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MULTILINEAR SINGULAR INTEGRAL OPERATORS WITH VARIABLE COEFFICIENTS

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Abstract. Some recent results for bilinear or multilinear singular integrals operators are presented. The focus is on some of the results that can be viewed as natural counterparts of classical theorems in Calderón-Zygmund theory, adding to the already existing extensive literature in the subject. In particular, two different classes of operators that can be seen as bilinear counterparts of linear Calderón-Zygmund operators are considered. Some highlights of the recent progress done for operators with variable coefficients are a modulation invariant bilinear T1-Theorem and some new weighted norm inequalities.

1. Introduction

In this expository article we want to recount a few recent results in the study of certain multilinear operators. Multilinear harmonic analysis is an active area of research that is still developing. We will limit ourselves to results that are, in a way, natural multilinear versions of well-known and powerful theorems in the study of linear singular integrals of Calderón-Zygmund type. These new results only arise after many important progresses have been done in related topics and by numerous authors. This presentation is far from being exhaustive in the sense that, for reasons of space, we will not be able to describe all existing contributions in multilinear analysis but only those most closely related to operators with *variable coefficients*. We will concentrate on some progresses done on a series of collaborations and some topics presented by the author at the 2008 CIMPA-UNESCO School on Real Analysis and its Applications. One of the focus points of the conference was precisely new aspects of the Calderón-Zygmund theory. We will assume that the interested reader has some familiarity with basic results about linear Calderón-Zygmund operators and linear pseudodifferntial operators, but refer to the book by Stein [65] for a comprehensive introduction. Also for brevity, we will not present the theorems in their greatest generality, but with hypotheses that simplify the narrative and still encapsulate the main mathematical aspects involved.

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One of the simplest bilinear expressions naturally appearing in analysis is the product of derivatives of two functions,

$$T(f,g) = \partial^\alpha f \cdot \partial^\beta g.$$

Using the Fourier transform, $\widehat{h}(\xi) = \int h(x)e^{-ix\cdot\xi}\,dx$, and its inverse, such an operator can be written as

$$T(f,g)(x) = \int_{\mathbf{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi\,d\eta,$$

where the *multiplier* σ is an appropriate polynomial function. Similarly, other product-like operations of m-linear nature can be represented in the pseudodifferential form,

$$T(f_1,\dots,f_m)(x) = \int_{(\mathbf{R}^n)^m} \sigma(x,\xi_1,\dots,\xi_m)\,\widehat{f_1}(\xi_1)\dots\widehat{f_m}(\xi_m)\,e^{ix\cdot(\xi_1+\dots+\xi_m)}d\xi_1\dots d\xi_m, \tag{1.1}$$

where now the *symbol* σ is allowed to depend on the space variable x (to include among other objects variable coefficients differential operators). Multilinear operators appear also as technical tools in the study of linear singular integral (through the method of rotations), the analysis of nonlinear operators (through power series and similar expansions), and the resolution of many nonlinear partial differential equations.

As in the linear case, the operators in (1.1) admit an integral representation in the space domain too. Formally, inverting the Fourier transform,

$$T(f_1,\dots,f_m)(x) = \langle K(x,y_1,\dots,y_m), f_1(y_1)\otimes\dots\otimes f_m(y_m)\rangle, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual pairing of distributions and test functions and the *kernel* of the operator, $K(x,y_1,\dots,y_m)$, is a distribution which often exhibits some symmetries (cancellation, modulation invariance, etc.) and also some singularities on certain varieties. The singularities appearing in both the symbols and kernels of interesting operators in the multilinear case are more diverse and harder to handle than in the linear one. This makes the multilinear theory very challenging and difficult. It requires both new tools and new ways to implement known ones.

We will concentrate on two main types of bilinear operators that naturally arise. The two types can be seen as multilinear extensions of linear Calderón-Zygmund operators, but they are very different objects. The first one consists of operators with *m-linear Calderón-Zygmund kernels.* That is,

$$T(f_1,\dots,f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x,y_1,\dots,y_m)f_1(y_1)\dots f_m(y_m)dy_1\dots dy_m, \tag{1.3}$$

for $x \notin \cap\,\mathrm{supp} f_j$, and where away form the diagonal in $(R^n)^m$ the kernel satisfies the estimates

$$|\partial^\alpha K(y_0,y_1,\dots,y_m)| \le C_\alpha\Big(\sum_{k,l=0}^{m}|y_k-y_l|\Big)^{-mn-|\alpha|}, \quad \text{for } |\alpha| = 0,1. \tag{1.4}$$

Typical examples of these operators are obtained when classical linear Calderón-Zygmund operators in $\mathbf{R}^{mn}$ are viewed as m-linear operator in $\mathbf{R}^n$. When bounded on product of Lebesgue spaces, we call these operators *multilinear Calderón-Zygmund operators.*

The second type is a class of bilinear operators in one dimension modeled after the bilinear Hilbert transform, which we will call *variable coefficients bilinear Hilbert transforms.* Their kernels have the singular form

$$K(y_0, y_1, y_2) = k(y_0, y_0 - y_1)\delta_{y_2 = 2y_0 - y_1},$$

where k is a linear Calderón-Zygmund kernel. More explicitly, for f_1 and f_2 with disjoint supports,

$$T(f_1, f_2)(x) = \int_{\mathbf{R}} k(x,t) f_1(x-t) f_2(x+t)\, dt. \tag{1.5}$$

These operators posses a certain modulation invariance that we will define in what follows and is already present in the bilinear Hilbert transform. Recall that the bilinear Hilbert transform corresponds to the case of x-independent kernel $k(x,t) = 1/t$ and x-independent symbol $\sigma(x, \xi, \eta) = \text{sign}(\xi - \eta)$.

The study of the first type of operators was launched by Coifman and Meyer [22], [23], [24], [25]. On the other hand, the bilinear Hilbert transform was introduced by Calderón, but its boundedness properties were not known until the groundbreaking results of Lacey and Thiele, [52], [53]. The study of multilinear operators picked up momentum then and attracted numerous investigators. The works by Gilbert and Nahmod [34], [35], [36]; Muscalu, Tao and Thiele [61]; Thiele [66]; Grafakos and Li [37], [38]; and Muscalu, Pipher, Tao and Thiele [60]; to name a few, were on translation invariant* singular multipliers. We refer to the work of Thiele [67] for an exposition about some of these results and the new time-frequency techniques driving them. We want, however, to focus here on results for non-translation invariant operators; i.e. with x-dependent symbols and kernels.

The capstone of the linear Calderón-Zygmund theory is the famous T1-Theorem of David and Journé [27] (and also the related Tb-Theorem by David Journé and Semmes [28]). Such a result gives a characterization of the boundedness of linear singular integral operators. In Section 2 we will present two different forms of this result for both types of bilinear operators (1.3) and (1.5). A reexamination after Coifman and Meyer of the operators as in (1.3) took place through works by Christ and Journé [18], Kenig and Stein [49], and the developments in the author's collaboration with Grafakos [41], [42], [43], and [44]. As a consequence a T1-Theorem for multilinear Calderón-Zygmund operators is by now well-understood. A more recent bilinear T1-Theorem that applies to variable coefficient bilinear Hilbert transforms is due to Bényi, Demeter, Nahmod, Thiele, Torres, and Villarroya [4]. We will also describe some collaboration with Bényi [10] , [11], Bényi and Nahmod [8] and Bényi, Maldonado and Nahmod [6] in the study of bilinear pseudodifferential operators and paraproducts that naturally lead, from the author's perspective, from one T1-Theorem to the other.

*With some abuse, translation invariance refers here to invariance under simultaneous translations by the same amount in each variable. This is the case of operators with kernels of the form $K(x - y_1, \ldots, x - y_m)$ and symbols which are x-independent multipliers.

The linear Calderón-Zygmund theory has proved to be very efficient in the solution of diverse problems in both real and complex analysis, operator theory, approximation theory, and partial differential equations. The methods employed in the Calderón-Zygmund theory have been broadly extended to different contexts. Some of the versatility of such extensions has been achieved through the development of a weighted theory. In Section 3 we present the weighted theory for multilinear Calderón-Zygmund operators. The first attempts to a weighted theory were done in collaborations of the author with Grafakos [45] and Pérez [63]. A more recent work addressing several problems raised in those work was carried out by Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [54]. This last work contains a truly multilinear weighted theory that we will summarize together with other related recent progresses. By comparison, it is worth mentioning that, as of the writing of this article, the construction of a weighted theory for modulation invariant bilinear operators, or even just for the bilinear Hilbert transform, remains a very attractive but extremely difficult open problem.

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2. Bilinear T1-Theorems and pesudodifferential operators

As it is nowadays well-known, the linear T1-Theorem states that a linear operator with a kernel satisfying (1.4) (wiht $m = 1$) is bounded on L^2 if and only if the functions $T(1)$ and $T^*(1)$ are in BMO and the *weak boundedness property* holds. Namely,

$$|\langle T\varphi_{r,z}, \psi_{r,z}\rangle| \leq Cr^{-n} \tag{2.1}$$

where φ and ψ range over a compact subset of $\mathcal{D}(\mathbf{R}^n)$ (the space of C^∞-functions with compact support) and where we use the notation $g_{r,z}(x) = r^{-n}g(r^{-1}(x-z))$. The condition (2.1) is a weak continuity on smooth bumps and can be interpreted by saying that the family of operators obtained from T by translation and dilations (T is no longer assumed to be translation and dilation invariant) can be (weakly) controlled in a uniform way. The condition that $T(1)$ and $T^*(1)$ are elements of BMO are usually interpreted as some sort of cancellation condition, since through the use of paraproducts they can be reduced to the case $T(1) = T^*(1) = 0$. It is important to note that all the hypotheses of the theorem are symmetric in T and its transpose T^*. The necessary and sufficient conditions for boundedness were also combined in the original work [27] in the simpler form

$$\sup_{\xi\in\mathbf{R}^n} \|T(e^{i\xi\cdot})\|_{BMO} + \sup_{\xi\in\mathbf{R}^n} \|T^*(e^{i\xi\cdot})\|_{BMO} \leq C. \tag{2.2}$$

This is a beautiful condition that says that *to understand the music played by T, one only needs to understand how T plays the pure tones $e^{i\xi x}$.*

Once L^2-boundedness is established, L^p results for $1 < p < \infty$ are obtained using the Calderón-Zygmund decomposition and interpolation. All this translates to the multilinear setting through a careful adaptation of the techniques in the linear case. The results have been know for some time now but we repeat them to motivate more recent ones. We state a bilinear T1-Theorem in a form convenient for our presentation.

Theorem 2.1. *([43]) Let $T : \mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ be a continuous bilinear operator with a Calderón-Zygmund kernel satisfying (1.4). Then T maps $L^4 \times L^4$ into L^2 if and only if $T = T^{*0}$ and its transposes, T^{*1} and T^{*2}, satisfy*

$$\sum_{j=0}^{2} \sup_{\xi_1,\xi_2 \in \mathbf{R}^n} \|T^{*j}(e^{i\xi_1 \cdot}, e^{i\xi_2 \cdot})\|_{BMO} \leq C.$$

Moreover, such a T maps $L^p \times L^q$ into L^r, for all $1 < p,q < \infty$ and $1/r = 1/p + 1/q < 2$.

A different $L^\infty \times L^2 \to L^2$ version of the result was first given in [18]. Note the symmetry in the hypotheses and the *p-independence* of the result as in the linear theory (in fact the starting point $L^4 \times L^4$ into L^2 could be replaced by other strong type estimate; see [43]). The full range of exponents, also obtained for translation invariant operators in [49], extends the previous work for such operators of Coifman and Meyer mentioned in the introduction and which covered the range $r > 1$. Moreover the operators in Theorem 2.1 also satisfy a weak-type estimate

$$T : L^1 \times L^1 \to L^{1/2,\infty}. \tag{2.3}$$

See [49] and [43].

Theorem 2.1 applies to the case of classical linear Calderón-Zygmund operators of convolution type in $\mathbf{R}^{2n}$ when seen as bilinear on $\mathbf{R}^n \times \mathbf{R}^n$ (see (3.10) in Section 3). It also applies to certain bilinear paraproduct operators of the form

$$T(f,g)(x) = \sum_Q |Q|^{-1/2} \langle f, \phi_Q^1 \rangle \langle g, \phi_Q^2 \rangle \phi_Q^3(x), \tag{2.4}$$

where the sum runs over all dyadic cubes in $\mathbf{R}^n$ and the functions ϕ_Q^i, $i = 1,2,3$, are families of *molecules* or *wavelets* as in the works by Meyer [57], Frazier and Jawerth [32] and Frazier, Jawerth and Weiss [33]. Essentially these are functions of the form $\phi_Q = \phi_{\nu k}(x) = 2^{\nu n/2}\phi(2^\nu x - k)$, where $Q = Q_{\nu k}$ is a dyadic cube and ϕ is a function with certain prescribed smoothness, decay at infinity, and cancellations. In fact, by fine tuning these properties of the functions ϕ^i one can check that the operators have Calderón-Zygmund kernels and are bounded, not only on Lebesgue spaces, but also in numerous other function spaces. The operators (2.4) satisfy *almost diagonal* estimates studied in [42] in the bilinear case based on the ideas in [32] (see the article of Bényi and Tzirakis [12] for higher degree of multilinearity too). We refer to [6] for details and an extensive list or references about paraproducts. See also the works of Lacey [50], Diestel [30], Lacey and Metcalf [51], and Maldonado and Naibo [55] for connections to bilinear Littlewood-Paley theory on Lebesgue and Besov spaces.

Theorem 2.1 is also readily applicable to pseudodifferntial operators. Consider operators of the form (1.1) with symbols σ in $BS^t_{\rho,\delta}(\mathbf{R}^{3n})$, $0 \le \delta \le \rho \le 1$. That is, σ satisfies the differential inequalities

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{t+\delta|\alpha|-\rho(|\beta|+|\gamma|)}. \tag{2.5}$$

The class $BS^0_{1,0}$ is usually called *the Coifman-Meyer class*, since those authors were the first to study it. The class $BS^0_{1,1}$ is the largest one with operators with kernels satisfying the estimates (1.4) and using the boundedness of their symbols it is easy to verify that the functions $T(e^{i\xi_1\cdot}, e^{i\xi_2\cdot})$ are uniformly in BMO (and actually L^∞). The symmetry in the conditions of Theorem 2.1 requires that the same must be true for the transposes if T is to be bounded. This, however, cannot be always satisfies. A parallel situation to the linear case arises and is described by the following result.

Theorem 2.2. *([10]) The class of bilinear pseudodifferential operators $BS^0_{1,0}$ is closed by transpositions but the class $BS^0_{1,1}$ is not.*

The first statement is proved constructing a symbolic calculus for T^{*1} and T^{*2}. It follows then that Theorem 2.1 applies to $BS^0_{1,0}$. To prove the second statement in Theorem 2.2 an example of an operators in $BS^0_{1,1}$ which is not bounded is exhibited in [10]. This prevents the symbols of the transposes to be even bounded, much less in the same class. In general the operators in $BS^0_{1,1}$ are not bounded on Lebesgue spaces because T^{*1} and T^{*2} fail to satisfy the BMO conditions.

Also paralleling the linear case there is a substitute Sobolev space bound for $BS^0_{1,1}$. One of the most powerful and commonly used bilinear estimates in PDEs is the generalized Leibniz rule for fractional derivatives of the product of two functions, as in the works by Kato and Ponce [47]; Christ and Weinstein [19]; and Kenig, Ponce, and Vega [48]. The following is a variable coefficient version involving $BS^0_{1,1}$.

Theorem 2.3. *([10]) Every operator with a symbol in $BS^0_{1,1}$ maps $L^p_s \times L^q_s$ into L^r_s for $1/p+1/q = 1/r$, $1 < r < \infty$, and $s > 0$ and, moreover, it satisfies the estimate*

$$\|T(f,g)\|_{L^r_s(\mathbf{R}^n)} \le C(\|f\|_{L^p_s(\mathbf{R}^n)}\|g\|_{L^q(\mathbf{R}^n)} + \|f\|_{L^p(\mathbf{R}^n)}\|g\|_{L^q_s(\mathbf{R}^n)}). \tag{2.6}$$

For another bilinear version involving (constant coefficient) mixed derivatives and the relation to multiparameter operators see [60]. We also recall that for translation invariant operators boundedness on Lebesgue spaces automatically gives boundeness on Sobolev spaces.

Theorem 2.4. *([8]) If the symbol σ of a bilinear operator T_σ is x-independent and T_σ is bounded from $L^p \times L^q$ into L^r, with $1/p+1/q = 1/r, r > 1$, then T_σ is also bounded from $L^p_s \times L^q_s$ into $L^{r_s}_s$, where $1/r_s = 1/p+1/q-s/n$ and $0 \le s \le (n/p \wedge n/q)$ (where $a \wedge b$ denotes the largest integer strictly smaller than $\min(a,b)$).*

This theorem applies in particular to the bilinear Hilbert transform and was motivated by bilinear estimates for derivatives as used in the collaboration with Brown [15] on the inverse conductivity problem. See that reference for the motivation.

It should be pointed out that not everything about classical linear pseudodifferntial operators translates to the bilinear setting. For example, unlike the linear case, the class $BS^0_{0,0}$ is anomalous too. It was shown in [11] that not every operator in $BS^0_{0,0}$ maps $L^2 \times L^2$ into L^1. Hence, there is no direct analog of the Calderón-Vaillancourt theorem [16], [17] for smooth linear symbols with bounded derivatives. Alternative substitutes were investigated in [11] too. Other results for the Coifman-Meyer classes on other functions spaces can be found in the works by Coifman, Dobyinsky and Meyer [20], Youssfi [68], and Bényi [2], [3]. A more general symbolic calculus for the transposes of operators in this classes has been developed by Bényi, Maldonado, Naibo, and Torres [7]. For multilinear pseudodifferential operators and modulation spaces see the works of Bényi, Gröchenig, Heil, and Okoudjou [5] and Bényi and Okoudjou [9].

In light of Leibniz's rule it is natural in the research of bilinear pseudodifferential operators to study a symbolic calculus generated by composition of linear and bilinear operators. Surprisingly, such composition reveals new classes of symbols. For θ in $(-\pi/2, \pi/2]$ let $BS^t_{\rho,\delta;\,\theta}$ denote the classes of symbols satisfying estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha\beta\gamma;\theta}(1 + |\eta - \xi \tan\theta|)^{t+\delta|\alpha|-\rho(|\beta|+|\gamma|)} \tag{2.7}$$

(with the convention that $\theta = \pi/2$ corresponds to the decay in terms of $1+|\xi|$ only). For example, the composition of a classical linear symbols in the Hörmander class $S^0_{1,0}$, i.e.

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \le C_{\alpha\beta}(1 + |\xi|)^{-\alpha}, \tag{2.8}$$

with bilinear ones in $BS^0_{1,0}$ gives rise to operators in $BS^0_{1,0;\,-\pi/4}$.

Once again, symmetric considerations motivated the need to establish a good understanding of the the transposes of operators in the classes $BS^t_{\rho,\delta;\,\theta}$. This was accomplished in [8] for $T \in BS^0_{1,0;\,\theta}$ with explicit formulae for T^{*1} and T^{*2}. The result is briefly stated here as follows.

Theorem 2.5. *([8]) The collection of classes of bilinear pseudodifferential operators $\{BS^0_{1,0;\,\theta}\}_\theta$ is closed under transpositions.*

For the rest of this section only the space dimension $n = 1$ will be considered. We will state boundedness results for some operators in these new classes again in the form of another bilinear T1-Theorem. In particular the result will apply to operators in the class $BS_{1,0;\,\pi/4}$ obtained by staring with a linear symbol $a \in S^0_{1,0}$, and defining $\sigma(x,\xi,\eta) = a(x,\xi-\eta)$. In fact, at a formal level the change in linear symbols and multipliers from ξ to $|\xi| + |\eta|$, respectively $\xi - \eta$, leads to bilinear Calderón-Zygmund operators, respectively variable coefficient bilinear Hilbert transforms. We illustrate this with Table 1, which puts in evidence that both classes can be seen as generalizations of the linear Calderón-Zygmund theory.

Like the bilinear Hilbert transform, operators in $BS^0_{1,0;\,\pi/4}$ of the form

$$T(f_1,f_2) = \int_{\mathbf{R}^2} \sigma(x,\xi-\eta)\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \tag{2.9}$$

Operator	**Symbol Type**
(linear) Hilbert Transform	$\mathrm{sign}(\xi)$
(linear) Hörmander–Mihlin multipliers	$\lvert\partial^\alpha m(\xi)\rvert \le C_\alpha \lvert\xi\rvert^{-\alpha}$
(linear) Classical PDOs ($S^0_{1,0}$)	$\lvert\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert)^{-\alpha}$
Bilinear Hilbert Transform	$\mathrm{sign}(\xi-\eta)$
Bilinear Singular Multipliers	$\lvert\partial^\alpha m(\xi-\eta)\rvert \le C_\alpha \lvert\xi-\eta\rvert^{-\alpha}$
Bilinear Coifman–Meyer PDOs ($BS^0_{1,0}$)	$\lvert\partial_x^\beta \partial_{\xi,\eta}^\alpha \sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert+\lvert\eta\rvert)^{-\lvert\alpha\rvert}$
Variable Coeff. Bilinear Hilbert Transforms	$\lvert\partial_x^\beta \partial_{\xi,\eta}^\alpha \sigma(x,\xi-\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi-\eta\rvert)^{-\lvert\alpha\rvert}$

TABLE 1. From the linear Hilbert transform to the x-dependent bilinear Hilbert transforms

trivially satisfy the modulation invariance

$$\langle T(f_1,f_2),f_3\rangle = \langle T(e^{iz\cdot}f_1, e^{iz\cdot}f_2), e^{-i2z\cdot}f_3\rangle \tag{2.10}$$

for all z in $\mathbf{R}$ and all Schwartz functions f_j. A simple computation inverting the Fourier transforms of the functions in (2.9) leads to a kernel representation of the operators as in (1.5). Note that the operators in (1.5) always satisfy the modulation invariance for functions with disjoint supports, but for arbitrary functions the condition needs to be imposed. Other directions of modulation invariance could be considered too as long as one avoids the same degenerate directions for the bilinear Hilbert transform. See [4] for more details. This modulation invariance is crucial in the analysis of operators given by a kernel representation and permits to use in the study of the operators the time-frequency techniques developed in [52], [53], [35], [36], [61], but adapted to a variable coefficient situation.

We can now state a simplified version of the new modulation invariant bilinear T1-Theorem.

Theorem 2.6. *([4]) Let $T : \mathcal{S}(\mathbf{R}) \times \mathcal{S}(\mathbf{R}) \to \mathcal{S}'(\mathbf{R})$ be a continuous bilinear operator satisfying the modulation invariance (2.10). Assume further that for f_1 and f_2 with disjoint supports*

$$T(f_1,f_2)(x) = \int_{\mathbb{R}} k(x,t) f_1(x-t) f_2(x+t)\,dt,$$

where for $t \neq 0$

$$|\partial^\alpha k(x,t)| \le C|t|^{-(1+|\alpha|)} \qquad |\alpha| = 0,1.$$

Then T, T^{*1}, and T^{*2} are bounded from $L^4 \times L^4$ into L^2 if and only if they satisfy the restricted boundedness conditions

$$|\langle T(\varphi_I,\psi_I),f\rangle| + |\langle T^{*1}(\varphi_I,\psi_I),f\rangle| + |\langle T^{*1}(\varphi_I,\psi_I),f\rangle| \le C|I|^{-1/2}\|f\|_{L^2}, \tag{2.11}$$

for all L^2-normalized bump functions[†] φ_I and ψ_I supported on an interval I. Moreover, in such a case, T maps $L^p \times L^q$ into L^r for all $1 < p,q < \infty$ and $1/r = 1/p + 1/q < 3/2$.

[†] A normalized bump has the form $\varphi_I(x) = |I|^{-1/2}\varphi_0((x-x_0)/|I|)$ where φ_0 is supported in the unit interval and all its derivatives up to a certain order are bounded by a fixed constant.

We note that this version of the theorem is modeled after a formulation of the linear T1-Theorem by Stein in [65]. It it shown there, that in the linear case, the classical conditions involving BMO and weak-boundedness property can also be combined into the equivalent ones

$$|\langle T(\varphi_I), f\rangle| + |\langle T^*(\varphi_I), f\rangle| \leq C\|f\|_{L^2}$$

for all normalized bumps. A version of the modulation invariant theorem involving $T(1,1)$, etc. and a weak-boundedness property is given in [4] too.

We also note by comparison with Theorem 2.1 that the range of exponents is smaller in Theorem 2.6. In fact, $r > 2/3$ and not $1/2$. This is the same known range of exponents for the bilinear Hilbert transform from the work of Lacey and Thiele. It remains an open problem (even for the bilinear Hilbert transform) whether the range can be pushed to $1/2$. Moreover, if one replace in Theorem 2.6 the conditions on the derivative of the kernel by the usual Lipschitz type one, then the range of exponents obtained may be further reduced. See [4] for more details.

Using Theorem 2.5 it is easy to see that operators in $BS^0_{1,0;\,\pi/4}$ satisfy (2.11), so the modulation invariant T1-Theorem immediately applies to the operators of the form (2.9).

It also remains an open problem whether the modulation invariant condition can be removed. Recently Bernicot [13] showed that the condition is not needed for smooth pseudodifferential operators, but the minimal conditions on the kernel assumed in Theorem 2.6 are not enough to obtain smooth symbols or even a pseudodifferntial representation with a function as symbol. For example, the theorem applies to operators given by kernels like the *bilinear Calderón commutators*

$$p.v. \int f(x+t)g(x-t)\frac{(A(x+t)-A(x))^m}{t^{m+1}}\, dt \tag{2.12}$$

with $\|A'\|_\infty < C$, whose symbols only exist as formal distributions.

Further progress on classes of bilinear pseudodifferential operators related to the new modulation invariant bilinear T1-Theorem can be found in the recent collaboration with Bernicot [14] where the class $BS^0_{1,1;\,\pi/4}$ is studied.

Complete higher multilinear version or higher dimensional versions of the theory for modulation invariant operators, however, have not been achieved yet. Some work in this direction was done by Pramanik and Terwilleger in [64]. In two space dimensions, progress for operators analogous of the bilinear Hilbert transform were very recently produced by Demeter and Thiele [29].

3. Weighted estimates for multilinear Calderón-Zygmund operators

The Muckenhoupt [59] A_p classes of weights characterize the boundedness of the Hardy-Littlewood maximal function,

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

on weighted $L^p(w)$ spaces. Recall that w is in A_p with $1 < p < \infty$ if for some constant C and all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p-1} < C,$$

$w \in A_1$ if

$$\frac{1}{|Q|}\int_Q w\,dx \leq C\,\mathrm{essinf}_Q\, w,$$

and $A_\infty = \bigcup A_p$. The classes also characterize the weights w for which the Hlbert transform is bounded on $L^p(w)$, as shown by Hunt, Muckenhoupt and Wheeden [46]. Through the work of Coifman and Fefferman [21] it is known that the A_p weights are the right weights for the boundedness of general linear Calderón-Zygmund operators too. Once a multlinear Calderón-Zygmund **theory** was developed it was natural then to investigate weighted estimates in the new setting.

In a first approach to several multilinear problems, it appears natural to predict that the role played by the Hardy-Littlewood maximal function in the linear Calderón-Zygmund theory, would be played in the m-linear version of the theory by the product operator

$$M^{(m)}(f_1,\ldots,f_m) = \prod_{j=1}^{m} M(f_j).$$

In fact, several maximal and weighted estimates were obtained in [45] and [63] using such product operator to control the operators in the m-linear version of Theorem 2.1. We will called such operators *m-linear Calderón-Zygmund operators.*

Theorem 3.1. *([45]) Let T be an m-linear Calderón-Zygmund operator and let $1 < p_1,\ldots,p_m < \infty$, and $\frac{1}{p_1}+\cdots+\frac{1}{p_m} = \frac{1}{p}$. If $w \in A_{p_0}$, $p_0 = \min\{p_1,\cdots,p_m\}$, then*

$$T : L^{p_1}(w)\times\cdots\times L^{p_m}(w) \to L^p(w).$$

The result was first proved using a good-λ inequality and establishing the estimate

$$\|T(f_1,\ldots,f_m)\|_{L^p(w)} \leq C\prod_{j=1}^{m}\|M(f_j)\|_{L^p(w)}. \tag{3.1}$$

This was then improved using a pointwise inequality. Recall the Fefferman-Stein sharp maximal function,

$$M^{\#}f(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy,$$

and also for $\delta > 0$

$$M_\delta(f) = M(|f|^\delta)^{1/\delta}$$

and

$$M_\delta^{\#}f = M^{\#}(|f|^\delta)^{1/\delta}.$$

Theorem 3.2. *([63]) Let T be an m-linear Calderón-Zygmund operator and let $0 < \delta < 1/m$. Then*

$$M_\delta^\#(T(f_1, \ldots, f_m))(x) \leq C \prod_{j=1}^m Mf_j(x) \tag{3.2}$$

for all bounded f_j with compact support.

This pointwise estimate is the multlinear version of

$$M_\delta^\#(T(f))(x) \leq C\, M(f)(x) \tag{3.3}$$

for a linear Calderón-Zygmund operator and $0 < \delta < 1$ obtained by Alvarez and Pérez [1]. Using Thereom 3.2 and Fefferman and Stein inequality [31] it is easy to obtain the estimate

$$\int |T(f_1, \ldots, f_m)(x)|^p w(x)\, dx \leq C \int \left(\prod_{j=1}^m M(f_j)(x) \right)^p w(x)\, dx,$$

which holds for any A_∞ weight. From this it is also easy to see that Theorem 3.1 can be extended to

$$T : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(\nu), \tag{3.4}$$

where $\nu = \prod_{j=1}^m w_j^{p/p_j}$ and w_j is in A_{p_j}. These type of weights were used by Grafakos and Martell to develop a general multilinear extrapolation theory analogous to the linear one [39]. Weighted results for a class of less regular Calderón-Zygmund operators of a so-called type w were investigated by Maldonado and Naibo [56].

Nevertheless, despite all these results it was not clear whether the control of m-linear Calderón-Zygmund operators by $M^{(m)}$ was really optimal. Recently, the answer to this question has come in the article [54]. This work has put in evidence that the right maximal function to consider is the smaller one

$$\mathcal{M}(f_1, \ldots, f_m)(x) = \sup_{Q \ni x} \prod_{ij=1}^m \frac{1}{|Q|} \int_Q |f_i(y_j)| dy_j.$$

Since clearly $\mathcal{M}$ is pointwise controlled by $M^{(m)}$ the boundeness of the new maximal function on Lebesgue spaces is immediate. A more careful analysis than the one used in Theorem 3.2 gives the following.

Theorem 3.3. *([54]) Let T be an m-linear Calderón-Zygmund operator and let $0 < \delta < 1/m$. Then*

$$M_\delta^\#(T(f_1, \ldots, f_m))(x) \leq C \prod_{j=1}^m \mathcal{M}f_j(x) \tag{3.5}$$

for all bounded f_j with compact support. As a consequence, for all $0 < p < \infty$ and all $w \in A_\infty$,

$$\int |T(f_1, \ldots, f_m)(x)|^p w(x)\, dx \leq C \int \left(\prod_{j=1}^m \mathcal{M}(f_1, \ldots, f_m)(x) \right)^p w(x)\, dx. \tag{3.6}$$

The smaller multi(sub)linear maximal function $\mathcal{M}$ uncovered the existence of a class of weights for m-linear Calderón-Zygmund operators much larger than the product of the classical Muckenhoupt A_p classes used in [45] and [63]. We first start with the weak-type characterization of the *two-weight* estimate for $\mathcal{M}$.

We will use the notation $\vec{f} = (f_1, \dots, f_m)$ for functions and $\vec{w} = (w_1, \dots, w_m)$ for weights. For m exponents $p_1, \dots, p_m$, and p such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ we will denote $\vec{P} = (p_1, \dots, p_m)$ and also write $\vec{1} = (1, \dots, 1)$ and $\vec{P} \geq \vec{Q}$ if $p_i \geq q_i$, $i = 1, \dots, m$.

Theorem 3.4. *([54]) Let $\vec{P} \geq \vec{1}$ and $\vec{w} = (w_1, \dots, w_m)$ and u be weights. Then the estimate*

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(u)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds if and only if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q u\Big)^{\frac{1}{p}} \prod_{j=1}^{m} \Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\Big)^{\frac{1}{p_j'}} < \infty.$$

It is understood that

$$\Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\Big)^{\frac{1}{p_j'}} = (\inf_Q w_j)^{-1}$$

when $p_j = 1$. Note that for $m = 1$ this is the usual two-weight A_p condition.

Differentiation suggests the following one-weight condition. Given $\vec{w} = (w_1, \dots, w_m)$, let $u_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}$. For $\vec{P} \geq \vec{1}$, we say that $\vec{w}$ satisfies the $A_{\vec{P}}$ *condition* if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q u_{\vec{w}}\Big)^{\frac{1}{p}} \prod_{j=1}^{m} \Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\Big)^{\frac{1}{p_j'}} < \infty \tag{3.7}$$

(with obvious interpretations if some $p_j = 1$). This definition clearly becomes the usual A_p condition if $m = 1$

Using Hölder's inequality it is not hard to see that if each w_j is in A_{p_j} then the corresponding $\vec{w}$ is in $A_{\vec{P}}$, so $\prod_{j=1}^{m} A_{p_j} \subset A_{\vec{P}}$. This inclusion, however, is strict. Examples are given in [54]. Also using Hölder's inequality one can verify that if w_j is in A_{p_j} then $u_{\vec{w}}$ is in A_{mp}. It turns out that something more general happens.

Theorem 3.5. *([54]) The vector weight $\vec{w}$ is in $A_{\vec{P}}$ if and only if $w_j^{1-p_j'}$ is in $A_{mp_j'}$ and $u_{\vec{w}}$ is in A_{mp}.*

Note that this reduces to the usual *dual weight* condition if $m = 1$. It is also important to note that although the w_j may not be usual A_p weights, the weights $w_j^{1-p_j'}$ and $u_{\vec{w}}$ always are. This a crucial result that allows to prove the following *one-weight* strong-type characterization of the boundedness of $\mathcal{M}$.

Theorem 3.6. *([54]) Let $\vec{w} = (w_1, \ldots, w_m)$ be a vector of weights and $\vec{P} > \vec{1}$. Then the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^p(u_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{3.8}$$

holds if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition.

Since by Theorem 3.5, for $\vec{w} = (w_1, \ldots, w_m)$ in $A_{\vec{P}}$ we have that $u_{\vec{w}}$ is an A_∞ weight, Thereom 3.3 gives new weighted norm inequalitties for m-linear Calderón-Zygmund operators.

Theorem 3.7. *([54]) Let T be an m-linear Calderón-Zygmund operator and $\vec{w} \in A_{\vec{P}}$ with $\vec{P} > \vec{1}$. Then,*

$$\|T(\vec{f})\|_{L^p(u_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}. \tag{3.9}$$

An endpoint weighted estimate m-linear analogous to (2.3) holds too. See again [54] for details.

That the $A_{\vec{P}}$ classes are the right classes for m-linear Calderón-Zygmund operators is given by a partial converse to Theorem 3.7. The $A_{\vec{P}}$ condition is at least necessary for the following particular operators.

For $i = 1, \cdots, n$, the *m-linear i-th Riesz transform* is defined by

$$R_i(\vec{f})\,(x) = \text{p.v.} \int_{(\mathbf{R}^n)^m} \frac{\sum_{j=1}^{m}(x_i - (y_j)_i)}{\left(\sum_{j=1}^{m} |x - y_j|^2\right)^{\frac{nm+1}{2}}} \vec{f}(\vec{y}) d\vec{y}. \tag{3.10}$$

Theorem 3.8. *([54]) If the estimate (3.9) holds for all of the m-linear Riesz transforms $R_i(\vec{f})$, then $\vec{w} \in A_{\vec{P}}$.*

We want to mention one application of the weighted theory for m-linear Calderón-Zygmund operators to the unweighted boundedness of certain commutators. Recall that in the linear case the commutator of a linear Calderón-Zygmund operator with a function b in BMO,

$$T_b(f)(x) = [b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

This was introduced by Coifman, Rochberg and Weiss [26], who showed that this operator satisfies

$$[b, T] : L^p \longrightarrow L^p,$$

for all $1 < p < \infty$. Such a result can be proved using the weighted estimates for T[‡]. Similarly in the multlinear setting one can consider for an m-linear Calderón-Zygmund operator

$$T_{\vec{b}}^{j}(\vec{f}) = b_j T(f_1, \cdots, f_j, \cdots, f_m) - T(f_1, \cdots, b_j f_j, \cdots, f_m)$$

[‡]A general version of this result was described by C. Pérez in his course at CIMPA.

and

$$T_{\vec{b}}(f_1,\cdots,f_m)=\sum_{j=1}^{m}T_{\vec{b}}^{j}(\vec{f}) \tag{3.11}$$

We will refer to such operator as *m-linear commutator*. Using the weighted estimates from [45] it was shown in [63] that if each b_j is in BMO, then the m-linear commutator satisfies

$$T_{\vec{b}}:L^{p_1}\times\cdots\times L^{p_m}\longrightarrow L^{p} \tag{3.12}$$

for all $1<p_1,\ldots p_m<\infty$ and $1<p<\infty$ with $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. This estimate was not the best expected, since it does not include the larger range of exponents that we have seen in other related results. That is, based on the results of the linear theory for commutators and the multilinear one for Calderón-Zygmund operators, one should expect (3.12) to hold for $1/m<p<\infty$.

The gap for $1/m<p\le 1$ was finally resolved in [54] in a very general way. The idea is to prove again a pointwise estimate using an appropriate maximal operator $\mathcal{M}_{L(\log L)}$ defined by

$$\mathcal{M}_{L(logL)}(\vec{f})(x)=\sup_{x\in Q}\prod_{j=1}^{m}\|f_j\|_{L(logL),Q}{}^{\S}.$$

Theorem 3.9. *([54]) Let T be a Calderón-Zygmund operator, $0<\delta<1/m$ and $0<\epsilon$. If $\vec{b}\in BMO^m$, then there exists a constant $C>0$ such that*

$$M_{\delta}^{\#}(T_{\vec{b}}(\vec{f}))(x)\le C\|\vec{b}\|_{BMO^m}\left[\mathcal{M}_{L(logL)}(\vec{f})(x)+M_{\delta+\epsilon}(T(\vec{f}))(x)\right]$$

From this pointwise estimate one can obtain the following strong type one.

Theorem 3.10. *([54]) If $\vec{w}\in A_{\vec{P}}$ with $\vec{P}>\vec{1}$, then*

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(u_{\vec{w}})}\le C\|\vec{b}\|_{BMO^m}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(w_j)}$$

In particular,

$$T_{\vec{b}}:L^{p_1}\times\cdots\times L^{p_m}\to L^{p},$$

for all $1<p_1,\ldots,p_m<\infty$, and $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$.

In the linear case, T_b does not satisfy a weak-L^1 estimate like a Calderón-Zygmund operator. It satisfies though a weak-$L(\log L)$ estimate as proved by Pérez in [62]. An analogous estimate also holds in the multilinear setting.

Theorem 3.11. *([54]) Let $\vec{b}\in BMO^m$. Then there exits a constant $C>0$, such that, for any $t>0$,*

$$|\{x\in\mathbf{R}^n:|T_{\vec{b}}(\vec{f})(x)|>t^m\}|\le C\prod_{j=1}^{m}\left(\int_{\mathbf{R}^n}\Phi(\frac{|f_j(x)|}{t})dx\right)^{1/m},$$

where $\Phi(t)=t(1+\log^+t)$.

§The definition of this norm requires the use of Orlicz spaces; we skip the details for brevity and refer once again to [54].

In the linear ($m = 1$) case, the above estimate can be used as an endpoint together with a strong bound for some $p > 1$ to interpolate. In a very recent collaboration of the author with Grafakos, Liu and Pérez [40] the same was achieved in the bilinear case, providing an alternative proof for the boundeness of the bilinear commutator in the whole range $1/2 < p < \infty$.

We would like to conclude mentioning that the new multilinear weighted theory has generated further research on the subject. In particular, Moen has looked recently at weighted estimates for multilinear versions of fractional singular integrals and fractional maximal functions. We remit the interested reader to [58] for details and further references to related work.

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