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HARMONIC ANALYSIS ON HEISENBERG NILMANIFOLDS

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Abstract. In these lectures we plan to present a survey of certain aspects of harmonic analysis on a Heisenberg nilmanifold $\Gamma\backslash\mathbb{H}^n$. Using Weil-Brezin-Zak transform we obtain an explicit decomposition of $L^2(\Gamma\backslash\mathbb{H}^n)$ into irreducible subspaces invariant under the right regular representation of the Heisenberg group. We then study the Segal-Bargmann transform associated to the Laplacian on a nilmanifold and characterise the image of $L^2(\Gamma\backslash\mathbb{H}^n)$ in terms of twisted Bergman and Hermite Bergman spaces.

1. Introduction

The standard theory of multiple Fourier series deals with functions on the n dimensional torus $\mathbb{T}^n$ which can be identified with the quotient group $\mathbb{R}^n/\mathbb{Z}^n$ where $\mathbb{Z}$ is the additive group of integers. If $L^2(\mathbb{T}^n)$ is the Hilbert space of square integrable functions on $\mathbb{T}^n$, formed using the Lebesgue measure, then $\mathbb{R}^n$ has a natural action defined by $R(x)f(y) = f(y+x),\ f \in L^2(\mathbb{T}^n)$. As is well known, the one dimensional subspaces $V_{\mathbf{m}}$ spanned by the functions $e_{\mathbf{m}}(x) = e^{2\pi i \mathbf{m}\cdot x}$ are invariant under $R(x)$ for all x and $L^2(\mathbb{T}^n)$ is the orthogonal direct sum of $V_{\mathbf{m}}$ as $\mathbf{m}$ runs through $\mathbb{Z}^n$. We can also replace the standard lattice $\mathbb{Z}^n$ by any discrete subgroup Γ for which $\mathbb{R}^n/\Gamma$ is compact.

In these lectures we plan to study a similar problem for the nilmanifolds which are $\Gamma\backslash\mathbb{H}^n$ where $\mathbb{H}^n$ is the Heisenberg group and Γ is a discrete co-compact subgroup of $\mathbb{H}^n$. We consider the representation R of $\mathbb{H}^n$ on $L^2(\Gamma\backslash\mathbb{H}^n)$ defined by $R(g)F(\Gamma h) = F(\Gamma hg)$ and address the problem of identifying all irreducible, invariant subspaces. As the group $\mathbb{H}^n$ is non-abelian this problem turns out to be more interesting and difficult. By performing a Fourier expansion in the central variable, we are lead to consider spaces of functions satisfying $F(\Gamma hz) = e^{i\lambda z}F(\Gamma h)$ for all z from the center of $\mathbb{H}^n$. Such spaces turn out to be invariant under R and can be further decomposed into finitely many irreducible, invariant subspaces. The number of such subspaces is known in terms of λ and on each of these spaces R is unitarily equivalent to the Schrödinger representation π_λ.

The spectrum of the standard Laplacian Δ acting on functions on $\mathbb{R}^n$ is continuous but on $L^2(\mathbb{T}^n)$ it is discrete. The functions $e_{\mathbf{m}}(x), \mathbf{m} \in \mathbb{Z}^n$ are all eigenfunctions of the Laplacian with eigenvalues $|\mathbf{m}|^2$. In Section 4 we study the spectrum of the

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sublaplacian $\mathcal{L}$ on a nilmanifold $\Gamma\backslash\mathbb{H}^n$. This operator $\mathcal{L}$ plays the role of Laplacian on the Heisenberg group. It has continuous spectrum on the full group $\mathbb{H}^n$ but when restricted to a (compact) nilmanifold it has discrete spectrum. The spectral theory of $\mathcal{L}$ on $L^2(\Gamma\backslash\mathbb{H}^n)$ is intimately connected with that of Hermite and special Hermite operators. We show that an orthonormal basis for $L^2(\Gamma\backslash\mathbb{H}^n)$ consisting of eigenfunctions of $\mathcal{L}$ can be constructed.

Another problem we address in Section 4 is the characterisation of the image of $L^2(\Gamma\backslash\mathbb{H}^n)$ under the Segal-Bargmann (or heat kernel) transform associated to the full Laplacian on $\mathbb{H}^n$. In the classical set up, the function

$$u(x,t) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \hat{f}(\mathbf{m}) e^{-4\pi^2 t|\mathbf{m}|^2} e^{2\pi i \mathbf{m}\cdot x}$$

solves the heat equation $\partial_t u(x,t) = \Delta u(x,t)$ with the initial condition $u(x,0) = f(x)$. Note that when $f \in L^2(\mathbb{T}^n)$ the above solution $u(x,t)$ extends to $\mathbb{C}^n$ as an entire function which is invariant under the action of $\mathbb{Z}^n$. It is a classical result of Segal and Bargmann that such entire functions are characterised by the property that they are square integrable on $\mathbb{C}^n/\mathbb{Z}^n$ with respect to the measure $e^{-\frac{1}{t}\pi|y|^2}dx\ dy$. For the Laplacian on the Heisenberg group we study the Segal-Bargmann transform acting on $L^2(\Gamma\backslash\mathbb{H}^n)$. The image turns out to be a direct sum of twisted Bergman spaces satisfying certain quasi-periodicity conditions. This is proved in Section 4.2 using another family of weighted Bergman spaces associated to the Hermite operator.

2. Heisenberg groups and their representations

Our main aim in this section is to introduce the Schrödinger and Fock- Bargmann representations of the Heisenberg group. We use the Schrödinger representations in defining the group Fourier transform. We use Bargmann transforms and twisted Bergman spaces in order to study the classical and generalised Fock-Bargmann representations. The latter spaces appear in connection with the Segal-Bargmann transform on nilmanifolds (see Section 4.2). The main references for Section 2.1 are the monographs of Folland [5] and Thangavelu [14]. For the twisted Bergman spaces we refer to Krötz, Thangavelu and Xu [9], [11] and Thangavelu [16].

2.1. **Fourier transform on $\mathbb{H}^n$.** The Heisenberg group $\mathbb{H}^n$ is $\mathbb{R}^{2n}\times\mathbb{R}$ equipped with the group law

$$(\xi,t)(\eta,s) = (\xi+\eta, t+s+\frac{1}{2}\omega(\xi,\eta))$$

where ω is the symplectic form on $\mathbb{R}^{2n}$ defined by $\omega(\xi,\eta) = (u\cdot y - x\cdot v)$ if $\xi = (x,u), \eta = (y,v)$. It is the simplest example of a non-abelian group from the realm of nilpotent Lie groups. Note that the center of $\mathbb{H}^n$ is $Z = \{0\}\times\mathbb{R}$. We briefly recall some basic results from the representation theory of $\mathbb{H}^n$.

If π is any irreducible unitary representation of $\mathbb{H}^n$ then its restriction $\pi(0,t)$ to the center of $\mathbb{H}^n$, viz. $Z = \{0\}\times\mathbb{R}$ commutes with every $\pi(\xi,s)$ and hence for some $\lambda\in\mathbb{R}$, $\pi(0,t) = e^{i\lambda t}Id$. When $\lambda = 0$ the representaion π defines a representation of $\mathbb{R}^{2n}$ and hence it is one dimensional given by a character. Such representations do not interest us. When $\lambda \neq 0$ there is a well known realisation of π on the Hilbert space $L^2(\mathbb{R}^n)$. This representation, denoted by π_λ is called the Schrödinger representation and is given explicitly by

$$\pi_\lambda(\xi, t)\varphi(v) = e^{i\lambda t} e^{i\lambda(x\cdot v + \frac{1}{2}x\cdot u)}\varphi(v+u)$$

for $\varphi \in L^2(\mathbb{R}^n)$. It is not difficult to show that this is indeed an irreducible unitary representation of $\mathbb{H}^n$. A celebrated theorem of Stone and von Neumann says that any irreducible unitary representation π whose restriction to the center is $e^{i\lambda t} Id$ with $\lambda \neq 0$ is unitarily equivalent to π_λ (see [5]).

We define the group Fourier transform of a function f on $\mathbb{H}^n$ in terms of π_λ. When $f \in L^1 \cap L^2(\mathbb{H}^n)$ set

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(g)\pi_\lambda(g)dg$$

for every $\lambda \neq 0$ as an operator valued function. The Plancherel theorem then reads as

$$\int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2 d\mu(\lambda) = \int_{\mathbb{H}^n} |f(g)|^2 dg$$

where $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$ is the Plancherel measure for $\mathbb{H}^n$. In the above $\|\hat{f}(\lambda)\|_{HS}$ stands for the Hilbert-Schmidt operator norm of $\hat{f}(\lambda)$. Recalling the definition of π_λ we see that $\hat{f}(\lambda)$ can be written as

$$\hat{f}(\lambda) = \int_{\mathbb{R}^{2n}} f^\lambda(\xi)\pi_\lambda(\xi, 0)d\xi$$

where $f^\lambda(\xi)$ is the inverse Fourier transform of f in the central variable,

$$f^\lambda(\xi) = \int_{-\infty}^{\infty} f(\xi, t)e^{i\lambda t}dt.$$

In view of this, we define $\pi_\lambda(F)$ for any function F on $\mathbb{R}^{2n}$ by

$$\pi_\lambda(F) = \int_{\mathbb{R}^{2n}} F(\xi)\pi_\lambda(\xi, 0)d\xi$$

and call it the Weyl transform of F. We also remark that

$$(f * g)^\lambda(\xi) = \int_{\mathbb{R}^{2n}} f^\lambda(\xi - \eta)e^{\frac{i}{2}\lambda\omega(\xi,\eta)}g(\eta)d\eta$$

which is usually denoted by $f^\lambda *_\lambda g^\lambda(\xi)$ and called the λ-twisted convolution of f^λ with g^λ. The relation $(f * g)\hat{}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$ which is easily checked translates into $\pi_\lambda(F *_\lambda G) = \pi_\lambda(F)\pi_\lambda(G)$. We also remark that for $F \in L^2(\mathbb{R}^{2n}), \pi_\lambda(F)$ is a Hilbert-Schmidt operator and

$$\|\pi_\lambda(F)\|_{HS}^2 = c_\lambda \int_{\mathbb{R}^{2n}} |F(\xi)|^2 d\xi$$

for an explicit constant c_λ which is the Plancherel theorem for the Weyl transform.

Given $\varphi, \psi \in L^2(\mathbb{R}^n)$ their Fourier-Wigner transform is defined to be the function $(\pi_\lambda(\xi, 0)\varphi, \psi)$ on $\mathbb{R}^{2n}$. Using the explicit formula for the representation π_λ and

Plancherel theorem for the Euclidean Fourier transform it is not difficult to show that

$$\int_{\mathbb{R}^{2n}} (\pi_\lambda(\xi,0)\varphi,\psi)\overline{(\pi_\lambda(\xi,0)f,g)}d\xi = (2\pi)^n|\lambda|^{-n}(\varphi,f)(g,\psi)$$

for functions $\varphi,\psi,f,g \in L^2(\mathbb{R}^n)$. We use the Fourier-Wigner transform to construct an orthonormal basis for $L^2(\mathbb{R}^{2n})$ starting from the Hermite basis for $L^2(\mathbb{R}^n)$.

If we let $H = -\Delta + |x|^2$ stand for the Hermite operator on $\mathbb{R}^n$ then the eigenfunctions are given by the normalised Hermite functions Φ_α indexed by $\alpha \in \mathbb{N}^n$. These are of the form $H_\alpha(x)e^{-\frac{1}{2}|x|^2}$ where H_α is a polynomial of degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Moreover, $H\Phi_\alpha = (2|\alpha|+n)\Phi_\alpha$ and the family $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. For each $\lambda \neq 0$ we define $\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}}\Phi_\alpha(|\lambda|^{\frac{1}{2}}x)$ so that they are eigenfunctions of the scaled Hermite operator $H(\lambda) = -\Delta + \lambda^2|x|^2$ with eigenvalues $(2|\alpha|+n)|\lambda|$. We remark that $\Phi_0^\lambda(x) = \pi^{-\frac{n}{4}}|\lambda|^{\frac{n}{4}}e^{-\frac{1}{2}|\lambda||x|^2}$.

We define the special Hermite functions $\Phi_{\alpha,\beta}^\lambda(\xi)$ in terms of the Fourier-Wigner transform as $(2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}}(\pi_\lambda(\xi,0)\Phi_\alpha^\lambda,\Phi_\beta^\lambda)$. Then it follows from the properties of the Fourier-Wigner transform that $\{\Phi_{\alpha,\beta}^\lambda : \alpha,\beta \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$. A simple calculation shows that

$$\pi_\lambda(\overline{\Phi_{\alpha,\beta}^\lambda})\varphi = (\varphi,\Phi_\alpha^\lambda)\Phi_\beta.$$

From this and the relation $\pi_\lambda(F *_\lambda G) = \pi_\lambda(F)\pi_\lambda(G)$ it follows that

$$\Phi_{\alpha,\beta}^\lambda *_\lambda \Phi_{\mu,\nu}^\lambda = (2\pi)^{\frac{n}{2}}|\lambda|^{-n}\delta_{\beta,\mu}\Phi_{\alpha,\nu}^\lambda.$$

This interesting orthogonality under $\lambda-$twisted convolution will be made use of in the next section. For more about special Hermite functions see [14].

2.2. **Fock-Bargmann representations.** We now describe another family of irreducible unitary representations ρ_λ of $\mathbb{H}^n$ known in the literature as Fock-Bargmann representations. More generally, we define a whole family of representations ρ_α^λ indexed by $\alpha \in \mathbb{N}^n$ with $\rho_0^\lambda = \rho_\lambda$. These representations may be called generalised Fock-Bargmann representations, see e.g. Szabo [13]. For each $\alpha \in \mathbb{N}^n$ we set $\mathbb{E}_\alpha^\lambda$ to be the subspace of $L^2(\mathbb{R}^{2n})$ spanned by $\Phi_{\alpha,\beta}^\lambda, \beta \in \mathbb{N}^n$. Define $\mathbb{F}_\alpha^\lambda$ to be the space of functions of the form $G(\xi,t) = e^{i\lambda t}F(\xi)$ where $F \in \mathbb{E}_\alpha^\lambda$ equipped with the inner product

$$(G_1,G_2) = \int_{\mathbb{R}^{2n}} G_1(\xi,0)\overline{G_2(\xi,0)}d\xi.$$

It is not difficult to show that each of these spaces $\mathbb{F}_\alpha^\lambda$ is invaraint under left translations and hence we can define representations ρ_α^λ by setting

$$\rho_\alpha^\lambda(\eta,s)G(\xi,t) = G((-\eta,-s)(\xi,t)).$$

Theorem 2.1. *For each $\alpha \in \mathbb{N}^n$ and $\lambda \neq 0$ the above representations are unitary, irreducible and equivalent to π_λ.*

We now consider the representation ρ_0^λ more closely. Writing $\xi = (x,u)$ the transformation $A : (x,u,t) \to (-u,x,t)$ is an automorphism of $\mathbb{H}^n$. We consider the representation $\rho_0^\lambda(A(x,u,t)) = \rho_0^\lambda(-u,x,t)$ and show that it can be realised on a Hilbert space of entire functions. Look at the intertwining operator $B_0^\lambda f(\xi) = (\pi_\lambda(\xi)\Phi_0^\lambda, \bar{f})$. Writing the definition and simplifying,we see that

$$B_0^\lambda f(-u,x) = e^{-\frac{1}{4}\lambda(x^2+u^2)} \int_{\mathbb{R}^n} f(v) e^{-\frac{1}{2}\lambda v^2} e^{-\lambda(x+iu)} e^{-\frac{1}{4}(x+iu)^2} dv.$$

Thus $e^{\frac{1}{4}\lambda(x^2+u^2)} B_0^\lambda f(-u,x)$ is an entire function of the complex variable $z = x+iu$. We define $\mathcal{F}_\lambda(\mathbb{C}^n)$ to be the space of entire functions F on $\mathbb{C}^n$ which are square integrable with respect to the measure $e^{-\frac{1}{2}\lambda(x^2+u^2)}$. This is the classical Fock-Bargmann space and the transformation $F(x,u) \to e^{\frac{1}{4}\lambda(x^2+u^2)} F(-u,x)$ sets up an isometric isomorphism from $\mathbb{E}_0^\lambda$ onto $\mathcal{F}_\lambda(\mathbb{C}^n)$. By setting

$$B_\lambda f(x+iu) = e^{\frac{1}{4}\lambda(x^2+u^2)} B_0^\lambda(-u,x)$$

we see that $B_\lambda \pi_\lambda(x,u,t) B_\lambda^*$ defines a representation ρ_λ of $\mathbb{H}^n$ on $\mathcal{F}_\lambda(\mathbb{C}^n)$ called the Fock-Bargmann representation.

2.3. **Twisted Bergman spaces.** We now introduce the twisted Bergman space $\mathcal{B}_t^\lambda$, for each $t > 0$, which is isometrically isomorphic to $L^2(\mathbb{R}^{2n})$ and show that it can be written as a direct sum of certain subspaces on which we can define representations of $\mathbb{H}^n$ unitarily equivalent to ρ_α^λ. Consider the weight function

$$W_t^\lambda(z,w) = e^{\lambda(u\cdot y - v\cdot x)} p_{2t}^\lambda(2y, 2v)$$

where

$$p_t^\lambda(y,v) = c_n \left(\frac{\lambda}{\sinh(t\lambda)}\right)^n e^{-\frac{1}{4}\lambda \coth(t\lambda)(y^2+v^2)}.$$

Later we will see that this is the heat kernel associated to a certain elliptic differential operator L_λ called the special Hermite operator. For each $t > 0$ we define the twisted Bergman space $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ to be the space of all entire functions F on $\mathbb{C}^{2n}$ which are square integrable with respect to the weight function W_t^λ. It is a Hilbert space equipped with the norm

$$\|F\|^2_{\mathcal{B}_t^\lambda} = \int_{\mathbb{C}^{2n}} |F(z,w)|^2 W_t^\lambda(z,w) dz dw.$$

We require the following facts about these spaces that we recall without proofs (see [9] and [16]).

The functions $\Phi_{\alpha,\beta}^\lambda(x,u)$ extend to $\mathbb{C}^{2n}$ as entire functions and they belong to $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ for every $t > 0$. In fact,

$$\int_{\mathbb{C}^{2n}} |\Phi_{\alpha,\beta}^\lambda(z,w)|^2 W_t^\lambda(z,w) dz dw = e^{2(2|\alpha|+n)|\lambda| t} \int_{\mathbb{R}^{2n}} |\Phi_{\alpha,\beta}^\lambda(x,u)|^2 dx du.$$

More generally,

$$(\Phi_{\alpha,\beta}^\lambda, \Phi_{\mu,\nu}^\lambda)_{\mathcal{B}_t^\lambda} = e^{2(2|\alpha|+n)|\lambda| t} \delta_{\alpha,\mu} \delta_{\beta,\nu}$$

and the functions $\tilde{\Phi}_{\alpha,\beta}^\lambda = e^{-(2|\alpha|+n)|\lambda| t} \Phi_{\alpha,\beta}^\lambda$ form an orthonormal basis for $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$. Therefore, for every $F \in \mathbb{E}_\alpha^\lambda$ which has the orthogonal expansion

$$F(x,u) = \sum_{\beta \in \mathbb{N}^n} (F, \Phi_{\alpha,\beta}^\lambda) \Phi_{\alpha,\beta}^\lambda(x,u)$$

the series

$$\tilde{F}(z,w) = \sum_{\beta\in\mathbb{N}^n} (F, \Phi^\lambda_{\alpha,\beta})\tilde{\Phi}^\lambda_{\alpha,\beta}(z,w)$$

converges in $\mathcal{B}^\lambda_t(\mathbb{C}^{2n})$. In other words, every $F \in \mathbb{E}^\lambda_\alpha$ is the restriction to $\mathbb{R}^{2n}$ of an element $(e^{(2|\alpha|+n)|\lambda|t}\tilde{F})$ of $\mathcal{B}^\lambda_t(\mathbb{C}^{2n})$. Note that $\|\tilde{F}\|_{\mathcal{B}^\lambda_t} = \|F\|_2$.

Every $F \in \mathcal{B}^\lambda_t(\mathbb{C}^{2n})$ can be written as $F = \sum_\alpha F_\alpha$ where

$$F_\alpha(z,w) = \sum_\beta (F, \tilde{\Phi}^\lambda_{\alpha,\beta})_{\mathcal{B}^\lambda_t}\tilde{\Phi}^\lambda_{\alpha,\beta}(z,w).$$

This means that $\mathcal{B}^\lambda_t(\mathbb{C}^{2n})$ is the orthogonal direct sum of the subspaces $\mathcal{B}^\lambda_{t,\alpha}(\mathbb{C}^{2n})$ spanned by $\{\tilde{\Phi}^\lambda_{\alpha,\beta} : \beta \in \mathbb{N}^n\}$. Hence we can identify $\mathbb{E}^\lambda_\alpha$ with $\mathcal{B}^\lambda_{t,\alpha}(\mathbb{C}^{2n})$ the isomorphism being given by the map $\Phi^\lambda_{\alpha,\beta} \to \tilde{\Phi}^\lambda_{\alpha,\beta}$. Thus we have the following result.

Theorem 2.2. *For every $t > 0$ the map $\tilde{B}^\lambda_\alpha : L^2(\mathbb{R}^n) \to \mathcal{B}^\lambda_{t,\alpha}(\mathbb{C}^{2n})$ given by $f \to e^{-(2|\alpha|+n)|\lambda|t}B^\lambda_\alpha \bar{f}(z,w)$ is an isometric isomorphism.*

This theorem shows that by conjugating with $\tilde{B}^\lambda_\alpha$ we can make π_λ into a representation of $\mathbb{H}^n$ realised on $\mathcal{B}^\lambda_{t,\alpha}(\mathbb{C}^{2n})$ which is unitarily equivalent to ρ^λ_α. From the above expansion for $F \in \mathcal{B}^\lambda_t(\mathbb{C}^{2n})$ we infer that its restriction to $\mathbb{R}^{2n}$ is given by

$$F(x,u) = \sum_{\alpha,\beta} (F, \tilde{\Phi}^\lambda_{\alpha,\beta})_{\mathcal{B}^\lambda_t}\tilde{\Phi}^\lambda_{\alpha,\beta}(x,u)$$

so that

$$(F, \Phi^\lambda_{\alpha,\beta}) = e^{-(2|\alpha|+n)|\lambda|t}(F, \tilde{\Phi}^\lambda_{\alpha,\beta})_{\mathcal{B}^\lambda_t}.$$

This means that

$$\sum_{\alpha,\beta} e^{2(2|\alpha|+n)|\lambda|t}|(F, \Phi^\lambda_{\alpha,\beta})|^2 = \|F\|^2_{\mathcal{B}^\lambda_t}.$$

Thus the twisted Bergman space $\mathcal{B}^\lambda_t$ can be identified with a certain subspace of $L^2(\mathbb{R}^{2n})$. As we will see later, this is precisely the image of $L^2(\mathbb{R}^{2n})$ under the semigroup generated by the operator L_λ.

Each of $\mathcal{B}^\lambda_{t,\alpha}(\mathbb{C}^{2n})$ is a Hilbert space of entire functions and hence is associated with a reproducing kernel which can be easily identified (see [6] and [13]). In fact, it is given by

$$K^t_{(z,w)}(z',w') = e^{-2(2|\alpha|+n)|\lambda|t}\Phi^\lambda_{\alpha,\alpha}(z-\bar{z}', w-\bar{w}')e^{-\frac{i}{2}\lambda(w\cdot\bar{z}' - \bar{w}'\cdot z)}.$$

3. Heisenberg nilmanifolds

In this section we introduce nilmanifolds as quotients of $\mathbb{H}^n$ by certain lattice subgroups. All such lattices have been characterised and we recall some results, without proof, from the paper by Tolimieri [19]. We are mainly interested in a standard lattice Γ and we study the right regular representation of $\mathbb{H}^n$ on $L^2(\Gamma\backslash\mathbb{H}^n)$. We obtain a decomposition of $L^2(\Gamma\backslash\mathbb{H}^n)$ into irreducible subspaces that are invariant under the right regular representation. General references for this section are Gelfand,Graeve and Piatetskii-Shapiro [7], Folland [5], Auslander and Brezin [1], Brezin [3] and Tolimieri [18].

3.1. **Lattices in $\mathbb{H}^n$.** A discrete subgroup Γ of $\mathbb{H}^n$ is said to be a lattice if the quotient $\Gamma\backslash\mathbb{H}^n$ is compact. When Γ is a lattice, the space $\Gamma\backslash\mathbb{H}^n$ of right cosets is called a nilmanifold. The standard example we study in details is $\Gamma = \mathbb{Z}^n \times \mathbb{Z}^n \times \frac{1}{2}\mathbb{Z}$. The quotient

$$\Gamma\backslash\mathbb{H}^n \simeq \mathbb{T}^{2n} \times \mathbb{S}^1$$

is clearly compact. It is a circle bundle over the torus $\mathbb{T}^{2n}$.

Let $\mathbb{A}_n$ stand for the group of automorphisms of $\mathbb{H}^n$ and let $\mathbb{A}_n^0$ be its identity component. This group $\mathbb{A}_n^0$ plays an important role in classifying all lattices in $\mathbb{H}^n$. Let $Sp(2n,\mathbb{R})$ stand for the symplectic group consisting of all $2n \times 2n$ matrices preserving the symplectic form $\omega(\xi,\eta)$. That is to say $A \in Sp(2n,\mathbb{R})$ if and only if $\omega(A\xi, A\eta) = \omega(\xi,\eta)$. Every element $A \in Sp(2n,\mathbb{R})$ defines an automorphism in $\mathbb{A}_n^0$ denoted by the same symbol, by $A(\xi,t) = (A\xi,t)$. Note that if Γ is a lattice and $A \in \mathbb{A}_n^0$ then $A(\Gamma)$ is another lattice. Thus we have an action of $\mathbb{A}_n^0$ on the set $L(\mathbb{H}^n)$ of all lattices in $\mathbb{H}^n$. We also note that $\Gamma \cap Z$ is a nontrivial discrete subgroup of Z and hence there is unique positive real number $\beta(\Gamma)$ such that

$$\Gamma \cap Z = \{(0, \beta(\Gamma)m) : m \in \mathbb{Z}\}.$$

Let $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$ be the projection $\pi(\xi,t) = \xi$. Then $\pi(\Gamma)$ is a lattice in $\mathbb{R}^{2n}$, meaning $\mathbb{R}^{2n}/\pi(\Gamma)$ is compact. But the lattices $\pi(\Gamma)$ thus obtained are somewhat special. The condition

$$[\Gamma,\Gamma] \subset \Gamma \cap Z = \{(0,\beta(\Gamma)m) : m \in \mathbb{Z}\}$$

imposes some extra conditions on $\pi(\Gamma)$. Indeed, when $g = (\xi,t), h = (\eta,s)$ then $[g,h] = ghg^{-1}h^{-1} = (0,\omega(\xi,\eta))$ and hence for any $\xi,\eta \in \pi(\Gamma)$ we have $\omega(\xi,\eta) \in \beta(\Gamma)Z$. Actually, it turns out that $\omega(\pi(\Gamma),\pi(\Gamma)) = \beta(\Gamma)Z$ whenever Γ is a lattice in $\mathbb{H}^n$. This motivates us to make the following definition. We call a lattice D in $\mathbb{R}^{2n}$ a Heisenberg lattice if $\omega(D,D) = lZ$ for some $l > 0$. The collection of such lattices will be denoted by $HL(\mathbb{R}^{2n})$. Thus $\pi(\Gamma) \in HL(\mathbb{R}^{2n})$ for any lattice Γ in $\mathbb{H}^n$. The converse is also true.

Theorem 3.1. *There is a one to one correspondence between lattices in $\mathbb{H}^n$ and Heisenberg lattices in $\mathbb{R}^{2n}$.*

The structure of Heisenberg lattices in $\mathbb{R}^{2n}$ have been well understood. To each such lattice D one can associate n positive real numbers $l_1, l_2, ..., l_n$ with the property that l_j divides l_{j+1}. Set

$$\mathbb{Z}_n^* = \{\mathbf{l} = (1, l_2, ..., l_n) \in \mathbb{N}^n : l_{j+1}l_j^{-1} \in \mathbb{Z}\}.$$

Let $e_j, 1 \leq j \leq 2n$ be the standard co-ordinate vectors in $\mathbb{R}^{2n}$. For $\mathbf{l} \in \mathbb{Z}_n^*$ denote by $D(\mathbf{l}) = [e_1, ..., e_n, e_{n+1}, l_2e_{n+2}, ..., l_ne_{2n}]$ be the $\mathbb{Z}$ module of $\mathbb{R}^{2n}$ spanned by the vectors $e_1, ..., e_n, e_{n+1}, l_2e_{n+2}, ..., l_ne_{2n}$. Then it is clear that $D(\mathbf{l}) \in HL(\mathbb{R}^{2n})$ and $\omega(D(\mathbf{l}), D(\mathbf{l})) = \mathbb{Z}$.

Theorem 3.2. *For each $D \in HL(\mathbb{R}^{2n})$ there exists a unique $\mathbf{l} \in \mathbb{Z}_n^*$, a unique $d > 0$ and an $A \in Sp(2n,\mathbb{R})$ such that $D = A(d.D(\mathbf{l}))$.*

Combining Theorems 3.2 and 3.3 we can obtain the following result which gives the structure of all lattices in $\mathbb{H}^n$. Given $\mathbf{l} \in \mathbb{Z}_n^*$, let $\Gamma(\mathbf{l})$ be the subgroup of $\mathbb{H}^n$ generated by

$$(e_1, 0), (e_2, 0), ..., (e_n, 0), (e_{n+1}, 0), (l_2 e_{n+2}, 0), ..., (l_n e_{2n}, 0).$$

We then have

Theorem 3.3. *For each* $\Gamma \in L(\mathbb{H}^n)$ *there exist a unique* $\mathbf{l} \in \mathbb{Z}_n^*$, *a unique* $d > 0$ *and an* $A \in \mathbb{A}_n$ *such that* $\Gamma = A(d.\Gamma(\mathbf{l}))$.

We refer to Tolimieri [19] for proofs of the above results. In the next subsection we consider function spaces on the nilmanifold $\Gamma \backslash \mathbb{H}^n$. When $D = A(d.D(\mathbf{l}))$ we remark that $\beta(D) = d^2$ where $\beta(D)$ is the unique constant introduced earlier. Similarly, $\beta(\Gamma) = d^2$ whenever $\Gamma = A(d.\Gamma(\mathbf{l}))$.

3.2. **Analysis on the standard nilmanifold** $\Gamma_n \backslash \mathbb{H}^n$**.** We first consider the standard lattice $\Gamma_n = \Gamma(1, 1, ..., 1)$ and the associated nilmanifold $M = \Gamma_n \backslash \mathbb{H}^n$. The Lebesgue measure on $\mathbb{H}^n$ induces a $\mathbb{H}^n$ invariant measure on M. Hence we get a unitary representation R of $\mathbb{H}^n$ on $L^2(M)$ defined by

$$R(g)F(\Gamma_n h) = F(\Gamma_n hg), \ F \in L^2(M), \ g, h \in \mathbb{H}^n.$$

We can identify functions on M with functions on $\mathbb{H}^n$ that are invariant under left translations by elements of Γ_n. (We may call them Γ_n periodic functions.) We are interested in identifying irreducible subspaces of $L^2(M)$ invariant under R.

Note that $\Gamma_n = \mathbb{Z}^{2n} \times \frac{1}{2}\mathbb{Z}$ so that $\beta(\Gamma_n) = \frac{1}{2}$. Therefore, every Γ_n periodic function is $\frac{1}{2}$ periodic in the central variable. Thus by defining

$$H_k(\Gamma_n) = \{F \in L^2(M) : F(\xi, t) = e^{4\pi i k t} F(\xi, 0)\}$$

we get the orthogonal direct sum decomposition

$$L^2(M) = \sum_{k \in \mathbb{Z}} \oplus H_k(\Gamma_n).$$

It is easy to see that each $H_k(\Gamma_n)$ is R invariant and hence for every $k \neq 0$ Stone-von Neumann theorem says that the restriction of R to $H_k(\Gamma_n)$ decomposes into a direct sum of irreducible representations each one unitarily equivalent to $\pi_{4\pi k}$. The task is to identify the irreducible subspaces and determine the intertwining operators. We also like to get the multiplicity with which $\pi_{4\pi k}$ occurs. All these questions can be answered easily in the case of standard lattice. For the rest of this subsection we simply write Γ instead of Γ_n.

The case $k = 0$ can be easily dispensed with. Note that $H_0(\Gamma)$ can be identified with $L^2(\mathbb{R}^{2n}/\pi(\Gamma))$ and hence it decomposes into an infinite direct sum of one dimensional R invariant subspaces.

A standard way of constructing Γ invariant functions on $\mathbb{H}^n$ is to start with a tempered distribution ν on $\mathbb{R}^n$ which is $\pi_\lambda(\Gamma)$ invariant and consider $F(x, u, t) = (\nu, \pi_\lambda(x, u, t)f)$ where f is a Schwartz function on $\mathbb{R}^n$ ([7]). Let ν be such a distribution; that is it verifies $(\nu, \pi_\lambda(h)f) = (\nu, f), \ h \in \Gamma$. Then taking $h = (0, 0, j/2) \in \Gamma, j \in \mathbb{Z}$ we are led to $\pi_\lambda(h)f = e^{i\lambda j/2} f$ and $(\nu, e^{i\lambda j/2} f) = (\nu, f)$. This holds for all $j \in \mathbb{Z}$ if and only if $\lambda = 4\pi k$ for some $k \in \mathbb{Z}$. Let us assume $k \neq 0$ and write $\rho_k = \pi_{4\pi k}$.

Proposition 3.4. *For $k \neq 0$ define $\mathbb{F}_k = \mathbb{Z}^n/(2k\mathbb{Z})^n$. Then every tempered distribution ν invariant under $\rho_k(\Gamma)$ is of the form $\nu = \sum_{\mathbf{j}\in\mathbb{F}_k} c_{\mathbf{j}}\nu_{\mathbf{j}}$ with $\nu_{\mathbf{j}}$ defined by*

$$(\nu_{\mathbf{j}}, f) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \hat{f}(2k\mathbf{m} + \mathbf{j})$$

where $\hat{f}$ denotes the Euclidean Fourier transform of the Schwartz class function f.

We refer to [11] for a proof of this result. We now show that the functions $(\nu_{\mathbf{j}}, f)$ can be expressed as Weil- Brezin- Zak transforms studied in [3]. Consider the operator V_k defined on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ by

$$V_k f(x,u,t) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \rho_k(x,u,t) f(\mathbf{m}).$$

Written explicitly

$$V_k f(x,u,t) = e^{4\pi kit} e^{2\pi kix\cdot u} \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{4\pi ki\mathbf{m}\cdot x} f(u+\mathbf{m}).$$

It is easy to see that $V_k f$ is Γ invariant. For each $\mathbf{j} \in \mathbb{F}_k$ we also define

$$V_{k,\mathbf{j}} f(x,u,t) = e^{2\pi i\mathbf{j}\cdot x} V_k f(x,u,t).$$

These are called the Weil-Brezin-Zak transforms in the literature.

Proposition 3.5. *(i) The transform $V_{k,\mathbf{j}}$ initially defined on $\mathcal{S}(\mathbb{R}^n)$ extends to the whole of $L^2(\mathbb{R}^n)$ as an isometry into $\mathcal{H}_k$.*
(ii) For each $f \in \mathcal{S}(\mathbb{R}^n)$ we have the relation

$$(\nu_{\mathbf{j}}, \rho_k(x,u,t)f) = V_{k,\mathbf{j}} g_{\mathbf{j}}(u,-x,t)$$

where f and $g_{\mathbf{j}}$ are related by $g_{\mathbf{j}}(x) = \hat{f}(2kx + \mathbf{j})$.

The above proposition shows that $(\nu_{\mathbf{j}}, \rho_k(x,u,t)f)$ can be defined on the whole of $L^2(\mathbb{R}^n)$. We can now prove the following result.

Theorem 3.6. *Denote by $\mathcal{H}_{k,\mathbf{j}}$ the span of $(\nu_{\mathbf{j}}, \rho_k(x,u,t)f)$ as f varies over $L^2(\mathbb{R}^n)$. Then $\mathcal{H}_k$ is the orthogonal direct sum of the spaces $\mathcal{H}_{k,\mathbf{j}}, \mathbf{j} \in \mathbb{F}_k$.*

The orthogonality part of the proof follows from a little group theory and direct computation. In fact, let us consider the finite group $\mathbb{F}_k := \mathbb{Z}^n/(2k\mathbb{Z})^n$. Let $\mathbf{M}_k$ be the linear span of $\nu_{\mathbf{j}}, \mathbf{j} \in \mathbb{F}_k$. Then the prescription

$$\Pi_k(a)(\nu) := \nu(\cdot + a) \qquad (\nu \in \mathbf{M}_k, a \in \mathbb{F}_k)$$

defines a representation of $\mathbb{F}_k$ on $\mathbf{M}_k$. Moreover, it is clear that $\nu_{\mathbf{j}}$ is a basis of eigenvectors for this action; explicitly:

$$\Pi_k(a)\nu_{\mathbf{j}} = e^{-2\pi ia\cdot\mathbf{j}}\nu_{\mathbf{j}} \qquad (\mathbf{j} \in \mathbb{F}_k)\,.$$

Futhermore for $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\nu, \mu \in \mathbf{M}_k$ we set $F_\nu(x,u,\xi) = (\nu, \rho_k(x, u.\xi)f)$ and $G_\mu(x,u,\xi) = (\mu, \rho_k(x,u,\xi)g)$. Then one immediately verifies that

$$(F_{\Pi_k(a)\nu}, G_\mu)_{L^2(\Gamma\backslash\mathbb{H}^n)} = (F_\nu, G_{\Pi_k(-a)\mu})_{L^2(\Gamma\backslash\mathbb{H}^n)} .$$

Therefore, the above gives

$$e^{-2\pi i a\cdot\mathbf{j}}(F_{\nu_\mathbf{j}}, G_{\nu_\mathbf{l}}) = e^{-2\pi i a\cdot\mathbf{l}}(F_{\nu_\mathbf{j}}, G_{\nu_\mathbf{l}}).$$

As this is true for all $a \in \mathbb{F}_k$ we get $(F_{\nu_\mathbf{j}}, G_{\nu_\mathbf{l}}) = 0$ whenever $\mathbf{j} \neq \mathbf{l}$.

From general theory ([7]) it follows that $\nu_\mathbf{j}$ is in fact an orthogonal basis for $\mathbf{M}_k$. By direct calculation which we omit we can show that $\|F_{\nu_\mathbf{j}}\|^2_{L^2(\Gamma H^n)} = 2\|f\|_2^2$. (One way to see this is to use Proposition 3.5.) The abstract theory can be used to prove the rest of the theorem. In what follows, we give a direct proof.

We only need to show that every $F \in \mathcal{H}_k$ can be decomposed as $F = \sum_{\mathbf{j}\in\mathbb{F}_k} F_\mathbf{j}$, $F_\mathbf{j} \in \mathcal{H}_{k,\mathbf{j}}$. In view of Proposition 3.5 it is enough to show that we have $F = \sum_{\mathbf{j}\in\mathbb{F}_k} V_{k,\mathbf{j}} f_\mathbf{j},\ f_\mathbf{j} \in L^2(\mathbb{R}^n)$. To this end we make use of the finite group $\mathbb{F}_k$.

We first note that if we define $G(x,u) = F(x,u,0)$ then the Γ -invariance of F translates into the condition

$$G(x+\mathbf{m}, u+\mathbf{n}) = e^{2\pi i k(u\cdot\mathbf{m} - x\cdot\mathbf{n})}G(x,u),\ \ \mathbf{m},\mathbf{n} \in \mathbb{Z}^n.$$

We show that every $G(x,u)$ satisfying the above condition can be further decomposed as $G(x,u) = \sum_{\mathbf{j}\in\mathbb{F}_k} G_\mathbf{j}$ where $G_\mathbf{j}$ satisfies some extra conditions. To prove this we define

$$G_{\mathbf{j},\mathbf{m}}(x,u) = e^{-\pi i\mathbf{m}\cdot u}e^{-\frac{1}{k}\pi i\mathbf{m}\cdot\mathbf{j}}G(x+\frac{1}{2k}\mathbf{m}, u)$$

and consider the sum

$$\sum_{\mathbf{j}\in\mathbb{F}_k}\sum_{\mathbf{m}\in\mathbb{F}_k} G_{\mathbf{j},\mathbf{m}}(x,u) = \sum_{\mathbf{m}\in\mathbb{F}_k} e^{-\pi i\mathbf{m}\cdot u}G(x+\frac{1}{2k}\mathbf{m},u)\left(\sum_{\mathbf{j}\in\mathbb{F}_k} e^{-\frac{1}{k}\pi i\mathbf{m}\cdot\mathbf{j}}\right)$$

Since the sum $\sum_{\mathbf{j}\in\mathbb{F}_k} e^{-\frac{1}{k}\pi i\mathbf{m}\cdot\mathbf{j}} = 0$ unless $\mathbf{m} = 0$ in which case it is $(2k)^n$ we get

$$\sum_{\mathbf{j}\in\mathbb{F}_k}\sum_{\mathbf{m}\in F_k} G_{\mathbf{j},\mathbf{m}}(x,u) = (2k)^n G(x,u).$$

By defining

$$G_\mathbf{j}(x,u) = (2k)^{-n}\sum_{\mathbf{m}\in\mathbb{F}_k} G_{\mathbf{j},\mathbf{m}}(x,u)$$

we get the decomposition $G = \sum_{\mathbf{j}\in\mathbb{F}_k} G_\mathbf{j}$. We now show that $G_\mathbf{j}$ satisfies the same condition as G and the extra condition

$$G_\mathbf{j}(x+\frac{1}{2k}\mathbf{m}, u) = e^{\pi i\mathbf{m}\cdot(u+\frac{1}{k}\mathbf{j})}G_\mathbf{j}(x,u)$$

for every $\mathbf{m} \in \mathbb{Z}^n$. To see this, consider

$$G_\mathbf{j}(x+\frac{1}{2k}\mathbf{n}, u) = (2k)^{-n}\sum_{\mathbf{m}\in\mathbb{F}_k} e^{-\pi i\mathbf{m}\cdot u}e^{-\frac{1}{k}\pi i\mathbf{m}\cdot\mathbf{j}}G(x+\frac{1}{2k}(\mathbf{m}+\mathbf{n}), u).$$

The right hand side simplifies to

$$e^{\pi i\mathbf{n}\cdot u}e^{\frac{1}{k}\pi i\mathbf{n}\cdot\mathbf{j}}(2k)^{-n}\sum_{\mathbf{m}\in\mathbb{F}_k}G_{\mathbf{j},\mathbf{m}+\mathbf{n}}(x,u).$$

Now observe that $G_{\mathbf{j},\mathbf{m}}(x,u)$ satisfies

$$G_{\mathbf{j},\mathbf{m}+2k\mathbf{n}}(x,u)=G(x+\frac{1}{2k}\mathbf{m}+\mathbf{n},u)e^{-\pi i\mathbf{m}\cdot u}e^{-\frac{1}{k}\pi i\mathbf{m}\cdot\mathbf{j}}e^{-2\pi ik\mathbf{n}\cdot u}.$$

Since G satisfies

$$G(x+\frac{1}{2k}\mathbf{m}+\mathbf{n},u)=e^{2\pi ik\mathbf{n}\cdot u}G(x+\frac{1}{2k}\mathbf{m}+,u)$$

we see that $G_{\mathbf{j},\mathbf{m}+2k\mathbf{n}}=G_{\mathbf{j},\mathbf{m}}$. Therefore,

$$\sum_{\mathbf{m}\in\mathbb{F}_k}G_{\mathbf{j},\mathbf{m}+\mathbf{n}}(x,u)=\sum_{\mathbf{m}\in\mathbb{F}_k}G_{\mathbf{j},\mathbf{m}}(x,u)$$

which proves

$$G_{\mathbf{j}}(x+\frac{1}{2k}\mathbf{n},u)=e^{\pi i\mathbf{n}\cdot u}e^{\frac{1}{k}\pi i\mathbf{n}\cdot\mathbf{j}}G_{\mathbf{j}}(x,u)$$

as claimed. Similarly we can show that

$$G_{\mathbf{j}}(x+\mathbf{m},u+\mathbf{n})=e^{2\pi ik(\mathbf{m}\cdot u-\mathbf{n}\cdot x)}G_{\mathbf{j}}(x,u).$$

It remains to show that $G_{\mathbf{j}}(x,u)=V_{k,\mathbf{j}}f_j(x,u,0)$ for some $f_{\mathbf{j}}\in L^2(\mathbb{R}^n)$ for each j. To prove this, consider

$$g_{\mathbf{j}}(x,u)=e^{-2\pi i\mathbf{j}\cdot x}e^{-2\pi ikx\cdot u}G_{\mathbf{j}}(x,u).$$

In view of the transformation properties of $G_{\mathbf{j}}$ the function $g_{\mathbf{j}}$ becomes $\frac{1}{2k}-$ periodic in the x-variables. Therefore, it admits an expansion of the form

$$g_{\mathbf{j}}(x,u)=\sum_{\mathbf{m}\in\mathbb{Z}^n}C_{\mathbf{m}}(u)e^{4\pi ik\mathbf{m}\cdot x}$$

where $C_{\mathbf{m}}(u)$ are the Fourier coefficients given by

$$C_{\mathbf{m}}(u)=\int_{[0,\frac{1}{2k})^n}g_{\mathbf{j}}(x,u)e^{-4\pi ik\mathbf{m}\cdot x}dx.$$

The properties of $G_{\mathbf{j}}(x,u)$ leads to $g_{\mathbf{j}}(x,u-\mathbf{m})=g_{\mathbf{j}}(x,u)e^{4\pi ik\mathbf{m}\cdot x}$ and hence $C_{\mathbf{m}}(u-\mathbf{m})=C_{\mathbf{0}}(u)$. Thus, we obtain

$$G_{\mathbf{j}}(x,u)=e^{2\pi i\mathbf{j}\cdot x}e^{2\pi ikx\cdot u}\sum_{\mathbf{m}\in\mathbb{Z}^n}C_{\mathbf{0}}(u+\mathbf{m})e^{i\lambda\mathbf{m}\cdot z}$$

which shows that $G_{\mathbf{j}}(x,u)=V_{k,\mathbf{j}}f_{\mathbf{j}}(x,u,0)$ where $f_{\mathbf{j}}=C_{\mathbf{0}}$. It is easily seen that $C_{\mathbf{0}}\in L^2(\mathbb{R}^n)$. This completes the proof of Theorem 3.6.

Remark 3.1. From the above proof it follows that the restriction of R to $\mathcal{H}_{k,\mathbf{j}}$ is unitarily equivalent to ρ_k. The intertwining operator is given by $V_{k,\mathbf{j}}$. The multiplicity of R restricted to $\mathcal{H}_k$ is easily seen to be $(2|k|)^n$.

4. Sublaplacian on nilmanifods

In this section we study the spectral theory of the sublaplacian $\mathcal{L}$ and the Segal-Bargmann transform associated to the full Laplacian on the standard nilmanifold. Section 4.1 deals with the spectrum of $\mathcal{L}$ on $L^2(\Gamma\backslash\mathbb{H}^n)$. We construct an explicit orthonormal basis for each $\mathcal{H}_k$ using special Hermite functions. In Section 4.2 we study the Segal-Bargmann transform and show that the image of $L^2(\Gamma\backslash\mathbb{H}^n)$ under this is a direct sum of weighted Bergman spaces related to twisted Bergman spaces. The main reference for this section is the paper by Krötz, Thangavelu and Xu [11]. For the Segal-Bargmann transform in other contexts see [2], [8], [12], [9] and [16].

4.1. **Spectrum of the sublaplacian.** On the Heisenberg group $\mathbb{H}^n$ with coordinates (x, u, t) we consider the left invariant vector filelds

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}u_j\frac{\partial}{\partial t},\ U_j = \frac{\partial}{\partial u_j} - \frac{1}{2}x_j\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}$$

for $j = 1, 2, ..., n$. They satisfy the relations $[X_j, U_j] = T$, all other commutators being zero. They form a basis for the Heisenberg Lie algebra $\mathfrak{h}^n$: the map

$$\mathbb{R}^{2n+1} \to \mathfrak{h}^n, \quad (x, u, t) \mapsto \sum_{j=1}^n x_jX_j + \sum_{j=1}^n u_jU_j + tT$$

is a linear isomorphism providing us with suitable coordinates for $\mathfrak{h}^n$. The sublaplacian on $\mathbb{H}^n$ is the operator $\mathcal{L} = -\sum_{j=1}^n(X_j^2 + U_j^2)$ and plays an important role in harmonic analysis on $\mathbb{H}^n$.

Another important operator closely related to $\mathcal{L}$ is the special Hermite operator, also called the twisted Laplacian and Landau Hamiltonian (see [6]). For each $\lambda \neq 0$ let L_λ be the operator defined by the relation $\mathcal{L}(e^{i\lambda t}F(x, u)) = e^{i\lambda t}L_\lambda F(x, u)$. It is explicitly given by

$$L_\lambda = -\sum_{j=1}^n(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial u_j^2}) + \frac{1}{4}\lambda^2(x^2 + u^2) + i\lambda\sum_{j=1}^n(x_j\frac{\partial}{\partial u_j} - u_j\frac{\partial}{\partial x_j}).$$

We remark that the scaled Hermite operator $H(\lambda) = -\Delta + \lambda^2x^2$ is also related to the sublaplacian: $(\hat{\mathcal{L}f})(\lambda) = \hat{f}(\lambda)H(\lambda)$. The spectral theory of the sublaplacian is intimately connected to those of Hermite and special Hermite operators.

Considering the standard lattice $\Gamma = \mathbb{Z}^{2n} \times \frac{1}{2}\mathbb{Z}$ let us look at the action of $\mathcal{L}$ on $\mathcal{H}_k$ appearing in the Fourier decomposition of $L^2(\Gamma\backslash\mathbb{H}^n)$. If $F \in \mathcal{H}_k$ and $G(x, u) = F(x, u, 0)$ then $\mathcal{L}F(x, u, t) = e^{4\pi ikt}L_{4\pi k}G(x, u)$. Thus it is enough to look at the spectral decomposition of $L_{4\pi k}$ acting on $\mathbb{R}^{2n}/\pi(\Gamma)$. When $k = 0, L_0$ is just the standard Laplacian and hence the eigenfunctions are provided by $e^{2\pi i\mathbf{m}\cdot\xi}, \mathbf{m} \in \mathbb{Z}^{2n}, \xi = (x, u)$. The corresponding eigenvalues are $4\pi^2|\mathbf{m}|^2$. The cases $k \neq 0$ are more interesting.

Consider the operators $U_{k,\mathbf{j}}$ defined on $L^2(\mathbb{R}^n)$ by

$$U_{k,\mathbf{j}}f(x, u, t) = (\nu_\mathbf{j}, \rho_k(x, u, t)f).$$

These are related to $V_{k,\mathbf{j}}$ and allowing f to run through an orthonormal basis of $L^2(\mathbb{R}^n)$ we can get an orthonormal basis for $L^2(\Gamma\backslash\mathbb{H}^n)$. It follows that the functions $\Psi^k_{\alpha,\mathbf{j}} = c_{\alpha,\mathbf{j}} U_{k,\mathbf{j}} \Phi^{4\pi k}_\alpha$, where $c_{\alpha,\mathbf{j}}$ are certain normalising constants, form an orthonormal basis for $L^2(\Gamma\backslash\mathbb{H}^n)$. The functions $\Psi^k_{\alpha,\mathbf{j}}$ are also eignefunctions of the sublaplacian with eigenvalues $4(2|\alpha|+n)\pi k$. Indeed, expanding the tempered distribution $\nu_{\mathbf{j}}$ in terms of $\Phi^{4\pi k}_\beta$ we have

$$U_{k,\mathbf{j}}\Phi^{4\pi k}_\alpha(x,u,t) = \sum_\beta (\nu_{\mathbf{j}}, \Phi^{4\pi k}_\beta)(\Phi^{4\pi k}_\beta, \rho_k(x,u,t)\Phi^{4\pi k}_\alpha).$$

It can be checked that the functions $(\Phi^{4\pi k}_\beta, \rho_k(x,u,t)\Phi^{4\pi k}_\alpha)$ are all eigenfunctions of the sublaplacian with eigenvalues $4(2|\alpha|+n)\pi k$.

Thus we have proved that the spectrum of the sublaplacian on $\Gamma\backslash\mathbb{H}^n$ consists of the points $4\pi^2|\mathbf{m}|^2, \mathbf{m} \in \mathbb{Z}^{2n}$ and $4(2m+n)\pi|k|, m \in \mathbb{N}, k \in \mathbb{Z}$. For any fixed $m \in \mathbb{N}$ the number of eigenfunctions in $\mathcal{H}_k$ having the eigenvalue $4(2m+n)\pi|k|$ is given by $(2|k|)^n \frac{(m+n-1)!}{m!(n-1)!}$. For another proof of this result see [6]. For the spectrum of $\mathcal{L}$ on a general nilmanifold we refer to [5].

4.2. **Segal-Bargmann transform on $\Gamma\backslash\mathbb{H}^n$.** The sublaplacian $\mathcal{L}$ generates a diffusion semigroup $T_t, t > 0$ which is given by a non-negative Schwartz class function p_t in the sense that $p_t * p_s = p_{t+s}$ and for $f \in L^2(\mathbb{H}^n)$ the function $u(g,t) = f * p_t(g)$ solves the heat equation $\partial_t u(g,t) = -\mathcal{L}u(g,t)$ with initial condition f. The heat kernel p_t is explicitly known: its Fourier transform in the central variable is given by

$$p^\lambda_t(x,u) = c_n \lambda^{-n}(\sinh(\lambda t))^n e^{-\frac{1}{4}\lambda\coth(\lambda t)(x^2+u^2)}.$$

Throughout this section we use (x,u,ξ) to denote elements of $\mathbb{H}^n$. When $f \in L^2(\Gamma\backslash\mathbb{H}^n)$ the convolution $f * p_t$ still makes sense and solves the heat equation on the nilmanifold. We also note that

$$f * p_t(g) = \int_{\Gamma\backslash\mathbb{H}^n} f(h) \sum_\gamma p_t(h^{-1}\gamma g) dh$$

as f is left Γ invariant. From the above it is clear that $f * p_t$ is again Γ invariant.

Let $\Delta = \mathcal{L} - T^2$ be the full Laplacian on $\mathbb{H}^n$ whose heat kernel is given by

$$q_t(x,u,\xi) = \int_{-\infty}^{\infty} e^{-i\lambda\xi} e^{-t\lambda^2} p^\lambda_t(x,u) d\lambda.$$

Owing to the factor $e^{-t\lambda^2}$ inside the integral this kernel can be extended to $\mathbb{C}^{2n+1}$ as an entire function. We remark that this property does not hold for the kernel p_t. Therefore, for any $f \in L^2(\Gamma\backslash\mathbb{H}^n)$ the function $f * q_t(x,u,\xi)$ also extends as an entire function. We denote this by $f * q_t(z,w,\zeta)$ where $z = x+iy, w = u+iv, \zeta = \xi + i\eta$. Note that $f * q_t(z,w,\zeta)$ is also invariant under the left action of Γ. This transform taking f into $f * q_t(z,w,\zeta)$ is known as the Segal-Bargmann transform (or heat kernel transform) in the literature. We are interested in characterising the image of $L^2(\Gamma\backslash\mathbb{H}^n)$ under this transform.

Any $f \in L^2(\Gamma\backslash\mathbb{H}^n)$ has the Fourier decomposition $f = \sum_k f_k,\ f_k \in \mathcal{H}_k$ and it is easily seen that the Segal-Bargmann transform preserves this decomposition. In fact, if $f_k(x,u,\xi) = e^{4\pi i k\xi} F_k(x,u)$ it follows that

$$f * q_t(z,w,\zeta) = \sum_k e^{-t(4\pi k)^2} e^{4\pi i k\zeta} F_k *_{-4\pi k} p_t^{-4\pi k}(z,w).$$

Thus, the image of $L^2(\Gamma\backslash\mathbb{H}^n)$ under the Segal-Bargmann transform will be a direct sum of spaces of functions of the form $e^{4\pi i k\zeta} F_k *_{-4\pi k} p_t^{-4\pi k}(z,w)$. Therefore, we can neglect the ξ part and consider the space of functions of the form $F *_{-4\pi k} p_t^{-4\pi k}(z,w)$ where F comes from $\mathcal{H}_k$. As we have remarked earlier the restriction of Δ on $\mathcal{H}_0$ is just the standard Laplacian. The image of $\mathcal{H}_0$ is well known, given by the classical Fock-Bargmann space. We therefore assume $k \neq 0$ from now on.

Let $\Lambda = \mathbb{Z}^{2n}$ and define $\mathcal{H}_k(\mathbb{R}^{2n}, \Lambda)$ to be the space of all functions $F(x,u)$ for which $e^{4\pi i k\xi} F(x,u) \in \mathcal{H}_k$. Note that these functions are characterised by the property

$$F(x + \mathbf{m}, u + \mathbf{n}) = e^{2\pi i k(u\cdot\mathbf{m} - x\cdot\mathbf{n})} F(x,u)$$

for all $(\mathbf{m}, \mathbf{n}) \in \Lambda$. With this notation we can say that $F \in L^2(\Gamma\backslash\mathbb{H}^n)$ if and only if for every $k \in \mathbb{Z}$ its Fourier component

$$F_k(x,u) = \int_0^{\frac{1}{2}} F(x,u,\xi) e^{-4\pi i k\xi} d\xi$$

belongs to $\mathcal{H}_k(\mathbb{R}^{2n}, \Lambda)$. For each $k \neq 0$ let $\mathcal{H}_k^t(\mathbb{C}^{2n}, \Lambda)$ be the space of all functions in $\mathcal{H}_k(\mathbb{R}^{2n}, \Lambda)$ having an entire extension to $\mathbb{C}^{2n}$ and satisfying

$$\|F\|_{k,t}^2 = \int_{\mathbb{R}^{2n}} \left(\int_{[0,1)^{2n}} |F(z,w)|^2 W_t^{-4\pi k}(z,w) dx du \right) dy dv < \infty.$$

When $k = 0$ we take

$$\|F\|_{0,t}^2 = t^{-2n} \int_{\mathbb{R}^{2n}} \left(\int_{[0,1)^{2n}} |F(z,w)|^2 dx du \right) e^{-\frac{1}{t}\pi(y^2+v^2)} dy dv$$

which corresponds to the classical Fock-Bargmann space.

Theorem 4.1. *An entire function $F(z,w,\zeta)$ belongs to the image of $L^2(\Gamma\backslash\mathbb{H}^n)$ under the Segal-Bargmann transform if and only if $F_k \in \mathcal{H}_k^t(\mathbb{C}^{2n}, \Lambda)$ for every k and*

$$\sum_k \|F_k\|_{k,t}^2 e^{2t(4\pi k)^2} < \infty.$$

Moreover, if $F = f * q_t,\ f \in L^2(\Gamma\backslash\mathbb{H}^n)$ then the above sum is a constant multiple of $\|f\|_2^2$.

This theorem will be proved by showing that $\mathcal{H}_k^t(\mathbb{C}^{2n}, \Lambda)$ is the direct sum of certain subspaces $\mathcal{H}_{k,\mathbf{j}}^t(\mathbb{C}^{2n}, \Lambda)$ which are defined as follows. For each $\lambda = 4\pi k$ and $\mathbf{j} \in \mathbb{F}_k$ let $\mathcal{H}_{k,\mathbf{j}}^t(\mathbb{C}^{2n}, \Lambda)$ be the subspace of $\mathcal{H}_k^t(\mathbb{C}^{2n}, \Lambda)$ satisfying the extra condition

$$G(z + \frac{1}{2k}\mathbf{m}, w) = e^{\pi i \mathbf{m}\cdot(w + \frac{1}{k}\mathbf{j})} G(z,w)$$

for all $\mathbf{m} \in \mathbb{Z}^n$.

Theorem 4.2. *An entire function F belongs to $\mathcal{H}^t_{k,\mathbf{j}}(\mathbb{C}^{2n},\Lambda)$ if and only if $F(z,w) = f * q_t(z,w,0)$ where f belongs to $\mathcal{H}_{k,\mathbf{j}}$.*

We postpone the proof of this theorem. We have already proved that $\mathcal{H}_k$ is the orthogonal direct sum of all $\mathcal{H}_{k,\mathbf{j}}, \mathbf{j} \in \mathbb{F}_k$. In view of this Theorem 4.1 follows from the above result once we show the following is true.

Theorem 4.3. *$\mathcal{H}^t_k(\mathbb{C}^{2n},\Lambda)$ is the orthogonal direct sum of $\mathcal{H}^t_{k,\mathbf{j}}(\mathbb{C}^{2n},\Lambda)$ as $\mathbf{j}$ varies over $\mathbb{F}_k$.*

It is easy to see that $\mathcal{H}^t_{k,\mathbf{j}}(\mathbb{C}^{2n},\Lambda), \mathbf{j} \in \mathbb{F}_k$ forms an orthogonal family of subspaces in the sense that

$$\int_{\mathbb{R}^{2n}} \int_{[0,1)^{2n}} F(z,w)\overline{G(z,w)} W_t^{-\lambda}(z,w)dzdw = 0$$

whenever $F \in \mathcal{H}^t_{k,\mathbf{j}}(\mathbb{C}^{2n},\Lambda)$ and $G \in \mathcal{H}^t_{k,\mathbf{l}}(\mathbb{C}^{2n},\Lambda)$ where $\mathbf{j},\mathbf{l} \in \mathbb{F}_k, \mathbf{j} \neq \mathbf{l}$. That every $G \in \mathcal{H}^t_k(\mathbb{C}^{2n},\Lambda)$ can be decomposed as a sum of $G_\mathbf{j} \in \mathcal{H}^t_{k,\mathbf{j}}(\mathbb{C}^{2n},\Lambda)$ follows as in the proof of Theorem 3.6. Indeed, we just have to define $G_\mathbf{j}(z,w) = (2k)^{-n}\sum_{\mathbf{m}\in\mathbb{F}_k} G_{\mathbf{j},\mathbf{m}}(z,w)$ where

$$G_{\mathbf{j},\mathbf{m}}(z,w) = e^{-\pi i \mathbf{m}\cdot w} e^{-\frac{1}{k}\pi i \mathbf{m}\cdot\mathbf{j}} G(z + \frac{1}{2k}\mathbf{m}, w)$$

and show that $G_\mathbf{j}$ has the required transformation properties and $G = \sum_{\mathbf{j}\in\mathbb{F}_k} G_\mathbf{j}$.

We now sketch the proof of Theorem 4.1 which uses a connection between twisted Bergman spaces and Hermite-Bergman spaces which are defined as follows. For each nonzero $\lambda \in \mathbb{R}$ let us consider the scaled Hermite operator $H(\lambda) = -\Delta + \lambda^2|x|^2$ on $\mathbb{R}^n$ whose eigenfunctions are provided by the Hermite functions

$$\Phi^\lambda_\alpha(x) = |\lambda|^{\frac{n}{4}}\Phi_\alpha(\sqrt{|\lambda|}x), x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n.$$

The operator $H(\lambda)$ generates the Hermite semigroup $e^{-tH(\lambda)}$ whose kernel is explicitly given by

$$K^\lambda_t(x,u) = \sum_{\alpha\in\mathbb{N}^n} e^{-(2|\alpha|+n)|\lambda|t}\Phi^\lambda_\alpha(x)\Phi^\lambda_\alpha(u).$$

Using Mehler's formula (see [14]) the above series can be summed to get

$$K^\lambda_t(x,u) = c_n(\sinh(\lambda t))^{-\frac{n}{2}}(\cosh(\lambda t))^{-\frac{n}{2}} \times e^{-\frac{\lambda}{4}\tanh(\lambda t)(x+u)^2} e^{-\frac{\lambda}{4}\coth(\lambda t)(x-u)^2}.$$

The image of $L^2(\mathbb{R}^n)$ under the Hermite semigroup has been studied by Byun [4] whose result is stated as follows. Let $\mathcal{H}^\lambda_t$ be the space of all entire functions on $\mathbb{C}^n$ for which

$$\int_{\mathbb{R}^{2n}} |F(x+iy)|^2 U^\lambda_t(x,y)dxdy < \infty$$

where the weight function U_t is given by

$$U_t(x,y) = c_n(\sinh(4\lambda t))^{-\frac{n}{2}} e^{\lambda\tanh(2\lambda t)x^2} e^{-\lambda\coth(2\lambda t)y^2}.$$

Theorem 4.4. *The image of $L^2(\mathbb{R}^n)$ under the Hermite semigroup is precisely the space $\mathcal{H}_t^\lambda$ and $e^{-tH(\lambda)}$ is a constant multiple of an isometry between these two spaces.*

Another proof of this theorem can be found in [17]. The relation between the Segal-Bargmann transform on $\Gamma\backslash\mathbb{H}$ and the Hermite semigroup is given in the following proposition.

Proposition 4.5. *Let $f \in L^2(\mathbb{R}^n)$ and $F = V_{k,\mathbf{j}}f$ for $\mathbf{j} \in \mathbb{F}_k$. Then*

$$F * k_t(x,u,\xi) = c_\lambda e^{-t\lambda^2+i\lambda\xi} e^{i\lambda(\mathbf{a}\cdot x+\frac{1}{2}x\cdot u)} \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{i\lambda x\cdot\mathbf{m}} \tau_{-\mathbf{a}} \left(e^{-tH(\lambda)}\tau_{\mathbf{a}} f\right)(u+\mathbf{m})$$

where $\lambda = 4\pi k$, $\mathbf{a} = \frac{1}{2k}\mathbf{j}$, $\tau_{\mathbf{a}} f(x) = f(x-\mathbf{a})$ and c_λ is a constant depending only on λ and n.

The above proposition is now used to get a characterisation of functions in $\mathcal{H}_k^t(\mathbb{C}^{2n},\Lambda)$. Theorem 4.2 follows easily from the next theorem.

Theorem 4.6. *An entire function $F(z,w)$ belongs to $\mathcal{H}_{k,\mathbf{j}}^t(\mathbb{C}^{2n},\Lambda)$ if and only if $F(z,w) = e^{t(4\pi k)^2}(V_{k,\mathbf{j}}f) * q_t(z,w,0)$ for some $f \in L^2(\mathbb{R}^n)$.*

If $F(z,w) = e^{t\lambda^2}(V_{k,\mathbf{j}}f) * q_t(z,w,0)$ where $f \in L^2(\mathbb{R}^n)$ then it is easy to check that F has the required transformation properties. From the previous proposition we know that

$$F(z,w) = c_\lambda e^{i\lambda(\mathbf{a}\cdot z+\frac{1}{2}z\cdot w)} \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{i\lambda z\cdot\mathbf{m}} \tau_{-\mathbf{a}} \left(e^{-tH(\lambda)}\tau_{\mathbf{a}} f\right)(w+\mathbf{m}).$$

The function $G(z,w) = e^{-i\lambda\mathbf{a}\cdot z}F(z,w-\mathbf{a})$ is given by the expansion

$$G(z,w) = e^{i\frac{\lambda}{2}z\cdot w} \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{i\lambda z\cdot\mathbf{m}} \left(e^{-tH(\lambda)}g\right)(w+\mathbf{m})$$

where $g = \tau_{\mathbf{a}} f$. Now

$$\int_{\mathbb{R}^{2n}} \int_{[0,1)^{2n}} |G(z,w)|^2 W_t^{-\lambda}(z,w)\,dxdudydv$$

$$= \int_{\mathbb{R}^{2n}} \int_{[0,1)^{2n}} \Big| \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{i\lambda x\cdot\mathbf{m}} e^{-\lambda y\cdot(u+\mathbf{m})} \left(e^{-tH(\lambda)}g\right)(w+\mathbf{m})\Big|^2$$

$$\times p_{2t}^\lambda(2y,2v)\,dxdudydv.$$

Applying Parseval's formula for the integral with respect to x variables and then summing the integrals with respect to u variables the above expression becomes

$$\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} |e^{-tH(\lambda)}g(w)|^2 e^{-2\lambda y\cdot u} p_{2t}^\lambda(2y,2v)\,dudydv.$$

Making use of the explicit formula for $p_{2t}^\lambda(2y,2v)$ and the fact that

$$\int_{\mathbb{R}^n} e^{-2\lambda y\cdot u} e^{-\lambda\coth(2\lambda t)y^2}\,dy = c_\lambda(\tanh(2\lambda t))^{\frac{n}{2}} e^{\lambda\tanh(2\lambda t)u^2}$$

integration with respect to y varaibles gives

$$(\sinh(4\lambda t))^{-\frac{n}{2}} \int_{\mathbb{R}^{2n}} |e^{-tH(\lambda)} g(w)|^2 e^{\lambda \tanh(2\lambda t) u^2} e^{-\lambda \coth(2\lambda t) v^2} du dv.$$

This means that $e^{-tH(\lambda)}g(w)$ is in the Hermite-Bergman space and we have, as $g = \tau_{\mathbf{a}} f$,

$$\int_{\mathbb{R}^{2n}} \int_{[0,1)^{2n}} |G(z,w)|^2 W_t^{-\lambda}(z,w) dx du dy dv = c_\lambda \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

Since $G(z,w) = e^{-i\frac{\lambda}{2}\mathbf{a}\cdot z} F(z, w - \mathbf{a})$ and

$$|e^{-i\frac{\lambda}{2}\mathbf{a}\cdot z}|^2 W_t^{-\lambda}(z, w + \mathbf{a}) = W_t^{-\lambda}(z,w)$$

the above simply means that

$$\int_{\mathbb{R}^{2n}} \int_{[0,1)^{2n}} |F(z,w)|^2 W_t^{-\lambda}(z,w) dx du dy dv = c_\lambda \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

This proves that $F(z,w)$ belongs to $\mathcal{B}^{\lambda}_{t,\mathbf{j}}$.

In order to prove the converse assume that $F \in \mathcal{H}^t_{k,\mathbf{j}}(\mathbb{C}^{2n}, \Lambda)$. If we can show that there exists $f \in L^2(\mathbb{R}^n)$ such that $V_{k,\mathbf{j}} f * q_t(z,w,0) = e^{-t\lambda^2} F(z,w)$ the converse follows. To this end we consider the function

$$G(z,w) = e^{-i\lambda \mathbf{a}\cdot z} e^{-i\frac{\lambda}{2} z\cdot w} F(z,w).$$

In view of the transformation properties of F the function G becomes $\frac{1}{2k}$-periodic in the x-variables. Therefore, it admits an expansion of the form

$$G(z,w) = \sum_{\mathbf{m}\in\mathbb{Z}^n} C_{\mathbf{m}}(w) e^{i\lambda \mathbf{m}\cdot z}$$

where $C_{\mathbf{m}}$ are the Fourier coefficients given by

$$C_{\mathbf{m}}(w) = \int_{[0,\frac{1}{2k})^n} G(x,w) e^{-i\lambda \mathbf{m}\cdot x} dx.$$

The transformation properties of F leads to $G(x, w - \mathbf{m}) = G(x,w) e^{i\lambda \mathbf{m}\cdot x}$ and hence $C_{\mathbf{m}}(w - \mathbf{m}) = C_{\mathbf{0}}(w)$. Thus, we obtain

$$F(z,w) = e^{i\lambda \mathbf{a}\cdot z} e^{i\frac{\lambda}{2} z\cdot w} \sum_{\mathbf{m}\in\mathbb{Z}^n} C_{\mathbf{0}}(w + \mathbf{m}) e^{i\lambda \mathbf{m}\cdot z}.$$

We now show that $C_{\mathbf{0}}$ belongs to the Hermite-Bergman space $\mathcal{H}^{\lambda}_t, \lambda = 4\pi k$. We look at

$$\int_{\mathbb{R}^{2n}} \int_{[0,1)^{2n}} |F(z,w)|^2 W_t^{-\lambda}(z,w) dx du dy dv$$

which is given to be finite. Using the above expansion and proceeding as in the previous part of the proof we can conclude that

$$\int_{\mathbb{R}^{2n}} |C_{\mathbf{0}}(w - \mathbf{a})|^2 U_t^{\lambda}(u,v) du dv < \infty.$$

Thus there exists $g \in L^2(\mathbb{R}^n)$ such that $C_{\mathbf{0}}(w) = e^{-tH(\lambda)}g(w+\mathbf{a})$. Taking $f = \tau_{-\mathbf{a}}g$ we see that

$$F(z,w) = e^{t\lambda^2} V_{k,\mathbf{j}} q * k_t(z,w,0).$$

This completely proves the theorem.

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