathfrak{z}$ denotes the reflection through the hiperplane orthogonal to z.

Also denote by $Spin(m)$ the subgroup generated by the even products $(\rho_z \rho_w, J_z J_w)$

Let us denote by l (respectively l_+, l_-) the multiplicity of the unique irreducible module (resp. V_+, V_-) in V.

Then U_0 is a classical group given by the following table

$U(l, \mathbb{C})$, ... $\mathbf{m} \equiv \mathbf{1} \pmod 8$
$U(l, \mathbb{H})$, ... $\mathbf{m} \equiv \mathbf{2} \pmod 8$
$U(l_+, H) \times U(l_-, H)$, ... $\mathbf{m} \equiv \mathbf{3} \pmod 8$
$U(l, H)$, ... $\mathbf{m} \equiv \mathbf{4} \pmod 8$
$O(2l, R)$, ... $\mathbf{m} \equiv \mathbf{5} \pmod 8$
$O(l, R)$, ... $\mathbf{m} \equiv \mathbf{6} \pmod 8$
$O(l_1, R) \times O(l_2, R)$, ... $\mathbf{m} \equiv \mathbf{7} \pmod 8$
$O(l, R)$ .... $\mathbf{m} \equiv 8 \pmod 8$

Also we have that $Pin(m)$ and U_0 commute, and their intersection contains at most four elements. Moreover, $A(N) = Pin(m) \times U_0$, unless $m \equiv 1 \pmod 4$. In this case $A(N) / Pin(m) \times U_0$ has two elements.

F. Ricci determined in [11] the groups N for which $(A(N), N)$ is a Gelfand pair. We give a sketch of the proof.

For $a \in Z, |a| = 1$, let us consider the complex space $V_a = (V, J_a)$ and the Lie algebra $\eta_a = \mathbb{R}a \oplus V_a$, with bracket given by

$$\langle a, [v, w] \rangle = \langle J_a v, w \rangle .$$

Then η_a is a Heisenberg algebra. Denote by N_a the corresponding Heisenberg group. Set $K = A(N)$ and $K_a = \{k \in A(N) : k(a) = a\}$. Since K_a acts trivially on the center of N_a, it is a subgroup of the unitary group $U(V_a)$. We know that $L^1_{K_a}(N_a)$ is commutative if and only if the metaplectic action of K_a on $P(V_a)$ is multiplicity free, that is by using the Kac list. Also

Theorem 3. *([11]) $L^1_K(N)$ is commutative if and only if $L^1_{K_a}(N_a)$ is commutative.*

Theorem 4. *([11]) The groups N such that $(A(N), N)$ are Gelfand pairs are those for which*

$m = 1, 2$ or 3,
$m = 5, 6$ or 7 and V irreducible,
$m = 7, V$ isotypic and $\dim V = 16$.

2. Examples of generalized Gelfand pairs

Coming to the general theory of Gelfand pairs, we recall that the following conditions are equivalent:

(i) $L^1_K(N)$ is commutative.
(ii) The algebra $D_K(N)$ of left and $K-$invariant differential operators on N, is commutative.
(iii) For each irreducible, unitary representation π of $K \rhd < N$, the space of vectors fixed by K is at most one dimensional.

The notion of Gelfand pair was extended to non compact, unimodular subgroups K of a unimodular Lie group G.

Given a representation (π, H) of a Lie group G, we say that v is a $C^\infty-$vector if the map

$$g \to \pi(g) v$$

is infinitely differentiable. We denote by H^∞ the space of $C^\infty-$vectors and by $H^{-\infty}$ the dual space of H^∞.

The elements of $H^{-\infty}$ are called distribution vectors. The action π on H induces a natural action on $H^{-\infty}$, given by

$$\langle \pi_{-\infty}(g) \mu, v \rangle = \langle \mu, \pi_\infty(g) v \rangle \tag{2.1}$$

for $v \in H^\infty$.

Definition 2. *We say that (G, K) is a generalized Gelfand pair if for each irreducible, unitary representation π of G, the space of distribution vectors fixed by K is at most one dimensional.*

A nice survey on the subject is in [14]. In particular, there are given examples of pair (G,K) such that $D_G(G/K)$ is commutative but (G,K) is not a generalized Gelfand pair, contrasting with the compact case.

In [10], Mokni and Thomas considered the cases (K,H_n) where K is a subgroup of $U(p,q) \subset Aut(H_n)$, $p+q=n$, extending the Carcano criterion. Indeed, their result states that for $K \subset U(p,q)$, (K,N) is a Gelfand pair if and only if the restriction of ω to K is multiplicity free.

Later on we will comment the idea of the proof.

With F. Levstein we considered in [8] the pairs (K,N) where N is of Heisenberg type and K is roughly the group of automorphisms that preserves the decomposition $\eta = V \oplus Z$.

We have that

$$K = Spin(m) \times U,$$

(direct product), where $U = \left\{g \in Aut(N) : g_{|Z} = Id\right\}$ is given by the following list (see [13]):

$$\begin{aligned}
Sp(l,\mathbb{R}), \ldots\ldots\ldots\ldots \mathbf{m} &\equiv \mathbf{1} \pmod 8\\
Sp(l,\mathbb{C}), \ldots\ldots\ldots\ldots \mathbf{m} &\equiv \mathbf{2} \pmod 8\\
(U(l_+,l_-),\mathbb{H}) \ldots\ldots\ldots\ldots \mathbf{m} &\equiv \mathbf{3} \pmod 8\\
(Gl(l),H), \ldots\ldots \mathbf{m} &\equiv \mathbf{4} \pmod 8\\
SO^*(2l), \ldots\ldots\ldots\ldots \mathbf{m} &\equiv \mathbf{5} \pmod 8\\
O(l,C), \ldots\ldots\ldots\ldots \mathbf{m} &\equiv \mathbf{6} \pmod 8\\
O((l_+,l_-),R), \ldots\ldots\ldots\ldots \mathbf{m} &\equiv \mathbf{7} \pmod 8\\
Gl(l,R) \ldots\ldots\ldots\ldots \mathbf{m} &\equiv 8 \pmod 8
\end{aligned}$$

Remark 1. *U_0 is the maximal, compact subgroup of U and when V is irreducible and $m \equiv 3,5,6,7 \pmod 8$ one has $U = U_0$.*

For the classical Heisenberg group H_n, that is, for $m=1$, we have $U = Sp(n,R)$.

Theorem 5. *([8]) Assume that N is irreducible. Then (K,N) is a generalized Gelfand pair if and only if $1 \le m \le 9$.*

To give a sketch of the proof we begin by describing the representations of $K \triangleright < N$. According to Mackey theory (see [9]), these are given in terms of the representations of N.

The irreducible, unitary, representations of a group of Heisenberg type N are:

* Infinite -dimensional representations, parametrized by the non zero elements of the centre Z : for $0 \neq a \in Z$, $|a| = 1$, the corresponding representation π_a is realized on the Fock space F_a of entire functions on (V, J_a).

* Unitary characters, $\chi_v(z,w) = e^{i\langle w,v\rangle}$, defined for each $v \in V$.

The representations of $K \triangleright < N$ coming from characters of N are irreducible, unitary representations of $K \triangleright < V$.

As observed in [10], since V is an abelian group, (K,V) is a generalized Gelfand pair and so the space of distribution vectors fixed by K is at most one dimensional.

Then, in order to determine when (K, N) it is a generalized Gelfand pair, it is enough to consider only those representations of $K \rhd < N$ associated to π_a, for $a \in Z$.

Let $K_a = \{k \in K : k(a) = a\}$,

We observe that

$$K_a = Spin_a(m)\,U,$$

where $Spin_a(m)$ is generated by $\{J_b J_c : b \perp a \perp c, |b| = |c| = 1\}$.

Since the elements of $Spin_a(m)$ are orthogonal transformations which commute with J_a, $K_a \subset Sp(V, J_a) = \{g \in Gl(V) : g^t J_a g = J_a\}$. Also $Sp(V, J_a)$ is the group of automorphims of the Heisenberg group $N_a = Ra \oplus V$, which fix the centre $\mathbb{R}a$.

According to [9], the representations of $K \rhd < N$ "coming" from π_a are *induced* by those of $K_a \rhd < N_a$. So we introduce the notion of induced representation:

Let H be a subgroup of Lie group G, and let (ρ, V_ρ) a unitary representation of H. Set

$$C(G; V_\rho) = \left\{f : G \to V_\rho \text{ continuos} : f(gh) = \rho(h^{-1})f(g)\right\}$$

for all $g \in G, h \in H$, and $\int_{K/H} |f(x)|^2\, dx < \infty$.

Then $Ind_H^G(V_\rho)$ is the completion of $C(G; V_\rho)$, and the action of G is by left translations.

Moreover, a C^∞–vector of $Ind_H^G(V_\rho)$ is an infinitely differentiable function $f \in C(G; W)$ (see [16], page 373.)

Theorem 6. *(see* [8]*, cfr* [11]*) (K, N) is a generalized Gelfand pair if and only if, for each $a \in Z, (K_a, N_a)$ is a generalized Gelfand pair.*

Sketch of the proof.

Let (ρ, V_ρ) be an irreducible representation of $K_a \rhd < N_a$ and assume that T is a distribution vector of V_ρ, fixed by K_a.

We know that $(\pi, H_\pi) := Ind_{K_a N}^{KN}(V_\rho)$ is an irreducible representation of $K \rhd < N$.

We define $\mu : H_\pi^\infty \to \mathbb{C}$ by

$$\langle \mu, f \rangle := \left\langle T, \int_{Spin(m)} f \right\rangle \tag{2.2}$$

For a non zero distribution vector T and $v \in V_\rho$ such that $\langle T, v \rangle \neq 0$, we construct some $f_v \neq 0$ such that

$$\left\langle T, \int_{Spin(m)} f_v \right\rangle \neq 0$$

Let us see that μ is $\pi(K)$–invariant. We recall that the action of π on H_π is by left translations. For $u \in U$,

$$\langle \mu, L_u f \rangle = \left\langle T, \int_{Spin(m)} L_u f \right\rangle.$$

Since $Spin\,(m)$ commutes with U, we have $\int_{Spin(m)} L_u f dk = \int_{Spin(m)} f(uk)dk = \int_{Spin(m)} f(ku)dk = \rho(u^{-1}) \int_{Spin(m)} f(k)dk$. So by the $U-$invariance of T we have $\left\langle T, \int_{Spin(m)} L_u f \right\rangle = \left\langle \rho_{-\infty}(u)T, \int_{Spin(m)} f \right\rangle = \left\langle T, \int_{Spin(m)} f \right\rangle$

Finally if $h \in Spin(m)$, $\langle \mu, L_h f \rangle = \left\langle T, \int_{Spin(m)} L_h f \right\rangle = \left\langle T, \int_{Spin(m)} f \right\rangle$ by the left invariance of the integral.

Replacing T by T_j and choosing $v_j \neq 0$ such that $\langle T_j, v_j \rangle \neq 0$, the above argument shows that there exist two non zero distribution vectors, fixed by K.

They are linearly independent: indeed, if $a\mu_1 + b\mu_2 = 0$ then $0 = \langle a\mu_1 + b\mu_2, f \rangle = \left\langle aT_1 + bT_2, \int_{Spin(m)} f \right\rangle$ for all $f \in C^{\infty}(K;\rho)$. But the above construction implies that $aT_1 + bT_2 = 0$ and so $a = b = 0$.

Conversely, let (π, H_π) be an irreducible representation of $K \rhd < N$ and assume that there exist two linearly independent distribution vectors μ_1, μ_2 fixed by K. So this representation can not be induced by a character. So

$$H_\pi = Ind_{K_a \lhd N_a}^{K \lhd N} (V_\rho).$$

Define $T_j \in V_\rho^{-\infty}$ by (2.2) :

$$\left\langle T_j, \int_{Spin(m)} f \right\rangle := \langle \mu_j, f \rangle .$$

We prove that T_j is well defined. Moreover that T_j is defined on a dense subset of V_ρ^{∞} , which is the subspace generated by the vectors $\rho\,(\psi)\,v$, $\psi \in C^{\infty}\,(K_a N_a)$, and finally, that T_i are K_a-invariant and linearly independent. □

Now we have reduced the problem to the pairs (K_a, N_a).

Again by Mackey theory, the irreducible unitary representations of $K_a \rhd < N_a$ are of the form

$$\rho = \tau \otimes \omega_\lambda \pi_\lambda, \tag{2.3}$$

where π_λ acts on the Fock space F_λ and τ is an irreducible representation of K_a. Thus

$$\rho / K_a = \tau \otimes \omega_\lambda$$

It is proved in [10] that $\tau \otimes \omega_\lambda$ has r linearly independent distributions vectors if and only if r is the *multiplicity* of τ in ω_λ.

According to this, we are interested in determining when the restriction of the metaplectic representation $\omega \downarrow_{K_a}^{Sp(V,J_a)}$ is multiplicity free, where $K_a = Spin_a\,(m) \times U$.

If N is an irreducible group of type H, the corresponding subgroup U is :

$$\begin{aligned}
SL(2,\mathbb{R}),\text{...........}\mathbf{m} &\equiv \mathbf{1}\pmod 8\\
SL(2,\mathbb{C}),\text{...........}\mathbf{m} &\equiv \mathbf{2}\pmod 8\\
\mathbb{H}^*,\text{....................}\mathbf{m} &\equiv \mathbf{3}\pmod 8\\
SU(2)\times R^*,\text{.......}\mathbf{m} &\equiv \mathbf{4}\pmod 8\\
U(1),\text{.................}\mathbf{m} &\equiv \mathbf{5}\pmod 8\\
O(1),\text{.................}\mathbf{m} &\equiv \mathbf{6}\pmod 8\\
O(1),\text{.................}\mathbf{m} &\equiv \mathbf{7}\pmod 8\\
\mathbb{R}^*,\text{....................}\mathbf{m} &\equiv \mathbf{8}\pmod 8
\end{aligned}$$

When $\mathbf{m}\equiv 3,5,6,7 \pmod 8$, U is compact and, by the results proved in [11], we know that $(Spin(m)\times U,)$ is a Gelfand pair if and only if $m=5,6,$ or 7.

Thus, we will study the restriction of the metaplectic representation $\omega\downarrow_{K_a}^{Sp(V,J_a)}$ for $\mathbf{m}\equiv 1,2,4,8 \pmod 8$.

To this end we will use the Kac list mentioned before.

Moreover, let us denote by $\mathbb{T}$ the one dimensional torus, and by $P_r(\mathbb{C}^n), r\in\mathbb{N}$, the space of homogeneous polynomials of degree α with $|\alpha|=r$. Then $\mathbb{T}$ acts on $P_r(\mathbb{C}^n)$ by e^{irt}, that is, by degree.

Remark 2. *Let H be a subgroup of $U(n)$. Then H acts without multiplicity on each $P_r(\mathbb{C}^l), r\in\mathbb{N}$, if and only if the action of $H_{\mathbb{C}}\times\mathbb{C}^*$ on $P(\mathbb{C}^l)$ is multiplicity free, if and only if $H_{\mathbb{C}}\times\mathbb{C}^*$ appear in the Kac list.*

Remark 3. *We recall that there are two models for the representations of the Heisenberg group. The Fock model realized on the space of holomorphic functions on (V,J_a) which are square integrable with respect to the measure $e^{-|z|^2}dz$ and the Schroedinger model realized on $L^2(R^N), N=\frac{\dim V}{2}$. An intertwining operator sends the monomials $z^\alpha=z_1^{i_1}z_2^{i_2}...z_N^{i_l}$ to the Hermite function $h_\alpha(x)=h_{i_1}(x_1)h_{i_2}(x_2)...h_{i_N}(x_N)$ where $h_i(t)=H_i(t)e^{-\frac{t^2}{2}}$ and $H_i(t)$ is the Hermite polynomial of degree i.*

Write $V=\mathbb{R}^N\oplus J_a\mathbb{R}^N$. Then the metaplectic action of $SO(N)$ on $P_r(V)$ corresponds to the natural action of $SO(N)$ on $P_r(\mathbb{R}^N)$.

The Mellin transform is the Fourier transform adapted to $\mathbb{R}_{>0}$ and it is defined by

$$Mf(\lambda)=\int_0^\infty f(s)\,s^{i\lambda}\frac{ds}{s} \tag{2.4}$$

The action of $\mathbb{R}_{>0}$ on $L^2(\mathbb{R}_{>0},\frac{ds}{s})$ given by $\delta_t f(s)=f(ts)$ decomposes, via the Mellin transform, as

$$L^2(\mathbb{R}_{>0},\frac{ds}{s})=\int_{-\infty}^\infty F_\lambda d\lambda \tag{2.5}$$

where F_λ is the $\mathbb{C}-$vector space generated by $s^{i\lambda}$.

We observe that the module generated by $g_r(s)=s^re^{-s},\ r\in\mathbb{N}$, is $L^2(\mathbb{R}_{>0},s^{-1}ds)$. Indeed, by a well known Wiener theorem, it is enough to prove that $Mg_r(s)\neq 0$

for all s, but this holds since $Mg_r(\lambda) = \int s^r e^{-s} s^{i\lambda} \frac{ds}{s} = \Gamma(r-1+i\lambda) \neq 0$, where Γ denotes the gamma function.

$\mathbf{m} \equiv \mathbf{4}\,(8)$.

First, we have to understand how $Spin_a(m) \times U$ is embedded in $Sp(J_a, V)$ and the corresponding metaplectic action. In this case

$$U = Gl(1,\mathbb{H}) = SU(2) \times \mathbb{R}_{>0}, \text{ and}$$

$$V = V_\Lambda \oplus J_a V_\Lambda,$$

where V_Λ is the real spin representation. Thus

$$Spin_a(m) \to SO(N)$$

via the spin representation. Also, $Gl(1,\mathbb{H}) \to Sp(V, J_a)$ as $q \to a_q = \left(R_q, R_{\overline{q}^{-1}}\right)$.

Thus $SU(2)$ acts by right multiplication by q and the metaplectic action of $Spin_a(m) \times SU(2)$ on $L^2\left(\mathbb{R}^N\right)$ is the natural one of $SO(N)$.

Setting $L^2\left(\mathbb{R}^N, dx\right) = L^2\left(S^{N-1}, d\sigma\right) \otimes L^2\left(\mathbb{R}_{>0}, r^{n-1}dr\right)$, we have that the action of $\mathbb{R}_{>0}$ is given by

$$\omega(a_t) f(x) = t^{\frac{N}{2}} f(tx), \ t \in \mathbb{R}_{>0}, x \in R^N. \tag{2.6}$$

This last action is equivalent to $\delta_t f(s) = f(ts)$ on $L^2(\mathbb{R}_{>0}, \frac{ds}{s})$.

Assume that the action of $Spin_a(m) \times SU(2)$ is multiplicity free on each $P_r(V)$ and let V_α be an irreducible representation of $Spin_a(m) \times SU(2)$ in $P_r(V)$. For $p \in V_\alpha$, we consider the function $p(x) e^{-\frac{|x|^2}{2}} = p\left(\frac{x}{|x|}\right) |x|^r e^{-\frac{|x|^2}{2}}$. Since $SO(N)$ acts on $p\left(\frac{x}{|x|}\right)$ in the natural way, and the action of $\mathbb{R}_{>0}$ on $s^r e^{-s}$ generates a space isomorphic to $L^2(\mathbb{R}_{>0}, \frac{ds}{s})$, we conclude that the K_a- module generated by V_α is $V_\alpha \otimes L^2\left(\mathbb{R}_{>0}, s^{n-1}ds\right)$. So

$$\omega \downarrow_{K_a}^{Sp(V,J_a)} = \oplus_\alpha \int_{-\infty}^{\infty} \alpha \otimes e^{i\lambda t} dt$$

and the decomposition is multiplicity free.

The converse follows the same lines.

Since $\mathbf{m} \equiv \mathbf{4}\,(8)$, we have that V is a complex irreducible $Spin_a(m) \times SU(2)$-module. By looking at the Kac list, we know that the action of $Spin_a(m) \times SU(2) \times T$ on $P(V)$ is multiplicity free only for $m = 4$. This case corresponds to the action of $GL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the decomposition of $\omega \downarrow_{K_a}^{Sp(V,J_a)}$ was given in [2].

$\mathbf{m} \equiv \mathbf{0}\,(8)$.

In this case $U = R^*$ and the action is given by

$$\omega(a_t) f(x) = |t|^{\frac{N}{2}} f(tx) \tag{2.7}$$

We observe that $-I \in Spin_a(m) \cap U$. Thus the action of K_a on $L^2\left(\mathbb{R}^N\right)$ is the same action of $Spin_a(m) \times \mathbb{R}_{>0}$ and we repeat the argument of the above proof to conclude that $\omega \downarrow_{K_a}^{Sp(V,J_a)}$ is multiplicity free only for $m = 8$.

$\mathbf{m} \equiv \mathbf{1}\,(8)$

In this case $U \simeq Sl(2,\mathbb{R})$ and $K_a \simeq Spin_a(m) \times Sl(2,\mathbb{R})$. Also, V can be decomposed as $Spin(m)-$ module in an orthogonal direct sum

$$V = V_\Lambda \oplus J_a V_\Lambda$$

where V_Λ is the real spin representation of $Spin(m)$. So $\dim V_\Lambda = N$ and $Spin_a(m)$ is embedded in $SO(N)$. But, as $Spin_a(m)-$module,

$$V_\Lambda = V_{\Lambda^+} \oplus V_{\Lambda^-}$$

where $V_{\Lambda^+}, V_{\Lambda^-}$ are the half spin representations. Thus

$$Spin_a(m) \hookrightarrow SO\left(\frac{N}{2}\right) \times SO\left(\frac{N}{2}\right) \hookrightarrow SO(N)$$

Besides, $Sl(2,\mathbb{R})$ is embedded in $Sp(V,J_a)$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} aI & -bQ \\ cQ & dI \end{pmatrix}$, where $Q = Q^t, QQ^t = I$ (see [6].)

It is well known that (see [15])

$$\omega \downarrow^{Sp(V,J_a)}_{SO(N)\times Sl(2,\mathbb{R})} = \oplus_k V_{k\Lambda} \otimes D_{l(k)} \tag{2.8}$$

where $V_{k\Lambda}$ denotes the irreducible representation of $SO(N)$ on the harmonic polynomials of degree k on V_Λ, and $D_{l(k)}$ is a discrete series representation of $SL(2,\mathbb{R})$ and $l(k) = \frac{k}{2} + \frac{N}{4}$ denotes the lowest K-type. Also

$$V_{k\Lambda} \downarrow^{SO(N)}_{SO\left(\frac{N}{2}\right)\times SO\left(\frac{N}{2}\right)} = \oplus_{r,s} V_{r\Lambda^+} \otimes V_{s\Lambda^-}, \tag{2.9}$$

where the sums runs over the integers r, s such that $k - r - s$ is an even, non negative integer .

We consider two possibilities for m.

Case $m \neq 9$.

We have that as $SO\left(\frac{N}{2}\right)$-modules, $P_r(V^+) = V_{r\Lambda^+} \oplus V_{(r-2)\Lambda^+} \oplus V_{(r-4)\Lambda^+} \oplus ...$ and $P_r(V^-) = V_{r\Lambda^-} \oplus V_{(r-2)\Lambda^-} \oplus V_{(r-4)\Lambda^-} \oplus ....$As $Spin_{\mathbb{C}}(m-1) \times \mathbb{C}^*$ does not appear in the Kac list, we deduce that there exists r for which the action of $Spin_a(m)$ on $P_r(V^+)$ can not be multiplicity free. Thus there exists an irreducible representation α that appears in $V_{(r-2i)\Lambda^+}$ and in $V_{(r-2j)\Lambda^+}$, for some i, j. Then $V_\alpha \otimes V_{r\Lambda^-}$ appears in $V_{(r-2i)\Lambda^+} \otimes V_{r\Lambda^-}$ and in $V_{(r-2j)\Lambda^+} \otimes V_{r\Lambda^-}$ concluding that $V_{k\Lambda} \downarrow^{SO\left(\frac{N}{2}\right)\times SO\left(\frac{N}{2}\right)}_{Spin_a(m)}$ is not multiplicity free.

Case $m = 9$.

In this case, $V_{j\Lambda^\pm}$ is irreducible for all j and the action of $Spin_a(m)$ on $P_r(V^+)$ is multiplicity free.

$\omega \downarrow^{Sp(V,J_a)}_{K_a}$ is still multiplicity free and the proof together with the corresponding decomposition was given in [2].

$\mathbf{m \equiv 2}\,(8)$

In this case $U \simeq Sl(2,\mathbb{C})$ and we can assume $m \geq 10$. Then $K_a \simeq Spin_a(m) \times Sl(2,\mathbb{C})$ and as $Spin_a(m)-$ module

$$V = V_\Lambda \oplus J_a J_b V_\Lambda \oplus J_a V_\Lambda \oplus J_b V_\Lambda$$

where a is orthogonal to b, and V_Λ denotes its real spin representation. Since $\dim V_\Lambda = \frac{N}{2}$, $Spin_a(m)$ is embedded in $SO\left(\frac{N}{2}\right)$. Besides, $Sl(2,\mathbb{C})$ is embedded in $Sp(V, J_a)$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} aI & -bQ \\ cQ & dI \end{pmatrix}$, where a, b, c, d belong to $\mathbb{C} = \{\alpha + \beta J_a J_b \text{ s.t. } \alpha, \beta \in \mathbb{R}\}$.

Adams and Barbasch proved that the restriction of ω to $O\left(\frac{N}{2}, \mathbb{C}\right) \times Sl(2,\mathbb{C})$ is multiplicity free and decomposes as $\omega \downarrow^{Sp(V,J_a)}_{O\left(\frac{N}{2},\mathbb{C}\right)\times Sl(2,\mathbb{C})} = \int_\oplus P_\lambda(L^2(\mathbb{R}^N))d\mu(\lambda)$, where $P_\lambda(L^2(\mathbb{R}^N)) \simeq \pi_\lambda \otimes \pi^\lambda$. Moreover they gave explicitely the correspondence $\pi_\lambda \to \pi^\lambda$. D. Barbasch pointed to us that we can consider a tempered representation π^λ of $SL(2,\mathbb{C})$, and in that case, the restriction to $SO\left(\frac{N}{2}, \mathbb{R}\right)$ of the corresponding π_λ is not multiplicity free.

Indeed, let π^λ be a tempered representation of $SL(2,\mathbb{C})$ then $\pi^k := \pi^\lambda$ is a unitary principal series of $SL(2,\mathbb{C})$ with lowest K-type, the $k+1-$dimensional irreducible module of $SU(2)$.

The corresponding $\pi_k := \pi_\lambda$ is the unitary principal series of $O\left(\frac{N}{2}, \mathbb{C}\right)$ with lowest K-type the irreducible representation of $SO\left(\frac{N}{2}, \mathbb{R}\right)$ given by the harmonic polynomials on V_Λ of degree k.

We proved that the restriction of π_k to $SO\left(\frac{N}{2}, \mathbb{R}\right)$ is not multiplicity free. First we recall that if $O\left(\frac{N}{2}, \mathbb{C}\right) = O\left(\frac{N}{2}, \mathbb{R}\right) AN$ denotes the Iwasawa decomposition, then the commutator M of A in $O\left(\frac{N}{2}, \mathbb{R}\right)$ is a maximal torus of it. Thus, by Frobenius reciprocity, the multiplicity of the representation with highest weight $2k\Lambda$ in π_k, $[\pi_k : V_{2k\Lambda}]$ is equal to $m_{2k\Lambda}(k\Lambda)$, the multiplicity of the weight $k\Lambda$ in $V_{2k\Lambda}$.

We compute $m_{2k\Lambda}(k\Lambda)$ by using Kostant multiplicity formula (see [3]).

Proposition 1. *(see [8])*

$$m_{2k\Lambda}(k\Lambda) = \binom{\frac{N}{4} + j - 1}{j} \quad \textit{for even } k = 2j, \tag{2.10}$$

$$m_{2k\Lambda}(k\Lambda) = 0 \quad \textit{otherwise}$$

□

References

[1] Benson, C., Jenkins, J., Ratcliff, G. *Bounded K-spherical functions on Heisenberg groups,* J. Funct. Analysis 105, (1992), 409-443.

[2] Galina, E., Kaplan, A., Levstein, F. *The oscillator representation and groups of Heisenberg type,* Comm. Math. Phys 210, (2000), 309-321.

[3] Humpreys, J. *Introduction to Lie algebras and representation theory.* Graduate texts in Math.9, Springer, (1972).

[4] Kac,V. *Some remarks on nilpotent orbits,* J. Algebra 64 (1980) , 190-213.

[5] Kaplan, A. *Fundamental solutions for a class of hipoelliptic P.D.E. generated by composition of quadratic forms*, Trans. Amer. Math. Soc. 29 (1980), 145-153.

[6] Kaplan, A., Levstein, F., Saal, L., Tiraboschi, A. *Horizontal submanifolds of groups of Heisenberg type.* Annali di Matematica 187, (2008), 67-91.

[7] Kaplan, A., Ricci, F. *Harmonic analysis on groups of Heisenberg type*, Lectures Notes in Math., vol 992, Springer, (1983), 416-435

[8] Levstein, F. Saal, L. *Generalized Gelfand pairs associated to Heisenberg type groups.* J. Lie Theory 18 (2008), no. 3, 503–515.

[9] Mackey, G. *Unitary group representations in physics, probability, and number theory.*, Probability and Number Theory, Benjamin-Cummings, 1978.

[10] Mokni, K., Thomas, E.G.F. *Paires de Guelfand généralisées associées au groupe d'Heisenberg* J. Lie Theory 8, (1998) , 325-334.

[11] Ricci, F. *Commutative algebras of invariant functions on groups of Heisenberg type,* J London Math. Soc. (2) 32, (1985) , 265-271.

[12] Riehm, C. *The automorphism group of a composition of quadratic form.* Trans. Amer. Math. Soc. 269, (1982), 403-415.

[13] Saal, L. *The automorphism group of a Lie algebra of Heisenberg type,* Rend. Sem. Mat. Univ. Pol. Torino 54, (1996), 101-113.

[14] Van Dijk, G. *Group representations on spaces of distributions,* Russian J. of Mathematical Physics, vol *2, (1994), 57-68.*

[15] Vilenkin, N. Ja.; Klimyk, A. U. *Representation of Lie groups and special functions.* Vol. 2. Class I representations, special functions, and integral transforms. Translated from the Russian by V. A. Groza and A. A. Groza. Mathematics and its Applications (Soviet Series), 74. Kluwer Academic Publishers Group, Dordrecht, 1993. xviii+607 pp. ISBN: 0-7923-1492-1

[16] Warner, G. *Harmonic analysis on semi-simple Lie groups. I.* Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, 1972. xvi+529 pp.

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