

THE GROWTH OF THE A_p CONSTANT ON CLASSICAL ESTIMATES

CARLOS PÉREZ

ABSTRACT. We survey on some recent results concerning weak and strong weighted L^p estimates for Calderón-Zygmund operators with sharp bounds when the weight satisfies the A_1 condition. These questions are related to a problem posed by Muckenhoupt and Wheeden in the seventies.

1. INTRODUCTION

In 1971, C. Fefferman and E.M. Stein [13] established the following extension of the classical weak-type $(1, 1)$ property of the Hardy-Littlewood maximal operator M :

$$\sup_{\lambda>0} \lambda w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq c \int_{\mathbb{R}^n} |f| Mwdx, \quad (1.1)$$

where a weight w is supposed to be a non-negative locally integrable function and $w(E) = \int_E w(x)dx$. This estimate yields some sort of duality for M . It was used in [13] to derive the vector-valued extension of the classical estimates for the Hardy-Littlewood maximal function which has many important applications: for every $1 < p, q < \infty$:

$$\left\| \left(\sum_j (Mf_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}$$

Assume now that T is a Calderón-Zygmund singular integral operator. It was conjectured by B. Muckenhoupt and R. Wheeden [24] many years ago that the full analogue of (1.1) holds for T , namely,

$$\sup_{\lambda>0} \lambda w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \leq c \int_{\mathbb{R}^n} |f| Mwdx. \quad (1.2)$$

We will refer to this as the **Muckenhoupt-Wheeden conjecture**.

The question whether inequality (1.2) holds is still open even for the Hilbert transform. Moreover, it is still unknown whether the following weaker variant of

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(1.2) is true: if w is an A_1 weight, then

$$\sup_{\lambda > 0} \lambda w \{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \leq c[w]_{A_1} \int_{\mathbb{R}^n} |f|w dx. \tag{1.3}$$

Recall that w is an A_1 weight if there is a finite constant c such that

$$Mw \leq cw \quad \text{a.e.}$$

and where $[w]_{A_1}$ denotes the smallest of these c .

We will refer to this new possible result as the **weak Muckenhoupt-Wheeden conjecture**. In this notes we shall be concerned with this conjecture. We don't prove the conjecture, namely the linear growth but we prove a logarithmic growth in (1.4) taken from [20] (see also[18]).

As far as we know, the first result along these line was proved by S. Buckley [3] as part of his PhD. Thesis. To estate this result we recall that a weight satisfy the Muckenhoupt A_p condition if

$$[w]_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

$[w]_{A_p}$ is called the A_p constant (or characteristic or norm) of the weight The case $p = 1$ is understood by replacing the right hand side by $(\inf_Q w)^{-1}$ which is equivalent to the one defined above.

Observe the duality relationship:

$$[w]_{A_p} = [w^{1-p'}]_{A_{p'}}$$

We establish now the result (with an improvement in the constant)

Theorem 1.1. *Let $w \in A_p$, then the Hardy-Littlewood maximal function satisfies the following operator estimate:*

$$\|M\|_{L^p(w)} \leq cp' [w]_{A_p}^{\frac{1}{p-1}}$$

namely,

$$\sup_{w \in A_p} \frac{1}{[w]_{A_p}^{\frac{1}{p-1}}} \|M\|_{L^p(w)} \leq cp' < \infty \tag{1.4}$$

Furthermore the result is sharp in the sense that: for any $\theta > 0$

$$\sup_{w \in A_p} \frac{1}{[w]_{A_p}^{\frac{1}{p-1} - \theta}} \|M\|_{L^p(w)} = \infty \tag{1.5}$$

In fact, we cannot replace the function $\psi(t) = t^{\frac{1}{p-1}}$ by a “smaller” function $\psi : [1, \infty) \rightarrow (0, \infty)$ in the sense that

$$\inf_{t > 1} \frac{\psi(t)}{t^{\frac{1}{p-1}}} = 0$$

(or $\lim_{t \rightarrow \infty} \frac{\psi(t)}{t^\beta} = 0$, or $\sup_{t > 1} \frac{t^\beta}{\psi(t)} = \infty$ or $\lim_{t \rightarrow \infty} \frac{t^\beta}{\psi(t)} = \infty$) since then

$$\sup_{w \in A_p} \frac{1}{\psi([w]_{A_p})} \|M\| = \infty \tag{1.6}$$

The original proof of Buckley is delicate because is based on sharp version of the so called Reverse Hölder Inequality. However, very, recently, A. Lerner [17] has found a very nice and simple proof of this result that we will given in section 3. It is based on the Besicovich lemma but the dyadic case is even simpler.

This result should be compared with weak-type bound is

$$\|M\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \leq c [w]_{A_p}^{1/p}. \tag{1.7}$$

We shall use several well-known facts about the A_p weights. First, it follows from the definition of the A_1 weights that if $w_1, w_2 \in A_1$, then

$$w = w_1 w_2^{1-p} \in A_p,$$

and furthermore

$$[w]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1} \tag{1.8}$$

This is the easy part of the celebrated P. Jones factorization theorem.

Another result we need is due Coifman-Rochberg [8], actually we need a quantitative version of it:

Let μ be a positive Borel, then for each $0 < \delta < 1$

$$(M\mu)^\delta \in A_1$$

and furthermore

$$[(Mf)^\delta]_{A_1} \leq \frac{c_n}{1-\delta}. \tag{1.9}$$

In fact they proved that any A_1 weight can be essentially written in this way.

These results and related ones (for instance for the square functions) have become important after the work of S. Petermichl and A. Volberg [28] for the Ahlfors-Beurling Transform. In this paper the authors proved a conjecture by Astala-Iwaniec-Saksman related to the border line regularity of the solutions of the Beltrami equation and which is connected to the theory of Quasirregular mappings as can be found in [2]. [28] opens up the possibility of considering some other operators such as the classical Hilbert Transform. Finally S. Petermichl [26, 27] has proved the corresponding results for the Hilbert transform and the Riesz Transforms.

To more precise, in [28] [26, 27] it has been shown that if T is either the Ahlfors-Beurling, Hilbert or Riesz Transforms and $1 < p < \infty$, then

$$\|T\|_{L^p(w)} \leq c_{p,n} [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}. \tag{1.10}$$

Furthermore the exponent $\max\{1, \frac{1}{p-1}\}$ is best possible by examples similar to the one related to Theorem 1.1.

The conjecture that we think should be true is the following:

Conjecture 1.2 (the A_2 conjecture). *Let $1 < p < \infty$ and let T be a Calderón-Zygmund singular integral operator. Then, there is a constant $c = c(n, T)$ such that for any A_p weight w ,*

$$\|T\|_{L^p(w)} \leq cp[w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}. \tag{1.11}$$

The concept of Calderón-Zygmund singular integral operator is the usual one as can be found for instance in [15]. The maximum in the exponent reflects the duality of T , namely that T^* is also a Calderón-Zygmund singular integral operator. In fact it can be shown that if T is selfadjoint (or essentially like Calderón-Zygmund operators) then if (1.11) is proved for $p \geq 2$ then the case $1 < p < 2$ is obtained by duality. What it is more interesting is that by the sharp Rubio de Francia extrapolation theorem obtained in [11] it is enough to prove (1.11) only for $p = 2$. This is the reason why is called the A_2 conjecture. Observe that in this case the growth of the constant is simply linear!!.

The proofs in the papers [28, 26, 27] are based on the Bellman function technique, and it is not clear whether they can be extended to the wider class of Calderón-Zygmund operators.

In order to study inequality (1.3), it is natural to ask first about the dependence of $L^p(w)$ operator norms of T on $[w]_{A_1}$ for $p > 1$. We discuss briefly the known results in this direction.

Denote by α the best possible exponent in the inequality

$$\|T\|_{L^p(w)} \leq c_{n,p}[w]_{A_1}^\alpha. \tag{1.12}$$

In the case when $p = 2$ and $T = H$ is the Hilbert transform, R. Fefferman and J. Pipher [12] established that $\alpha = 1$. The proof is based on sharp A_1 bounds for appropriate square functions on $L^2(w)$ from the works [4, 5], in particular, the following celebrated inequality of Chang-Wilson-Wolff was used:

$$\int_{\mathbb{R}^n} (Sf)^2 w \, dx \leq C \int_{\mathbb{R}^n} |f|^2 M(w) \, dx$$

One can show that this approach yields $\alpha = 1$ also for $p > 2$. However, for $1 < p < 2$ the same approach gives the estimate $\alpha \leq 1/2 + 1/p$. Also, that approach works only for classical singular integrals.

Recall that $A_1 \subset A_p$, and

$$[w]_{A_p} \leq [w]_{A_1}.$$

Therefore, (1.10) clearly gives that $\alpha = 1$ in (1.12) when $p \geq 2$. However, (1.10) cannot be used in order to get the sharp exponent α in the range $1 < p < 2$, becoming the exponent worst when p gets close to 1.

In [18] and [20] we use a different approach to show that for any Calderón-Zygmund operator, the sharp exponent in (1.12) is $\alpha = 1$ for all $1 < p < \infty$. Our method is more closely related to the classical Calderón-Zygmund techniques but refining some known estimates.

We hope that some of these ideas may lead to a proof of the A_2 conjecture 1.2.

We state now our main theorems. From now on T will always denote any Calderón-Zygmund operator.

Theorem 1.3. *[the linear growth theorem] Let T be a Calderón-Zygmund operator. Then*

$$\|Tf\|_{L^p(w)} \leq c p p' [w]_{A_1} \|f\|_{L^p(w)} \quad (1 < p < \infty) \tag{1.13}$$

where $c = c(n, T)$.

Theorem 1.4. *[the logarithmic growth theorem] Let T be a Calderón-Zygmund operator. Then*

$$\|Tf\|_{L^{1,\infty}(w)} \leq c [w]_{A_1} (1 + \log [w]_{A_1}) \|f\|_{L^1(w)}, \tag{1.14}$$

where $c = c(n, T)$.

The result in (1.13) is best possible. However, we don't know about the second result.

If we could improve (1.14) by removing the log term, namely if the weak Muckenhoupt-Wheeden conjecture were true then we had the following following result.

Conjecture 1.5. *Let $1 < p < \infty$ and let T be a Calderón-Zygmund singular integral operator. There is a constant $c = c(n, T)$ such that for any A_p weight w ,*

$$\|T\|_{L^{p,\infty}(w)} \leq c p [w]_{A_p}. \tag{1.15}$$

This result would be the best possible.

In section 4 we prove this conjecture assuming the weak Muckenhoupt-Wheeden conjecture is true. The same argument yields the following result

Corollary 1.6. *Let $1 < p < \infty$ and let T be a Calderón-Zygmund operator. Also let $w \in A_p$, then*

$$\|Tf\|_{L^{p,\infty}(w)} \leq c_p [w]_{A_p} (1 + \log [w]_{A_p}) \|f\|_{L^p(w)}, \tag{1.16}$$

where $c = c(n, p, T)$.

Observe that for p close to one, the behaviour of the constant is much better than the one in Pettermichl result (1.10). Our advantage is that our method works for any Calderón-Zygmund operator.

2. THE FRACTIONAL INTEGRAL CASE

It is a very natural question weather similar results mentioned in the introduction would hold for Fractional Integral Operators. In this small section we survey some result recently obtained in collaboration with M. Lacey, K. Moen and R. Torres in [22] and will be part of K. Moen PhD's thesis at the University of Kansas.

For $0 < \alpha < n$, the fractional integral operator or Riesz potential I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy,$$

while the related fractional maximal operator M_α is given by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| \, dy.$$

These operators play an important role in analysis, particularly in the study of differentiability or smoothness properties a functions. See Grafakos [14] for the basic properties of these operators.

Weighted inequalities for these operators and more general potential operators have been studied in depth. See e.g. the works of Muckenhoupt and Wheeden [23], Sawyer [29], and Pérez [9], [10]. Such estimates naturally appear in many problems of mathematics.

In [23], B. Muckenhoupt and R. Wheeden characterized the weighted strong-type inequality for fractional operators in terms of the so-called $A_{p,q}$ condition. For $1 < p < n/\alpha$ and q defined by $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, they showed that for all $f \geq 0$,

$$\left(\int_{\mathbb{R}^n} (wT_\alpha f)^q \, dx \right)^{1/q} \leq c \left(\int_{\mathbb{R}^n} (wf)^p \, dx \right)^{1/p}, \tag{2.1}$$

where $T_\alpha = I_\alpha$ or M_α , if and only if $w \in A_{p,q}$. That is,

$$[w]_{A_{p,q}} \equiv \sup_Q \left(\frac{1}{|Q|} \int_Q w^q \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-p'} \, dx \right)^{q/p'} < \infty.$$

It is not obvious a priori what the analogous of (1.10) should be for I_α . A possible guess is

$$\|w I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq C [w]_{A_{p,q}}^{\max\{1, \frac{p'}{q}\}} \|wf\|_{L^p(\mathbb{R}^n)}. \tag{2.2}$$

That was the first estimate we were able to obtain when we started our investigations. Note that if we formally put $\alpha = 0$ in (2.2), then we obtain (1.10) suggesting that (2.2) could be sharp as well. Simple examples, however, show that this is not the case. In fact, we prove in [22] the estimate

$$\|w I_\alpha f\|_{L^{q_0}(\mathbb{R}^n)} \leq c [w]_{A_{p_0, q_0}} \|wf\|_{L^{p_0}(\mathbb{R}^n)} \tag{2.3}$$

for an appropriate pair of exponents (p_0, q_0) such that $\frac{q_0}{p_0} = 1 - \frac{\alpha}{n}$. This result combined with an appropriate off diagonal extrapolation theorem immediately yields

$$\|w I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq c [w]_{A_{p,q}}^{\max(1, (1-\frac{\alpha}{n})\frac{p'}{q})} \|wf\|_{L^p(\mathbb{R}^n)}. \tag{2.4}$$

Note that this new estimate also yields (1.10) if we formally put $\alpha = 0$ obtaining a better result than (2.2). The end of this search is given by the following theorem.

Theorem 2.1. *Let $1 < p < n/\alpha$ and q be defined by the equation $1/q = 1/p - \alpha/n$, and let $w \in A_{p,q}$. Then,*

$$\|w I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq c [w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1, \frac{p'}{q}\}} \|wf\|_{L^p(\mathbb{R}^n)}, \tag{2.5}$$

where $\eta(x) = \min\{\max(1 - \alpha/n, x), \max(1, (1 - \alpha/n)x)\}$. Furthermore this estimate is sharp.

This result can be found in [22] where some examples are given proving the sharpness of the exponent range.

Our method in [22] avoids completely the good- λ method used by Muckenhoupt and Wheeden [23]. We prove first a sharp weak type estimate in the spirit of (2.5) and then we use a characterization of two-weight norm inequalities for I_α which is a deep result due to E. Sawyer [29]. We also use a procedure to discretize the operator which is interesting on its own right. We combine ideas from the work of Sawyer and Wheeden in [30], together with some techniques from [10] (see also [9]).

We do not know if our results in [22] could be derived avoiding the deep result from Sawyer [29] using other techniques. As we already mentioned, even in the Calderón-Zygmund case the known results for full range of exponents are obtained via extrapolation from one estimate. Our results illustrate again the power of extrapolation techniques, this time in an off-diagonal setting.

3. THE MAXIMAL FUNCTION

In this section we give a proof of Theorem 1.1, but first we will show the result for the weak type case that can be derived very easily from standard methods.

For the "proof" we try we will use the following result for the maximal function.

Lemma 3.1. *Let $w \in A_p$, $1 < p < \infty$. There is a constant $c = c_n$ such that for any A_p weight w ,*

$$\|M\|_{L^{p,\infty}(w)} \leq c_n [w]_{A_p}^{\frac{1}{p}}, \tag{3.1}$$

Observe the constant is better than in (1.15). Also observe that if we consider the dual estimate, the constant is essentially the same, namely:

$$\|M\|_{L^{p',\infty}(\sigma)} \leq c_n [\sigma]_{A_{p'}}^{\frac{1}{p'}} = c_{p'} [w]_{A_p}^{\frac{1}{p-1} \frac{1}{p'}} = c_{p'} [w]_{A_p}^{\frac{1}{p}}. \tag{3.2}$$

where recall $\sigma = w^{1-p'}$.

Proof of the Lemma. Since $w \in A_p$ for each cube Q and nonnegative function f

$$\left(\frac{1}{|Q|} \int_Q f(y) dy \right)^p w(Q) \leq [w]_{A_p} \int_Q f(y)^p w(y) dy$$

and hence

$$Mf(x) \approx M^c f(x) \leq [w]_{A_p}^{\frac{1}{p}} M_w^c(f^p)(x)^{\frac{1}{p}},$$

and Besicovitch:

$$\begin{aligned} \|Mf\|_{L^{p,\infty}(w)} &\leq c_n [w]_{A_p}^{\frac{1}{p}} \|M_w^c(f^p)^{\frac{1}{p}}\|_{L^{p,\infty}(w)} \leq c_n [w]_{A_p}^{\frac{1}{p}} \|M_w^c(f^p)\|_{L^{1,\infty}(w)}^{\frac{1}{p}} \\ &\leq c_n [w]_{A_p}^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} f^p w dx \right)^{1/p} \end{aligned}$$

□

We remark that there is another proof of this theorem without appealing the besicovitch lemma, just by a Vitali type covery lemma. We leave the proof to the interested reader.

We now give Lerner 's proof of Buckley's improvement Theorem 1.1.

Proof of Theorem 1.1. To prove (1.4) we set

$$A_p(Q) = \frac{w(Q)}{|Q|} \left(\frac{\sigma(3Q)}{|Q|} \right)^{p-1}$$

then, we have using that

$$A_p(Q) \leq 3^{np} [w]_{A_p}$$

and that for any $x \in Q$ then $Q \subset Q(x, 2\ell(Q)) \subset 3Q$

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f| &= A_p(Q)^{\frac{1}{p-1}} \left\{ \frac{|Q|}{w(Q)} \left(\frac{1}{\sigma(3Q)} \int_Q |f| \right)^{p-1} \right\}^{\frac{1}{p-1}} \\ &\leq 3^{np'} [w]_{A_p}^{\frac{1}{p-1}} \left\{ \frac{1}{w(Q)} \int_Q M_\sigma^c(f\sigma^{-1})^{p-1} dx \right\}^{\frac{1}{p-1}}. \end{aligned}$$

where M_σ^c is the weighted centered maximal function.

Using again that

$$Mf(x) \leq 2^n M^c f(x)$$

we get

$$Mf(x) \leq 2^n 3^{np} [w]_{A_p}^{\frac{1}{p-1}} \left\{ M_w^c(M_\sigma^c(f\sigma^{-1})^{p-1}w^{-1})(x) \right\}^{\frac{1}{p-1}}$$

We conclude using that both

$$\|M_w^c\|_{L^{p'}_w} \quad \text{and} \quad \|M_\sigma^c\|_{L^p_\sigma}$$

are finite with constants uniformly in w . This follows from the Besicovitch covering Lemma.

For the sharpness we consider $n = 1$ and $0 < \varepsilon < 1$. Let

$$u(x) = |x|^{(1-\varepsilon)(p-1)}.$$

It is easy to check that

$$[w]_{A_p}^{\frac{1}{p-1}} \approx \frac{1}{\varepsilon}$$

and hence as in Buckley 's paper

$$f(y) = y^{-1+\varepsilon(p-1)} \chi_{(0,1)}(y)$$

Observe that:

$$\|f\|_{L^p(w)}^p \approx \frac{1}{\varepsilon}$$

To estimate now $\|Mf\|_{L^p(w)}$ we pick $0 < x < 1$, hence

$$Mf(x) \geq \frac{1}{x} \int_0^x f(y) dy = c_p \frac{1}{\varepsilon} f(x)$$

and hence

$$\|Mf\|_{L^p(w)} \geq c_p \frac{1}{\varepsilon} \|f\|_{L^p(w)}$$

From which the rest follows easily. □

4. WEAK (1, 1) IMPLIES WEAK (p, p)

We need the following lemma which is a variation of the **Rubio de Francia iteration scheme**.

Lemma 4.1. *Let $1 < q < \infty$ and let $w \in A_q$. Then there exists a nonnegative sublinear operator D bounded on $L^{q'}(w)$ such that for any nonnegative $h \in L^{q'}(w)$:*

- (a) $h \leq D(h)$
- (b) $\|D(h)\|_{L^{q'}(w)} \leq 2\|h\|_{L^{q'}(w)}$
- (c) $D(h) \cdot w \in A_1$ with

$$\|D(h) \cdot w\|_{A_1} \leq cq [w]_{A_q}$$

where the constant c is a dimensional constant.

The sharpness of the constant q in (c) it is not needed for the theorem.

Proof. To define D we consider the operator

$$S_w(f) = \frac{M(fw)}{w}$$

and observe that for any $1 < q < \infty$, by Muckenhoupt's theorem

$$S_w : L^{q'}(w) \rightarrow L^{q'}(w) \quad w \in A_q$$

However we need the sharp version in both the constant and the A_q constant (1.4):

$$\|S_w\|_{L^{q'}(w)} \leq cq [w^{1-q'}]_{A_{q'}}^{q-1} = cq [w]_{A_q}$$

Define now for any nonnegative $h \in L^{q'}(w)$

$$D(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S_w^k(h)}{\|S_w\|_{L^{q'}(w)}^k}$$

Hence properties (a) and (b) are immediate and for (c) simply observe that

$$S_w(D(h)) \leq 2\|S_w\|_{L^{q'}(w)} D(h) \leq 2cq [w]_{A_q} D(h)$$

or what is the same $D(h) \cdot w \in A_1$ with

$$\|D(h) \cdot w\|_{A_1} \leq 2cq [w]_{A_q}$$

□

We know prove Conjecture 1.5 assuming that the weak Muckenhoupt-Wheeden conjecture holds.

Proof of Conjecture 1.5. Let $w \in A_p$ and let $f \in C^\infty(\mathbb{R}^n)$ with compact support. For each $t > 0$, let

$$\Omega_t = \{x \in \mathbb{R}^n : |Tf(x)| > t\}.$$

This set is bounded, so $w(\Omega_t) < \infty$. By duality, there exists a non-negative function $h \in L^{p'}(w)$ such that $\|h\|_{L^{p'}(w)} = 1$ and

$$w(\Omega_t)^{1/p} = \|\chi_{\Omega_t}\|_{L^p(w)} = \int_{\Omega_t} h w dx.$$

We consider now the operator D associated to this weight from Lemma 4.1. Hence the operator D satisfies

- (a) $h \leq D(h)$
- (b) $\|Dh\|_{L^{p'}(w)} \leq 2 \|h\|_{L^{p'}(w)} = 2$
- (c) $\|D(h) w\|_{A_1} \leq c p [w]_{A_p}$

hence assuming that the weak Muckenhoupt conjecture holds, then

$$\begin{aligned} w(\Omega_t)^{1/p} &\leq \int_{\Omega_t} D(h) w dx = (D(h) w)(\Omega_t) \\ &\leq c \|D(h) w\|_{A_1} \int_{\mathbb{R}^n} \frac{|f|}{t} D(h) w dx \\ &\leq \frac{c}{t} p [w]_{A_p} \left(\int_{\mathbb{R}^n} |f|^p w dx \right)^{1/p} \left(\int_{\mathbb{R}^n} D(h)^{p'} w dx \right)^{\frac{1}{p'}} \\ &\leq \frac{cp}{t} [w]_{A_p} \left(\int_{\mathbb{R}^n} |f|^p w dx \right)^{1/p}. \end{aligned}$$

This completes the proof. □

5. THE SHARP REVERSE HÖLDER INEQUALITY FOR A_1 WEIGHTS

In the classical situation, if $w \in A_1$, then there is a constant $r > 1$ such that

$$\left(\frac{1}{|Q|} \int_Q w^r \right)^{1/r} \leq \frac{c}{|Q|} \int_Q w$$

However there is a **bad** dependence on the constant $c = c(r, [w]_{A_1})$. To prove our results we need a more precise estimate.

Lemma 5.1. *Let $w \in A_1$, and let $r_w = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$. Then for each Q*

$$\left(\frac{1}{|Q|} \int_Q w^{r_w} \right)^{1/r_w} \leq \frac{2}{|Q|} \int_Q w$$

i.e.

$$M_{r_w} w(x) \leq 2 [w]_{A_1} w(x). \tag{5.1}$$

Recall that $M_r w = M(w^r)^{1/r}$

Proof. Let $w_Q = \frac{1}{|Q|} \int_Q w$, we have by the converse weak-type estimate for M that for $\lambda > w_Q$,

$$w |\{x \in Q : M_Q^d w(x) > \lambda\}| \leq 2^n \lambda |\{x \in Q : M_Q^d w(x) > \lambda\}|$$

where M_Q^d is the dyadic maximal operator restricted to a cube Q . Multiplying both parts of this inequality by $\lambda^{\delta-1}$ and then integrating and using Fubini's theorem, we get

$$\int_Q (M_Q^d w)^\delta w dx \leq (w_Q)^\delta \int_Q w dx + \frac{2^n \delta}{\delta + 1} \int_Q (M_Q^d w)^{\delta+1} dx$$

Setting here $\delta = \frac{1}{2^{n+1}[w]_{A_1}}$, we obtain

$$\frac{1}{|Q|} \int_Q w^{\delta+1} dx \leq \frac{1}{|Q|} \int_Q (M_Q^d w)^\delta w dx \leq 2(w_Q)^{\delta+1}$$

□

6. THE KEY (TRICKY) LEMMA

Recall the classical situation: by the Theorem of Coifman-Fefferman Let $0 < p < \infty$ and $w \in A_\infty$, there is a constant c depending of the A_∞ constant of w such that

$$\|Tf\|_{L^p(w)} \leq c \|Mf\|_{L^p(w)}$$

However, we need a more precise result for very specific weights

Lemma 6.1 (the tricky lemma). *Let w be any weight and let $1 \leq p < \infty$. Then, there is a constant $c = c(n, T)$ such that:*

$$\|Tf\|_{L^p(M_r w)^{1-p}} \leq cp \|Mf\|_{L^p(M_r w)^{1-p}}$$

This is the main improvement in [20] of [18] where we had obtained logarithmic growth on p

The classical proof by good λ Coifman-Fefferman is not sharp i.e gives:

$$C(p) \approx 2^p$$

because

$$[(M_r w)^{1-p}]_{A_p} \approx (r')^{p-1}$$

There is another proof by **Bagby-Kurtz** (using rearrangements) given in the mid 80's that is more optimal from the point of view of the L^p constant but NOT in terms of the weight constant.

The proof of this lemma is tricky, it combines another variation the of **Rubio de Francia algorithm** together with a **sharp L^1 Coifman-Fefferman estimate**:

Let $w \in A_q$, $1 \leq q < \infty$. Then, there is a dimensional constant c such that:

$$\|Tf\|_{L^1(w)} \leq c[w]_{A_q} \|Mf\|_{L^1(w)}$$

The original proof we had in [20] of last estimate was based on an idea by Fefferman-Pipher from [12] using a sharp version of the good- λ inequality of S. Buckley together with a sharp reverse Holder property of the weights. Both results

are interesting on its own. However we have found a better proof based on the following estimate:

Lemma 6.2 (sharp Bagby-Kurtz). *Let $0 < p < \infty$, $0 < \delta < 1$ and let $w \in A_q$, $1 \leq q < \infty$, then*

$$\|f\|_{L^p(w)} \leq c p[w]_{A_q} \|M_\delta^\#(f)\|_{L^p(w)}$$

for any function f such that $|\{x : |f(x)| > t\}| < \infty$.

Here,

$$M_\delta^\# f(x) = M^\#(|f|^\delta)(x)^{1/\delta}$$

and $M^\#$ is the usual sharp maximal function of Fefferman-Stein:

$$M^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

$$f_Q = \frac{1}{|Q|} \int_Q f(y) dy$$

To prove the final result for singular integrals we will use the following pointwise estimate [1]:

Lemma 6.3. *Let T be any Calderón-Zygmund singular integral operator and let $0 < \delta < 1$ then there is a constant c such that*

$$M_\delta^\#(T(f))(x) \leq c M f(x)$$

Theorem 6.4. *Let $0 < p < \infty$ and let $w \in A_q$. Then*

$$\|Tf\|_{L^p(w)} \leq c [w]_{A_q} \|Mf\|_{L^p(w)}$$

for any f such that $|\{x : |Tf(x)| > t\}| < \infty$.

Assuming this result we prove the tricky lemma

Proof of Lemma 6.1. We need a sublemma whose proof is based on using another variant of the “**Rubio de Francia iteration scheme**”.

Sublemma: Let $1 < s < \infty$ and let w be a weight. Then there exists a nonnegative sublinear operator R satisfying the following properties:

- 1) $h \leq R(h)$
- 2) $\|R(h)\|_{L^s(w)} \leq 2\|h\|_{L^s(w)}$
- 3) $R(h)w^{1/s} \in A_1$ with

$$[R(h)w^{1/s}]_{A_1} \leq cs'$$

We consider the operator

$$S(f) = \frac{M(f w^{1/s})}{w^{1/s}}$$

Since $\|M\|_{L^s} \sim s'$, we have

$$\|S(f)\|_{L^s(w)} \leq cs' \|f\|_{L^s(w)}.$$

Now, define the Rubio de Francia operator R by

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{(\|S\|_{L^s(w)})^k}.$$

It is very simple to check that R satisfies the required properties.

We are now ready to give the proof of the “tricky” Lemma, namely to prove

$$\left\| \frac{Tf}{M_r w} \right\|_{L^p(M_r w)} \leq cp \left\| \frac{Mf}{M_r w} \right\|_{L^p(M_r w)}$$

By duality we have,

$$\left\| \frac{Tf}{M_r w} \right\|_{L^p(M_r w)} = \left| \int_{\mathbb{R}^n} Tf h \, dx \right| \leq \int_{\mathbb{R}^n} |Tf| h \, dx$$

for some $\|h\|_{L^{p'}(M_r w)} = 1$. By the sublemma with $s = p'$ and $v = M_r w$ there exists an operator R such that

- 1) $h \leq R(h)$
- 2) $\|R(h)\|_{L^{p'}(M_r w)} \leq 2\|h\|_{L^{p'}(M_r w)}$
- 3) $[R(h)(M_r w)^{1/p'}]_{A_1} \leq cp$.

We want to make use of property 3) combined with the following two facts: First, if $w_1, w_2 \in A_1$, and $w = w_1 w_2^{1-p} \in A_p$, then by (1.8)

$$[w]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1}$$

Second, if $r > 1$ then $(Mf)^{\frac{1}{r}} \in A_1$ by Coifman-Rochberg theorem, furthermore we need to be more precise (1.9)

$$[(Mf)^{\frac{1}{r}}]_{A_1} \leq c_n r'.$$

Hence combining we obtain

$$\begin{aligned} [R(h)]_{A_3} &= [R(h)(M_r w)^{1/p'} ((M_r w)^{1/2p'})^{-2}]_{A_3} \\ &\leq [R(h)(M_r w)^{1/p'}]_{A_1} [(M_r w)^{1/2p'}]_{A_1}^2 \\ &\leq cp. \end{aligned}$$

□

7. THE MAIN LEMMA AND THE LINEAR GROWTH THEOREM

Lemma 7.1. *Let w be **any weight** and let $1 < p < \infty$ and $1 < r < 2$.*

Then, there is a $c = c_n$ such that:

$$\|Tf\|_{L^p(w)} \leq cp' \left(\frac{1}{r-1} \right)^{1-1/pr} \|f\|_{L^p(M_r w)}$$

These three lemmas combined give the linear growth theorem 1.3. Indeed, if we choose $w \in A_1$ with sharp Reverse Holder’s inequality $r = r_w = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$ and plug this in the inequality we prove the linear growth theorem:

$$\|T\|_{L^p(w)} \leq cp' [w]_{A_1}$$

Proof. We consider to the equivalent dual estimate:

$$\|T^* f\|_{L^{p'}(M_r w)^{1-p'}} \leq cp' \left(\frac{1}{r-1} \right)^{1-1/pr} \|f\|_{L^{p'}(w^{1-p'})}$$

Then use the **key lemma** since T^* is also a Calderón-Zygmund operator

$$\left\| \frac{T^*f}{M_r w} \right\|_{L^{p'}(M_r w)} \leq p' c \left\| \frac{Mf}{M_r w} \right\|_{L^{p'}(M_r w)}$$

Next we note that by Hölder's inequality with exponent pr ,

$$\frac{1}{|Q|} \int_Q f w^{-1/p} w^{1/p} \leq \left(\frac{1}{|Q|} \int_Q w^r \right)^{1/pr} \left(\frac{1}{|Q|} \int_Q (f w^{-1/p})^{(pr)'} \right)^{1/(pr)'}$$

and hence,

$$(Mf)^{p'} \leq (M_r w)^{p'-1} M \left((f w^{-1/p})^{(pr)'} \right)^{p'/(pr)'}$$

From this, and by the classical unweighted maximal theorem with the sharp constant,

$$\begin{aligned} \left\| \frac{Mf}{M_r w} \right\|_{L^{p'}(M_r w)} &\leq c \left(\frac{p'}{p' - (pr)'} \right)^{1/(pr)'} \left\| \frac{f}{w} \right\|_{L^{p'}(w)} \\ &= c \left(\frac{rp - 1}{r - 1} \right)^{1-1/pr} \left\| \frac{f}{w} \right\|_{L^{p'}(w)} \leq cp \left(\frac{1}{r - 1} \right)^{1-1/pr} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}. \end{aligned}$$

□

8. PROOF OF THE LOGARITHMIC GROWTH THEOREM:

Proof of Theorem 1.4. The proof is based on ideas from [25]. Applying the Calderón-Zygmund decomposition to f at level λ , we get a family of pairwise disjoint cubes $\{Q_j\}$ such that

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^n \lambda$$

Let $\Omega = \cup_j Q_j$ and $\tilde{\Omega} = \cup_j 2Q_j$. The “good part” is defined by

$$g = \sum_j f_{Q_j} \chi_{Q_j}(x) + f(x) \chi_{\Omega^c}(x)$$

and the “bad part” b as

$$b = \sum_j b_j$$

where

$$b_j(x) = (f(x) - f_{Q_j}) \chi_{Q_j}(x)$$

Then, $f = g + b$.

However, it turns out that b is “**excellent**” and g is really “**ugly**”.

It is so good the b part that we have the full Muckenhoupt-Wheeden conjecture:

$$w\{x \in (\tilde{\Omega})^c : |Tb(x)| > \lambda\} \leq \frac{c}{\lambda} \int_{\mathbb{R}^n} |f| M w dx$$

by a more or less well known argument using the cancelation of the b_j .

Also the term $w(\tilde{\Omega})$ is the level set of the maximal function and the Fefferman-Stein applies (again we have the full Muckenhoupt conjecture).

Combining we have

$$w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \leq w(\tilde{\Omega}) + w\{x \in (\tilde{\Omega})^c : |Tb(x)| > \lambda/2\} \\ + w\{x \in (\tilde{\Omega})^c : |Tg(x)| > \lambda/2\}.$$

and the first two terms are already controlled:

$$w(\tilde{\Omega}) + w\{x \in (\tilde{\Omega})^c : |Tb(x)| > \lambda/2\} \leq \frac{c}{\lambda} \int_{\mathbb{R}^n} |f| M w dx \leq \frac{c[w]_{A_1}}{\lambda} \int_{\mathbb{R}^n} |f| w dx$$

Now, by Chebyshev and the Lemma, for any $p > 1$ we have

$$w\{x \in (\tilde{\Omega})^c : |Tg(x)| > \lambda/2\} \\ \leq c(p')^p \left(\frac{1}{r-1}\right)^{p-\frac{1}{r}} \frac{1}{\lambda^p} \int_{\mathbb{R}^n} |g|^p M_r(w\chi_{(\tilde{\Omega})^c}) dx \\ \leq c(p')^p \left(\frac{1}{r-1}\right)^{p-\frac{1}{r}} \frac{1}{\lambda} \int_{\mathbb{R}^n} |g| M_r(w\chi_{(\tilde{\Omega})^c}) dx.$$

By more or less standard arguments we have

$$\int_{\mathbb{R}^n} |g| M_r(w\chi_{(\tilde{\Omega})^c}) dx \leq c \int_{\mathbb{R}^n} |f| M_r w dx.$$

Combining this estimate with the previous one, and then taking the sharp reverse Holder's exponent $r = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$, by the RHI Lemma we get

$$w\{x \in (\tilde{\Omega})^c : |Tg(x)| > \lambda/2\} \leq \frac{c(p'[w]_{A_1})^p}{\lambda} \int_{\mathbb{R}^n} |f| w dx.$$

Setting here

$$p = 1 + \frac{1}{\log(1 + [w]_{A_1})}$$

gives

$$w\{x \in (\tilde{\Omega})^c : |Tg(x)| > \lambda/2\} \leq \frac{c[w]_{A_1}(1 + \log[w]_{A_1})}{\lambda} \int_{\mathbb{R}^n} |f| w dx.$$

This estimate combined with the previous one completes the proof. □

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Carlos Pérez

Departamento de Análisis Matemático,
Facultad de Matemáticas,
Universidad de Sevilla,
41080 Sevilla, Spain
`carlosperez@us.es`

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