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ENTIRE SOLUTIONS OF THE ALLEN-CAHN EQUATION AND COMPLETE EMBEDDED MINIMAL SURFACES

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ABSTRACT. We review some recent results on construction of entire solutions to the classical semilinear elliptic equation $\Delta u + u - u^3 = 0$ in $\mathbb{R}^N$. In various cases, large dilations of an embedded, complete minimal surface approximate the transition set of a solution that connects the equilibria ± 1. In particular, our construction answers negatively a celebrated conjecture by E. De Giorgi in dimensions $N \geq 9$.

1. INTRODUCTION AND MAIN RESULTS

1.1. **The Allen-Cahn equation and minimal surfaces.** The Allen-Cahn equation in $\mathbb{R}^N$ is the semilinear elliptic problem

$$\Delta u \,+\, f(u) \,=\, 0 \quad \text{in } \mathbb{R}^N\,, \tag{1.1}$$

where $f(s) = -W'(s)$ and W is a "double-well potential", bi-stable and balanced, namely

$$W(s) > 0 \text{ if } \; s \neq 1, -1, \quad W(1) = 0 = W(-1), \quad W''(\pm 1) \,=\, f'(\pm 1) \,=:\, \sigma_\pm^2 > 0. \tag{1.2}$$

A typical example of such a nonlinearity is

$$f(u) \,=\, (1-u^2)u \qquad \text{for } W(u) \,=\, \frac{1}{4}(1-u^2)^2. \tag{1.3}$$

Equation (1.1) is a prototype for the continuous modeling of phase transition phenomena. Let us consider the energy in a subregion region Ω of $\mathbb{R}^N$

$$J_\alpha(v) = \int_\Omega \frac{\alpha}{2}\,|\nabla v|^2 + \frac{1}{4\alpha} W(v),$$

whose Euler-Lagrange equation is a scaled version of (1.1),

$$\alpha^2 \Delta v \,+\, f(v) \,=\, 0 \quad \text{in } \Omega\,. \tag{1.4}$$

We observe that the constant functions $u = \pm 1$ minimize J_α. They are idealized as two *stable phases* of a material in Ω. It is of interest to analyze stationary configurations in which the two phases coexist. Given any subset Λ of Ω, any discontinuous function of the form

$$v_* = \chi_\Lambda - \chi_{\Omega\setminus\Lambda} \tag{1.5}$$

minimizes the second term in J_ε. The introduction of the gradient term in J_α makes an α-regularization of u_* a test function for which the energy gets bounded and proportional to the surface area of the *interface* $M = \partial\Lambda$, so that in addition to minimizing approximately the second term, stationary configurations should also select asymptotically interfaces M that are stationary for surface area, namely (generalized) minimal surfaces. This intuition on the Allen-Cahn equation gave important impulse to the calculus of variations, motivating the development of the theory of Γ-*convergence* in the 1970's. Modica [30] proved that a family of local minimizers u_α of J_α with uniformly bounded energy must converge in suitable sense to a function of the form (1.5) where $\partial\Lambda$ minimizes perimeter. Thus, intuitively, for each given $\lambda \in (-1,1)$, the level sets $[v_\alpha = \lambda]$, collapse as $\alpha \to 0$ onto the interface $\partial\Lambda$. Similar result holds for critical points not necessarily minimizers, see [26]. For minimizers this convergence is known in very strong sense, see [3, 4].

If, on the other hand, we take such a critical point u_α and scale it around an interior point $0 \in \Omega$, setting $u_\alpha(x) = v_\alpha(\alpha x)$, then u_α satisfies equation (1.1) in an expanding domain,

$$\Delta u_\alpha + f(u_\alpha) = 0 \quad \text{in } \alpha^{-1}\Omega$$

so that letting formally $\alpha \to 0$ we end up with equation (1.1) in entire space. The "interface" for u_α should thus be around the (asymptotically flat) minimal surface $M_\alpha = \alpha^{-1}M$. Modica's result is based on the intuition that if M happens to be a smooth surface, then the transition from the equilibria -1 to 1 of u_α along the normal direction should take place in the approximate form $u_\alpha(x) \approx w(z)$ where z designates the normal coordinate to M_α. Then w should solve the ODE problem

$$w'' + f(w) \;=\; 0 \quad \text{in } \mathbb{R}, \quad w(-\infty) = -1,\; w(+\infty) = 1\;. \tag{1.6}$$

This *heteroclinic solution*, indeed exists thanks to assumption (1.2). It is defined, uniquely up to a constant translation $a \in \mathbb{R}$, by the identity

$$\int_0^{w(t)} \frac{ds}{\sqrt{2W(s)}} = t - a,$$

which follows from the fact that

$$w'^2 - 2\,W(w) = 0 \quad \text{in } \mathbb{R}.$$

We fix in what follows the unique w for which

$$\int_{\mathbb{R}} t\, w'(t)^2\, dt \;=\; 0\;. \tag{1.7}$$

For the nonlinearity (1.3), we have $w(t) = \tanh\left(t/\sqrt{2}\right)$. In general w approaches its limits at exponential rates,

$$w(t) - \pm 1 \;=\; O(\,e^{-\sigma_\pm |t|}\,) \quad \text{as } t \to \pm\infty\;.$$

Observe then that

$$J_\alpha(u_\alpha) \approx Area\,(M) \int_{\mathbb{R}} [\frac{1}{2}w'^2 + W(w)]$$

which is what makes it plausible that M is critical for area, namely a minimal surface.

1.2. **De Giorgi's conjecture.** The above considerations led E. De Giorgi [10] to formulate in 1978 a celebrated conjecture on the Allen-Cahn equation (1.1) for the nonlinearity (1.3).

(DG) *Let u be a bounded solution of the equation*

$$\Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

such that $\partial_{x_N} u > 0$. Then the level sets $[u=\lambda]$ are all hyperplanes, at least for dimension $N \le 8$. Equivalently, u must have the form $u(x) = w(\,(x-x_0)\cdot e\,)$, for some $x_0, e \in \mathbb{R}^N$, where $|e| = 1$.

De Giorgi's conjecture has been fully established in dimensions $N = 2$ by Ghoussoub and Gui [16] and for $N = 3$ by Ambrosio and Cabré [1]. Savin [37] proved its validity for $4 \le N \le 8$ under a mild additional assumption. (DG) is a statement parallel to *Bernstein's theorem* for minimal graphs which in its most general form, due to Simons [40], states that any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N-1$ variables, must be a hyperplane if $N \le 8$. Bombieri, De Giorgi and Giusti [2] proved that this fact is false in dimension $N \ge 9$, by constructing a nontrivial solution to the problem

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8. \tag{1.9}$$

by means of the super-subsolution method. Let us write

$$x' = (x_1, \dots, x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \cdots + x_4^2}, \quad v = \sqrt{x_5^2 + \cdots + x_8^2}.$$

The BDG solution has the form $F(x') = F(u,v)$ with the symmetry property $F(u,v) = -F(v,u)$ if $u \ge v$. In addition we can show that F becomes asymptotic to a function homogeneous of degree 3 that vanishes on the cone $u = v$. The result in [2] has made plausible the existence of a counterexample to statement (DG) for $N \ge 9$. This has been recently found in [11], see also [7, 27]. See [15] for a recent survey on the state of the art of this question. Let us describe the result in [11].

Let $M = \{x_9 = F(x')\}$ be the minimal BDG graph so predicted, and let us consider for $\alpha > 0$ its dilation $M_\alpha = \alpha^{-1} M$, which is also a minimal graph. We have the following result, which disproves statement (DG) in dimensions 9 or higher.

Theorem 1. [11]. *Let $N \ge 9$. For all $\alpha > 0$ sufficiently small there exists a bounded solution $u_\alpha(x)$ of equation* (1.8) *such that*

$$u_\alpha(0) = 0, \quad \partial_{x_N} u_\alpha(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}^N,$$

and

$$|u_\alpha(x)| \;\to\; 1 \text{ as } \operatorname{dist}(x, M_\alpha) \;\to\; +\infty,$$

uniformly in all small $\alpha > 0$.

Sketch of the proof. The proof in [11] provides accurate information on u_α. If $t = t(y)$ denotes a choice of signed distance to the graph M_α then, for a small fixed number $\delta > 0$, the solution looks like $u_\alpha(x) \sim w(t), \quad \text{if } |t| < \frac{\delta}{\alpha}$. with $w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$, the one-dimensional heteroclinic solution to (1.8). Let us us assume $N = 9$ (which is sufficient), and consider Fermi coordinates to describe points in $\mathbb{R}^9$ near M_α, $x = y + z\nu_\alpha(y), \quad y \in M_\alpha, \quad |z| < \frac{\delta}{\alpha}$ where ν_α is the unit normal to M_α for which $\nu_{\alpha 9} > 0$. Then we choose as a first approximation $\mathtt{w}(x) := w(z + h(\alpha y))$ where h is a smooth, small function on $M = M_1$, to be determined. Looking for a solution of the form $\mathtt{w} + \phi$, it turns out that the problem becomes essentially reduced to

$$\Delta_{M_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi + E + N(\phi) = 0 \quad \text{in } M_\alpha \times \mathbb{R}$$

where $S(\mathtt{w}) = \Delta\mathtt{w} + f(\mathtt{w})$, $E = \chi_{|z|<\alpha^{-1}\delta}\, S(\mathtt{w})$, $N(\phi) = f(\mathtt{w}+\phi) - f(\mathtt{w}) - f'(\mathtt{w})\phi + B(\phi)$, $f(\mathtt{w}) = \mathtt{w}(1-\mathtt{w}^2)$, and $B(\phi)$ is a second order linear operator with small coefficients, also cut-off for $|z| > \delta\alpha^{-1}$. Rather than solving the above problem directly we consider a projected version of it:

$$L(\phi) := \Delta_{M_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi = -E - N(\phi) + c(y)w'(z) \quad \text{in } M_\alpha \times \mathbb{R} \tag{1.10}$$

$$\int \phi(y,z)w'(z)\,dz \;=\; 0 \quad \text{for all} \quad y \in M_\alpha \tag{1.11}$$

A solution to this problem can be found in such a way that it respects the size and decay rate of the error E, which is roughly of the order $\sim r(\alpha y)^{-3}e^{-|z|}$, this is made precise with the use of a linear theory for the projected problem in weighted Sobolev norms and an application of contraction mapping principle. Finally h s found so that $c(y) \equiv 0$. We have $c(y)\int {w'}^2 dz = \int (E + N(\phi))w'\,dz$ and thus we get reduced to a (nonlocal) nonlinear PDE in M of the form

$$\mathcal{J}(h) := \Delta_M h + |A|^2 h = O(\alpha)r(y)^{-3} + M_\alpha(h) \quad \text{in } M, \quad h = 0 \quad \text{in } M \cap [u = v], \tag{1.12}$$

where $M(h)$ is a small operator which includes nonlocal terms. A solvability theory for the Jacobi operator in weighted Sobolev norms is then devised, with the crucial ingredient of the presence of explicit barriers for inequalities involving the linear operator above, and asymptotic curvature estimates by Simon [39]. Using this theory, problem (1.12) is finally solved by means of contraction mapping principle. The monotonicity property follows from maximum principle applied to the linear equation satisfied by $\partial_{x_9} u$. □

The assumption of monotonicity in one direction for the solution u in De Giorgi's conjecture implies a form of stability, locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, if $Z = \partial_{x_N} u > 0$, then the linearized operator $L = \Delta + f'(u)$, satisfies maximum principle. This implies stability of u, in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$\mathcal{Q}(\psi,\psi) := \int |\nabla\psi|^2 - f'(u)\psi^2 \tag{1.13}$$

satisfies $\mathcal{Q}(\psi,\psi) > 0$ for all $\psi \neq 0$ smooth and compactly supported. Stability is a basic ingredient in the proof of the conjecture dimensions $2, 3$ in [1, 16], based on finding a control at infinity of the growth of the Dirichlet integral. In dimension $N = 3$ it turns out that

$$\int_{B(0,R)} |\nabla u|^2 \;=\; O(R^2) \tag{1.14}$$

which intuitively means that the embedded level surfaces $[u = \lambda]$ must have a finite number of components outside a large ball, which are all "asymptotically flat". The question whether stability alone suffices for property (1.14) remains open. More generally, it is believed that this property is equivalent to *finite Morse index* of the solution u (which means essentially that u is stable outside a bounded set). The Morse index $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathcal{Q}(\psi,\psi) < 0 \quad \text{for all} \quad \psi \in E \setminus \{0\}.$$

Rather surprisingly, basically no examples of finite Morse index entire solutions of the Allen-Cahn equation seem known in dimension $N = 3$. Great progress has been achieved in the last decades, both in the theory of semilinear elliptic PDE like (1.1) and in minimal surface theory in $\mathbb{R}^3$. While this link traces back to the very origins of the study of (1.1) as discussed above, it has only been partially explored in producing new solutions.

In the paper [14], we construct a new class of entire solutions to the Allen-Cahn equation in $\mathbb{R}^3$ which have the characteristic (1.14), and also finite Morse index, whose level sets resemble a large dilation of a given complete, embedded minimal surface M, asymptotically flat in the sense that it has *finite total curvature*, namely

$$\int_M |K|\, dV \;<\; +\infty$$

where K denotes Gauss curvature of the manifold, which is also *non-degenerate* in a sense that we will make precise below.

As pointed out by Dancer [8], Morse index is a natural element to attempt classification of solutions of (1.1). Beyond De Giorgi's conjecture, classifying solutions with given Morse index should be a natural step towards the understanding of the structure of the bounded solutions of (1.1). Our main results show that, unlike the stable case, the structure of the set of solutions with finite Morse index is highly complex. On the other hand, we believe that our construction contains germs of generality, providing elements to extrapolate what may be true in general, in analogy with classification of embedded minimal surfaces We elaborate on these issues in §2. We point out that solutions with exactly k nodal lines, for any given $k \geq 1$, have been found in $\mathbb{R}^2$ in [13]. These solutions are very to have finite Morse index.

We shall describe next the main results in [14].

1.3. **Embedded minimal surfaces of finite total curvature.** The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$, has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa, [5, 6]. The *Costa surface* is a genus one minimal surface, complete and properly embedded, which outside a large ball has exactly three components (its *ends*), two of which are asymptotically catenoids with the same axis and opposite directions, the third one asymptotic to a plane perpendicular to that axis. The complete proof of embeddedness is due to Hoffman and Meeks [21]. In [22, 24] these authors generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same look as Costa's far away, but with an array of tunnels that provides arbitrary genus $k \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus k.

Many other examples of multiple-end embedded minimal surfaces have been found since, see for instance [28, 41] and references therein. In general all these surfaces look like parallel planes, slightly perturbed at their ends by asymptotically logarithmic corrections with a certain number of catenoidal links connecting their adjacent sheets. In reality this intuitive picture is not a coincidence. Using the Eneper-Weierstrass representation, Osserman [34] established that any embedded, complete minimal surface with finite total curvature can be described by a conformal diffeomorphism of a compact surface (actually of a Riemann surface), with a finite number of its points removed. These points correspond to the ends. Moreover, after a convenient rotation, the ends are asymptotically all either catenoids or plane, all of them with parallel axes, see Schoen [38]. The topology of the surface is thus characterized by the genus of the compact surface and the number of ends, having therefore "finite topology".

1.4. **Results.** In what follows M designates a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature (to which below we will make a further nondegeneracy assumption). As pointed out in [25], M is orientable and the set $\mathbb{R}^3 \setminus M$ has exactly two components S_+, S_-. In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take it to point towards S_+.

For $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we denote

$$r = r(x) = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

After a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0, then M decomposes into a finite number m of unbounded components $M_1, \ldots, M_m$, its *ends*. From a result in [38], we know that asymptotically each end of M_k either resembles a plane or a catenoid. More precisely, M_k can be represented as the graph of a function F_k of the first two variables,

$$M_k = \{\, y \in \mathbb{R}^3 \ / \ r(y) > R_0,\ y_3 = F_k(y') \,\}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + b_{ik}\frac{y_i}{r^2} + O(r^{-3}) \quad \text{as } r \to +\infty, \tag{1.15}$$

for certain constants a_k, b_k, b_{ik}, and this relation can also be differentiated. Here

$$a_1 \le a_2 \le \ldots \le a_m \ , \qquad \sum_{k=1}^{m} a_k \ = \ 0\,. \tag{1.16}$$

The direction of the normal vector $\nu(y)$ for large $r(y)$ approaches on the ends that of the x_3 axis, with alternate signs. We use the convention that for $r(y)$ large we have

$$\nu(y) \ = \ \frac{(-1)^k}{\sqrt{1+|\nabla F_k(y')|^2}}\,(\nabla F_k(y')\,,\,-1\,) \quad \text{if } y \in M_k. \tag{1.17}$$

Let us consider the Jacobi operator of M

$$\mathcal{J}(h) \ := \ \Delta_M h + |A|^2 h \tag{1.18}$$

where $|A|^2 = -2K$ is the Euclidean norm of the second fundamental form of M. $\mathcal{J}$ is the linearization of the mean curvature operator with respect to perturbations of M measured along its normal direction. A smooth function $z(y)$ defined on M is called a *Jacobi field* if $\mathcal{J}(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields: Associated to respectively translations along coordinates axes and rotation around the x_3-axis, are the functions

$$\begin{aligned} z_1(y) &= \nu(y)\cdot e_i, \quad y \in M, \quad i = 1,2,3, \\ z_4(y) &= (-y_2, y_1, 0)\cdot \nu(y), \quad y \in M. \end{aligned} \tag{1.19}$$

We assume that M is *non-degenerate* in the sense that these functions are actually *all* the bounded Jacobi fields, namely

$$\{\ z \in L^\infty(M)\ /\ \mathcal{J}(z) = 0\,\} \ = \ \operatorname{span}\{\,z_1, z_2, z_3, z_4\,\}\,. \tag{1.20}$$

We denote in what follows by J the dimension (≤ 4) of the above vector space.

This assumption, expected to be generic for this class of surfaces, is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface which is an example of a three ended M whose genus may be of any order. See Nayatani [32, 33] and Morabito [31]. Note that for a catenoid, $z_{04} = 0$ so that $J = 3$. Non-degeneracy has been used as a tool to build new minimal surfaces for instance in Hauswirth and Pacard [20], and in Pérez and Ros [36]. It is also the basic element, in a compact-manifold version, to build solutions to the small-parameter Allen-Cahn equation in Pacard and Ritoré [35].

In this paper we will construct a solution to the Allen Cahn equation whose zero level sets look like a large dilation of the surface M, with ends perturbed logarithmically. Let us consider a large dilation of M,

$$M_\alpha \ := \ \alpha^{-1} M.$$

This dilated minimal surface has ends parameterized as

$$M_{k,\alpha} \ = \ \{\, y \in \mathbb{R}^3 \ /\ r(\alpha y) > R_0,\ y_3 = \alpha^{-1} F_k(\alpha y')\,\}\ .$$

Let β be a vector of given m real numbers with

$$\beta \;=\; (\beta_1,\dots,\beta_m), \qquad \sum_{i=1}^{m} \beta_i = 0 \,. \tag{1.21}$$

Our first result asserts the existence of a solution $u = u_\alpha$ defined for all sufficiently small $\alpha > 0$ such that given $\lambda \in (-1,1)$, its level set $[u_\alpha = \lambda]$ defines an embedded surface lying at a uniformly bounded distance in α from the surface M_α, for points with $r(\alpha y) = O(1)$, while its k-th end, $k = 1,\dots,m$, lies at a uniformly bounded distance from the graph

$$r(\alpha y) > R_0,\; y_3 \;=\; \alpha^{-1}\, F_k(\alpha y') + \beta_k \log|\alpha y'| \,. \tag{1.22}$$

The parameters β must satisfy an additional constraint. It is clear that if two ends are parallel, say $a_{k+1} = a_k$, we need at least that $\beta_{k+1} - \beta_k \geq 0$, for otherwise the ends would eventually intersect. Our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate. We require

$$\beta_{k+1} - \beta_k \;>\; 4\,\max\{\sigma_-^{-1}, \sigma_+^{-1}\} \quad \text{if} \quad a_{k+1} = a_k \;. \tag{1.23}$$

Let us consider the smooth map

$$X(y,z) = y + z\nu(\alpha y), \quad (y,t) \in M_\alpha \times \mathbb{R}. \tag{1.24}$$

$x = X(y,z)$ defines coordinates inside the image of any region where the map is one-to-one. In particular, let us consider a function $p(y)$ with

$$p(y) = (-1)^k \beta_k \log|\alpha y'| + O(1), \quad k = 1,\dots,m,$$

and β satisfying $\beta_{k+1} - \beta_k > \gamma > 0$ for all k with $a_k = a_{k+1}$. Then the map X is one-to-one for all small α in the region of points (y,z) with

$$|z - q(y)| < \frac{\delta}{\alpha} + \gamma \log(1 + |\alpha y'|)$$

provided that $\delta > 0$ is chosen sufficiently small.

Theorem 2. *Let $N = 3$ and M be a minimal surface embedded, complete with finite total curvature which is nondegenerate. Then, given β satisfying relations* (1.21) *and* (1.23)*, there exists a bounded solution u_α of equation* (1.1)*, defined for all sufficiently small α, such that*

$$u_\alpha(x) = w(z - q(y)) + O(\alpha) \quad \textit{for all} \quad x = y + z\nu(\alpha y), \quad |z - q(y)| < \frac{\delta}{\alpha}, \tag{1.25}$$

where the function q satisfies

$$q(y) \;=\; (-1)^k \beta_k\, \log|\alpha y'| + O(1) \quad y \in M_{k,\alpha}, \quad k = 1,\dots,m.$$

In particular, for each given $\lambda \in (-1,1)$, the level set $[u_\alpha = \lambda]$ is an embedded surface that decomposes for all sufficiently small α into m disjoint components (ends) outside a bounded set. The k-th end lies at $O(1)$ distance from the graph

$$y_3 \;=\; \alpha^{-1}\, F_k(\alpha y) + \beta_k \log|\alpha y'|.$$

The solution predicted by this theorem depends, for fixed α, on m parameters. Taking into account the constraint $\sum_{j=1}^{m} \beta_j = 0$ this gives $m-1$ independent parameters corresponding to logarithmic twisting of the ends of the level sets. Let us observe that consistently, the combination $\beta \in \mathrm{Span}\,\{(a_1, \dots, a_m)\}$ can be set in correspondence with moving α itself, namely with a dilation parameter of the surface. We are thus left with $m-2$ parameters for the solution in addition to α. Thus, besides the trivial rigid motions of the solution, translation along the coordinates axes, and rotation about the x_3 axis, this family of solutions depends exactly on $m-1$ "independent" parameters. Part of the conclusion of our second result is that the bounded kernel of the linearization of equation (1.1) about one of these solutions is made up exactly of the generators of the rigid motions, so that in some sense the solutions found are L^∞-isolated, and the set of bounded solutions nearby is actually $m-1+J$-dimensional. A result parallel to this one, in which the *moduli space* of the minimal surface M is described by a similar number of parameters, is found in [36].

Next we discuss the connection of the Morse index of the solutions of Theorem 2 and the *index* of the minimal surface M, $i(M)$, which has a similar definition relative to the quadratic form for the Jacobi operator: The number $i(M)$ is the largest dimension for a vector spaced E of compactly supported smooth functions in M with

$$\int_M |\nabla k|^2 \, dV \; - \int_M |A|^2 k^2 \, dV \; < 0 \quad \text{for all} \quad k \in E \setminus \{0\}.$$

We point out that for complete, embedded surfaces, finite index is equivalent to finite total curvature, see [19] and also §7 of [25] and references therein. Thus, for our surface M, $i(M)$ is indeed finite. Moreover, in the Costa-Hoffmann-Meeks surface it is known that $i(M) = 2l - 1$ where l is the genus of M. See [32], [33] and [31].

Our second result is that the Morse index and non-degeneracy of M are transmitted into the linearization of equation (1.1).

Theorem 3. *Let u_α the solution of problem* (1.1) *given by Theorem 2. Then for all sufficiently small α we have*

$$m(u_\alpha) \;=\; i(M).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\phi + f'(u_\alpha)\phi = 0 \qquad \text{in } \mathbb{R}^3$$

must be a linear combination of the functions Z_i, $i = 1, 2, 3, 4$ defined as

$$Z_i = \partial_i u_\alpha, \quad i = 1, 2, 3, \quad Z_4 = -x_2 \partial_1 u_\alpha + x_1 \partial_2 u_\alpha.$$

We will devote the rest of this paper to the proofs of Theorems 1 and 2.

2. Further comments and open questions

2.1. **Symmetries.** As it is natural, the invariances of the surface are at the same time inherited from the construction. If M is a catenoid, revolved around the x_3 axis, the solution in Theorem 2 is radial in the first two variables,

$$u_\alpha(x) = u_\alpha\left(\,|x'|,\, x_3\right).$$

This is a consequence of the construction. The invariance of the Laplacian under rotations and the autonomous character of the nonlinearity imply that the entire proof can be carried out in spaces of functions with this radial symmetry. More generally, if M is invariant a group of linear isometries, so will be the solution found, at least in the case that $f(u)$ is odd. This assumption allows for odd reflections. The Costa-Hoffmann-Meeks surface is invariant under a discrete group constituted of combination of dihedral symmetries and reflections to which this remark apply.

2.2. **Towards a classification of finite Morse index solutions.**
Understanding bounded, entire solutions of nonlinear elliptic equations in $\mathbb{R}^N$ is a problem that has always been at the center of PDE research. This is the context of various classical results in PDE literature like the Gidas-Ni-Nirenberg theorems on radial symmetry of one-signed solutions, Liouville type theorems, or the achievements around De Giorgi's conjecture. In those results, the geometry of level sets of the solutions turns out to be a posteriori very simple (planes or spheres). More challenging seems the problem of classifying solutions with finite Morse index, in a model as simple as the Allen-Cahn equation. While the solutions predicted by Theorem 2 are generated in an asymptotic setting, it seems plausible that they contain germs of generality, in view of parallel facts in the theory of minimal surfaces. In particular we believe that the following two statements hold true for a bounded solution u to equation (1.1) in $\mathbb{R}^3$.

(1) *If u has finite Morse index and $\nabla u(x) \neq 0$ outside a bounded set, then each level set of u must have outside a large ball a finite number of components, each of them asymptotic to either a plane or to a catenoid. After a rotation of the coordinate system, all these components are graphs of functions of the same two variables.*

(2) *If u has Morse index equal to one. Then u must be axially symmetric, namely after a rotation and a translation, u is radially symmetric in two of its variables. Its level sets have two ends, both of them catenoidal.*

It is worth mentioning that a *balancing formula* for the "ends" of level sets to the Allen-Cahn equation is available in $\mathbb{R}^2$, see [18]. An extension of such a formula to $\mathbb{R}^3$ should involve the configuration (1) as its basis. The condition of finite Morse index can probably be replaced by the energy growth (1.14).

On the other hand, (1) should not hold if the condition $\nabla u \neq 0$ outside a large ball is violated. For instance, let us consider the octant $\{x_1, x_2, x_3 \geq 0\}$ and the odd nonlinearity $f(u) = (1-u^2)u$. Problem (1.1) in the octant with zero boundary data can be solved by a super-subsolution scheme (similar to that in [9]) yielding a positive solution. Extending by successive odd reflections to the remaining octants, one generates an entire solution (likely to have finite Morse index), whose zero level set does not have the characteristics above: the condition $\nabla u \neq 0$ far away corresponds to *embeddedness of the ends.*

Various rather general conditions on a minimal surface imply that it is a catenoid. For example, R. Schoen [38] proved that a complete embedded minimal surface in $\mathbb{R}^3$ with two ends must be catenoid (and hence it has index one). One may wonder if a bounded solution to (1.1) whose zero level set has only two ends is radially symmetric in two variables. On the other hand a one-end minimal surface is forced to be a plane [23]. We may wonder whether or not the zero level set lies on a half space implies that the solution depends on only one variable.

These questions seem rather natural generalizations of that by De Giorgi, now on the classification finite Morse index entire solutions of (1.1). The case in which the minimal surfaces have finite topology but infinite total curvature, like the helicoid, are natural objects to be considered. While results parallel to that in Theorem 2 may be expected possible, they may have rather different nature. The condition of diverging ends in β is not just technical. If it fails a solution may still be associated to the manifold but interactions between neighboring interfaces, which are inherent to the Allen-Cahn equation but not to the minimal surface problem, will come into play. The case of infinite topology may also give rise to very complicated patterns, we refer to Pacard and Hauswirth [20] and references therein for recent result on construction of minimal surfaces in this scenario.

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