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SCHWARTZ FUNCTIONS ON THE HEISENBERG GROUP, SPECTRAL MULTIPLIERS AND GELFAND PAIRS

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ABSTRACT. We review recent results proved jointly with B. Di Blasio and F. Astengo. On the Heisenberg group H_n, consider the two commuting self-adjoint operators L and $i^{-1}T$, where L is the sublaplacian and T is the central derivative. Their joint L^2-spectrum is the so-called Heisenberg fan, contained in $\mathbb{R}^2$. To any bounded Borel function m on the fan, we associate the operator $m(L, i^{-1}T)$. The main result that we describe says that the convolution kernel of $m(L, i^{-1}T)$ is a Schwartz function if and only if m is the restriction of a Schwartz function on $\mathbb{R}^2$. We point out that this result can be interpreted in terms of the spherical transform for the convolution algebra of $U(n)$-invariant functions on H_n. We also describe extensions to more general situations.

1. Introduction

The problem addressed in these notes can be looked upon from two points of view, one concerning spectral multipliers for families of commuting differential operators, the other concerning the spherical transform in certain Gelfand pairs. Both problems can be posed in greater generality, but my presentation is restricted to the context of the Heisenberg group.

The results we present have been obtained in collaboration with Francesca Astengo and Bianca Di Blasio, and they appear in [1, 2].

It is a basic fact in Fourier analysis on $\mathbb{R}^n$ that the translation invariant operator $T_m f = \mathcal{F}^{-1}(m\hat{f})$ defined by the Fourier multipliers m is the operator $m(i^{-1}\nabla)$, in the sense of the joint spectral theory of the n partial derivatives.

The fundamental property of the Fourier transform $\mathcal{F}$ in $\mathbb{R}^n$, of being an isomorphism of the Schwartz space onto itself, can then be read in the following terms: *the convolution kernel of the multiplier operator $m(i^{-1}\nabla)$ is a Schwartz function if and only if m itself is a Schwartz function.*

A similar statement is true for spectral multipliers $m(\Delta)$ of the Laplacian Δ, where m is now defined only on $[0, +\infty)$ (we take Δ as a positive operator, $\Delta = -\sum_{j=1}^n \partial_{x_j}^2$, so that its L^2-spectrum is the positive half-line): *the convolution kernel of the operator $m(\Delta)$ is a Schwartz function if and only if m is the restriction of a Schwartz function on the line.*

In greater generality, given a finite number k of self-adjoint, constant-coefficient differential operators on $\mathbb{R}^n$, $D_1, \dots, D_k$, we associate to them the *joint spectrum*

$\Sigma \subset \mathbb{R}^k$. Assuming that one of the D_j (or some polynomial in the D_j's) is hypoelliptic, it is a simple fact to prove that if m is the restriction to Σ of a Schwartz function on $\mathbb{R}^k$, then $m(D_1, \ldots, D_k)$ has a Schwartz convolution kernel. As we will see in Section 5, the converse is also true if the D_j are invariant under the action of a compact group of linear transformations of $\mathbb{R}^n$.

The first formulation of our result is that an analogous statement holds for families of commuting differential operators on the Heisenberg group H_n, all being invariant under left translations and under the action of an appropriate compact group of automorphisms.

The other point of view is the following. There exist various compact groups K of automorphisms of H_n with the property that convolution on H_n is commutative when restricted to K-invariant functions. If this is the case, the convolution Banach algebra $L^1_K(H_n)$ of K-invariant integrable functions on H_n is commutative, and Gelfand theory gives a notion of Gelfand spectrum, Σ_K, and a continuous linear embedding, the *Gelfand transform*, or *spherical transform*,

$$\mathcal{G} : L^1_K(H_n) \longrightarrow C_0(\Sigma_K) ,$$

satisfying the identity $\mathcal{G}(f * g) = (\mathcal{G}f)(\mathcal{G}g)$.

As proved in [13] (cf. also [7]), the spectrum Σ_K admits a natural realization (in fact more than one) as a closed subset of some $\mathbb{R}^m$. Call $\Sigma'_{\mathcal{D}}$ one of these realizations (the notation is explained in Section 2). Interpreting $\mathcal{G}$ as an analogue of the Fourier transform, it is natural to pose the following problem:

Problem. *How is the image under $\mathcal{G}$ of the K-invariant Schwartz space $\mathcal{S}_K(H_n)$ related to the Schwartz space of the ambient space $\mathbb{R}^m$?*

Our result says that $f \in L^1_K(H_n)$ *is Schwartz if and only if $\mathcal{G}f$ is the restriction to $\Sigma'_{\mathcal{D}}$ of a Schwartz function on $\mathbb{R}^m$*. More precisely, $\mathcal{G}$ is a topological isomorphism of $\mathcal{S}_K(H_n)$ onto

$$\mathcal{S}(\Sigma'_{\mathcal{D}}) \overset{\text{def}}{=} \mathcal{S}(\mathbb{R}^m)/\{f : f_{|_{\Sigma'_{\mathcal{D}}}} = 0\} .$$

In this note we describe the main ideas behind this result and explain the relation between the two formulations of the problem.

Historically, the problem has its origin in D. Geller's work on the characterization of the image of the entire Schwartz space $\mathcal{S}(H_n)$ under the group Fourier transform [14]. His results were later used by L. De Michele and G. Mauceri to obtain Mihlin-Hörmander-type conditions on Fourier multipliers implying L^p-boundedness for $1 < p < \infty$ [12, 19].

More recently, a Marcinkiewicz-type theorem was proved by D. Müller, E. Stein and the author [20, 21] for spectral multipliers of the sublaplacian L and the central derivative. The problem posed above resents very much of the point of view taken this earlier work.

Early work on commutative convolution algebras on nilpotent groups associated to actions of compact groups is due to A. Hulanicki and the author [17] and to G. Carcano [11]. A systematic analysis was late developed by C. Benson and G. Ratcliff, partially with J. Jenkins and other co-authors [3, 4, 5, 6, 7, 9, 8, 10]. Among their results, we mention the classification of all subgroups K for which $L^1_K(H_n)$ is commutative [9], and a characterization of spherical tranforms of K-invariant Schwartz functions in terms of estimates on difference-differential operators in the parameters [6]. Hence the conditions given there turn out to be equivalent to the extension properties of [1, 2].

2. The Heisenberg group and Gelfand pairs

We adopt the usual description of the Heisenberg group H_n as $\mathbb{C}^n \times \mathbb{R}$ with product

$$(z,t)(z',t') = \Big(z+z', t+t' - \frac{1}{2}\Im\mathrm{m} \sum_{j=1}^n z_j \overline{z'_j}\Big) .$$

For $j = 1, \ldots, n$, we denote by Z_j, resp. $\overline{Z_j}$, the complex left-invariant vector field which coincides with ∂_{z_j}, resp. $\partial_{\overline{z_j}}$, at the origin. Together with the central derivative $T = \partial_t$, these vector fields span the complexified Lie algebra of H_n.

If K is a compact group of automorphisms of H_n, the space $L^1_K(H_n)$ of K-invariant integrable functions is closed under convolution, and hence it is a Banach algebra. The best known case is that of the unitary group $K = U(n)$, acting on H_n by

$$k \cdot (z,t) = (kz, t) , \qquad \big(k \in U(n) , (z,t) \in H_n\big) .$$

In fact $U(n)$ is maximal among connected compact groups of automorphisms, and every connected K is conjugate to a subgroup of $U(n)$ within the full group of automorphisms.

If $L^1_K(H_n)$ is a commutative Banach algebra, one says that $(K \ltimes H_n, K)$ is a *Gelfand pair*. This is the case for $K = U(n)$. A good reference for the general theory fo Gelfand pairs is J. Wolf's book [27].

From now on we assume that K is a compact subgroup of $U(n)$ and that $L^1_K(H_n)$ is commutative.

It follows from a theorem proved independently by S. Helgason and E. Thomas [15, 25] that $L^1_K(H_n)$ is commutative if and only if the left-invariant, K-invariant differential operators on H_n commute among themselves. We denote by $\mathbb{D}_K(H_n)$ the algebra of such operators.

The Gelfand spectrum Σ_K of $L^1_K(H_n)$ is the space of *bounded K-spherical functions* φ on H_n, characterized by the following equivalent conditions:

(i) they are the bounded functions satisfying the equation

$$\int_K \varphi\big((z,t)(kz',t')\big)\, dk = \varphi(z,t)\varphi(z',t') ,$$

for all $(z,t), (z',t') \in H_n$;

(ii) they are the bounded eigenfunctions of all operators in $\mathbb{D}_K(H_n)$.

It follows from the Hilbert basis theorem that $\mathbb{D}_K(H_n)$ is finitely generated. We fix a finite generating subset $\mathcal{D} = \{D_1, \dots, D_m\}$ of $\mathbb{D}_K(H_n)$ with all the D_j's self-adjoint. To each bounded spherical function φ we associate the m-tuple $\xi = \xi(\varphi) = (\xi_1, \dots, \xi_m)$, where

$$D_j \varphi = \xi_j \varphi$$

for $j = 1, \dots, m$. By a result of C. Benson, J. Jenkins and G. Ratcliff [4], all bounded K-spherical functions are of positive type, and therefore each ξ_j is real.

The following topologies coincide on Σ_K:

(i) the Gelfand topology, induced on Σ_K from the weak* topology of $L^\infty_K(H_n)$;
(ii) the topology of uniform convergence on compact sets;
(iii) the topology induced from $\mathbb{R}^m$ on

$$\Sigma'_{\mathcal{D}} = \big\{\xi(\varphi) : \varphi \in \Sigma_K\big\} . \tag{2.1}$$

The equivalence of (i) and (ii) is classical, cf. [15], and the equivalence of (iii) with the other two conditions has been proved by F. Ferrari Ruffino [13]. Notice that the immersion (2.1) in Euclidean space depends on the choice of the generating set of differential operators.

The *spherical transform* $\mathcal{G}_K f$ of $f \in L^1_K(H_n)$ is the function in $C_0(\Sigma_K)$ defined as

$$\mathcal{G}_K f(\varphi) = \int_{H_n} f(z,t)\varphi(-z,-t)\, dz\, dt .$$

In the same way as for the Fourier transform on locally compact abelian groups, $\mathcal{G}_K$ is linear, injective, norm-decreasing and tranforms convolution into pointwise product. It will be implicit in the sequel that, for any given set $\mathcal{D}$, we regard $\mathcal{G}_K f$ as defined on $\Sigma'_{\mathcal{D}}$.

We can state now our main theorem precisely.

Main Theorem. *Let K be a compact group of automorphisms of H_n such that $L^1_K(H_n)$ is commutative. Let $\mathcal{D} = \{D_1, \dots, D_m\}$ be a set of self-adjoint generators of the algebra $\mathbb{D}_K(H_n)$, and $\Sigma'_{\mathcal{D}}$ the associated immersion of Σ_K in $\mathbb{R}^m$. Then $\mathcal{G}_K$ establishes a topological isomorphism between $\mathcal{S}_K(H_n)$ and $\mathcal{S}(\Sigma'_{\mathcal{D}})$.*

It is worth noticing that Σ_K only has, by definition, the structure of a topological space, and only continuous functions are intrinsically defined on Σ_K. Whichever notion of "smooth function" on it necessarily depends on some given homeomorphic immersion into a Euclidean space. Requiring the identification of the "Schwartz space" on Σ_K with $\mathcal{G}_K\big(\mathcal{S}_K(H_n)\big)$ imposes a considerable rigidity in the choice of the immersion. The theorem states that the immersion as $\Sigma'_{\mathcal{D}}$ is good enough (for every $\mathcal{D}$) to yield this identification.

It is convenient to discuss this point more concretely in the case $K = U(n)$.

The usual realization of the Gelfand spectrum for $K = U(n)$ is the so-called *Heisenberg fan*, see Figure 1 below (the name is due to R. Strichartz, cf. [23]). It is the union of infinitely many half-lines exiting from the origin in $\mathbb{R}^2$ for nonnegative abscissas, with slopes equal to 0 and to $\pm(2k+n)^{-1}$, $k \in \mathbb{N}$. It corresponds to the realization (iii) relative to the differential operators

$$D_1 = L = -2\sum_{j=1}^{n}(Z_j\overline{Z_j} + \overline{Z_j}Z_j) \ ,$$

(the $U(n)$-invariant *sublaplacian*) and $D_2 = i^{-1}T$.

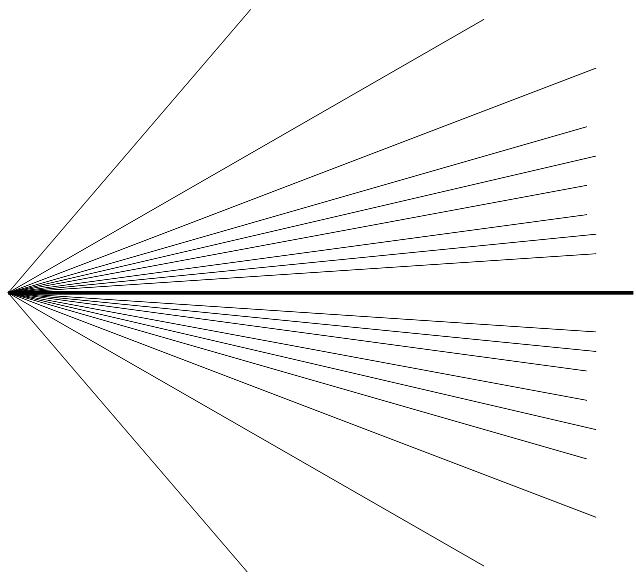

FIGURE 1. $\Sigma_{U(n)}$ as the Heisenberg fan

The spherical functions have different expressions, depending on whether they correspond to points $(\xi, 0)$, with $\xi \geq 0$, on the horizontal half-line, or to points $\big(|\lambda|(2k+n), \lambda\big)$ on the other half-lines (the letters ξ, λ will refer to horizontal and vertical components respectively). We denote by $\varphi_{\xi,0}$, resp. $\varphi_{k,\lambda}$, the two types of spherical functions. Then

$$\varphi_{\xi,0}(z,t) = \frac{(n-1)!}{\sqrt{\xi}|z|/2} J_{n-1}\big(\sqrt{\xi}|z|\big) \ ,$$

where J_{n-1} denotes the Bessel function of order $n-1$ (in particular, $\varphi_{0,0}(z,t) = 1$), and

$$\varphi_{k,\lambda}(z,t) = \binom{k+n-1}{n-1}^{-1} e^{i\lambda t} \ell_k^{n-1}\Big(\frac{|\lambda||z|^2}{2}\Big) \ ,$$

where ℓ_k^{n-1} is the Laguerre function of order $n-1$ and degree k, normalized by the condition $\ell_k^{n-1}(0) = \binom{k+n-1}{n-1}$, cf. [24].

We show that the choice of the angular coefficients for the non-horizontal half-lines is very much restricted by the only requirement that the spherical transform of any Schwartz function extends to a function on $\mathbb{R}^2$ which is differentiable at 0.

Take $f \in \mathcal{S}_K(H_n)$. Writing shortly $\mathcal{G}$ for $\mathcal{G}_{U(n)}$, for fixed $k \in \mathbb{N}$, the function

$$\begin{aligned}\mathcal{G}f\big(|\lambda|(2k+n),\lambda\big) &= \int_{H_n} f(z,t)\varphi_{k,\lambda}(-z,-t)\\ &= \binom{k+n-1}{n-1}^{-1} \int_{\mathbb{C}^n} \mathcal{F}_t f(z,\lambda)\ell_k^{n-1}\Big(\frac{|\lambda||z|^2}{2}\Big)\,dz\ ,\end{aligned}$$

is smooth in λ for $\lambda \neq 0$ and

$$\begin{aligned}\partial_\lambda \mathcal{G}f\big(|\lambda|(2k+n),\lambda\big) = \binom{k+n-1}{n-1}^{-1} \bigg(&\int_{\mathbb{C}^n} \partial_\lambda \mathcal{F}_t f(z,\lambda)\ell_k^{n-1}\Big(\frac{|\lambda||z|^2}{2}\Big)\,dz\\ &+ \frac{1}{2}\operatorname{sgn}\lambda \int_{\mathbb{C}^n} \mathcal{F}_t f(z,\lambda)|z|^2(\ell_k^{n-1})'\Big(\frac{|\lambda||z|^2}{2}\Big)\,dz\bigg)\ .\end{aligned}$$

From this and the identity $(\ell_k^{n-1})'(0) = -\frac{2k+n}{2n}$,cf. [20], we obtain that

$$\lim_{\lambda\to 0^\pm} \partial_\lambda \mathcal{G}f\big(|\lambda|(2k+n),\lambda\big) = \int_{\mathbb{C}^n} \partial_\lambda \mathcal{F}_t f(z,0)\,dz \mp \frac{2k+n}{4n}\int_{\mathbb{C}^n} \mathcal{F}_t f(z,0)|z|^2\,dz\ . \tag{2.2}$$

Let g be an extension of $\mathcal{G}f$ which is differentiable at 0. Then these limits must be directional derivatives of g at the origin. Precisely,

$$\lim_{\lambda\to 0^\pm} \partial_\lambda \mathcal{G}f\big(|\lambda|(2k+n),\lambda\big) = v_k^\pm \cdot \nabla g(0)\ ,$$

for $v_k^\pm = \big(\pm(2k+n),1\big)$. This is compatible with (2.2), and implies that

$$\nabla g(0) = \Big(-\frac{1}{4n}\int_{\mathbb{C}^n} \mathcal{F}_t f(z,0)|z|^2\,dz\ ,\ \int_{\mathbb{C}^n} \partial_\lambda \mathcal{F}_t f(z,0)\,dz\Big)\ .$$

It is also clear that the choice $\pm(2k+n)^{-1}$ for the angular coefficients is essentially imposed by this compatibility condition.

3. Spectral multipliers

The set $\Sigma'_\mathcal{D}$ plays another important role: it is the joint L^2-spectrum of the operators $D_j \in \mathcal{D}$. We remark that, for general Gelfand pairs (G,K), and any set $\mathcal{D}$ of self-adjoint generators of the algebra of G-invariant differential operators on G/K, one only has a proper inclusion of the joint L^2-spectrum of $\mathcal{D}$ into the Gelfand spectrum $\Sigma'_\mathcal{D}$; see, e.g., the case of symmetric pairs [27]. This special feature of the Gelfand pairs that we are considering relies on the fact that all bounded K-spherical functions are of positive type [6].

In general one says that two self-adjoint (unbounded) operators on a Hilbert space $\mathcal{H}$ commute if their spectral projections do so (in the ordinary sense for bounded operators). If $\mathcal{D} = \{D_1, \dots, D_m\}$ is a set of commuting self-adjoint operators, and E_j denotes the spectral measure for D_j defined on the real line, one defines the resolution of the identity E on $\mathbb{R}^m$ by setting

$$E(\omega_1 \times \omega_2 \times \cdots \times \omega_m) = E_1(\omega_1)E_2(\omega_2)\cdots E_m(\omega_m) ,$$

where the ω_j are Borel subsets of the line. The joint spectrum $S_{\mathcal{D}}$ of $D_1, \dots, D_m$ is, by definition, the support of the measure E.

Given a bounded Borel function m on $S_{\mathcal{D}}$, the *spectral multiplier*, one defines the operator

$$m(D) = m(D_1, \dots, D_m) = \int_{S_{\mathcal{D}}} m(\xi)\, dE(\xi) ,$$

which is bounded on $\mathcal{H}$.

Going back to our context, since the D_j commute with left translations and with the action of K, so does $m(D)$. Being bounded on L^2, $m(D)$ is given by convolution with a K-invariant distribution Φ,

$$m(D)f = f * \Phi . \tag{3.1}$$

Proposition 3.1. *The joint L^2-spectrum $S_{\mathcal{D}}$ coincides with $\Sigma'_{\mathcal{D}}$ and Φ in (3.1) satisfies $\mathcal{G}_K\Phi = m$.*

In view of this, the following proposition provides one of the two implications in the Main Theorem.

Proposition 3.2. *If m is the restriction to $\Sigma'_{\mathcal{D}}$ of a Schwartz function on $\mathbb{R}^m$, then $\Phi \in \mathcal{S}_K(H_n)$ and $\mathcal{G}_K\Phi = m$.*

This statement is based on a theorem of A. Hulanicki [16], concerning spectral multipliers of a single positive Rockland operator. An extension to multipliers of L and $i^{-1}T$ was provided in [26], and it has been extended to the general case in [1].

4. Taylor expansions on the spectrum

In order to prove the converse implication in the Main Theorem, it is important to discuss some preliminary regularity properties of spherical transforms of Schwartz functions. We first present these properties in the case $K = U(n)$, where they describe the behaviour of $\mathcal{G}_K f$ near the horizontal half-line. Writing, as before $\mathcal{G}$ for $\mathcal{G}_{U(n)}$, at a point $(\xi, 0)$ with $\xi \geq 0$ we have

$$\begin{aligned}\mathcal{G}f(\xi,0) &= \int_{H_n} f(z,t)\frac{(n-1)!}{\sqrt{\xi}|z|/2}J_{n-1}\big(\sqrt{\xi}|z|\big)\, dz\, dt \\ &= \int_{H_n} f(z,t)e^{-i\Re\mathrm{e}\,\langle w,z\rangle}\, dz\, dt = \hat{f}(w,0) ,\end{aligned}$$

for any $w \in \mathbb{C}^n$ with $|w|^2 = \xi$, with $\hat{f}$ denoting the ordinary Fourier transform on $\mathbb{C}^n \times \mathbb{R}$.

Since f is radial in the z-variable, so is $\hat{f}$ as a function of w. It follows from Whitney's theorem that $\hat{f}(w,0)$ is a smooth function of $|w|^2$. More precisely, there exists $g \in \mathcal{S}(\mathbb{R})$ such that

$$\hat{f}(w,0) = g\big(|w|^2\big) \ .$$

It is clear that for $\xi \geq 0$, $g(\xi) = \mathcal{G}f(\xi,0)$. So, to begin with, $\mathcal{G}f$ admits a Schwartz extension to the horizontal coordinate axis. It then remains to extend $\mathcal{G}f$ in the vertical direction.

The other aspect to take into account is that, due to the presence of the infinitely many half-lines clustering toward the horizontal one, the Taylor development in λ of any extension of $\mathcal{G}f$ is uniquely determined, if it exists, at each point $(\xi,0)$, with $\xi \geq 0$.

Such a Taylor development is provided by a result of D. Geller [14], which gives the following formula.

Lemma 4.1. *Given $f \in \mathcal{S}_{U(n)}(H_n)$, there exist $U(n)$-invariant Schwartz functions f_j, $j \in \mathbb{N}$, with $f_0 = f$, such that, for every $N \in \mathbb{N}$ and $(\xi,\lambda) \in \Sigma'_{\mathcal{D}}$,*

$$\mathcal{G}f(\xi,\lambda) = \sum_{j=0}^{N} \frac{\lambda^j}{j!}\mathcal{G}f_j(\xi,0) + \frac{\lambda^{N+1}}{(N+1)!}\mathcal{G}f_{N+1}(\xi,\lambda) \ .$$

Now that we have all these f_j, we first use the previous argument to extend each restricted spherical transform $\mathcal{G}f_j(\xi,0)$ to a Schwartz function $g_j(\xi) \in \mathcal{S}(\mathbb{R})$. The family $\{g_j\}$ so obtained forms a *Whitney jet* on the horizontal line. According to Whitney's extension theorem, cf. [18], there exists a smooth function G on $\mathbb{R}^2$ such that

$$\partial_\lambda^j G(\xi,0) = g_j(\xi)$$

for every $\xi \in \mathbb{R}$. The construction shows that such a G can be taken in $\mathcal{S}(\mathbb{R}^2)$.

However, the construction does not guarantee that $G = \mathcal{G}f$ at the points of $\Sigma'_{\mathcal{D}}$ with $\lambda \neq 0$. But, since the error is rapidly decreasing to zero as $\lambda \to 0$ and $\mathcal{G}f$ is smooth on each of the non-horizontal half-lines in the fan, it is not hard to correct the error by means of another Schwartz function vanishing of infinite order on the ξ-axis.

This concludes our sketch of the proof of the Main Theorem for $K = U(n)$. The general case requires some preliminary analysis on the pair $(K \ltimes \mathbb{C}^n, K)$.

5. K-invariant Schwartz functions on Euclidean spaces

More generally, we consider the convolution algebra $L^1_K(\mathbb{R}^d)$ of functions invariant under the action of a compact group $K \subset SO(d)$. This means dealing with the

Gelfand pair $(K \ltimes \mathbb{R}^d, K)$. The K-spherical functions are the *generalized Bessel functions*

$$\varphi_\eta(x) = \int_K e^{i\eta\cdot kx}\, dk\ , \qquad (\eta \in \mathbb{R}^d)\ .$$

We denote by Δ_K the Gelfand spectrum of $L^1_K(\mathbb{R}^d)$. Since $\varphi_\eta = \varphi_{k\eta}$ for every $k \in K$, the K-spherical functions are in one-to-one correspondence with the orbits of K in $\mathbb{R}^d$. Therefore Δ_K coincides, as a set, with the set $\mathbb{R}^d/K$ of K-orbits in $\mathbb{R}^d$. Furthermore, the Gelfand topology coincides with the quotient topology on $\mathbb{R}^d/K$ inherited from the Euclidean topology on $\mathbb{R}^d$.

There is a standard way of embedding homeomorphically $\mathbb{R}^d/K$ in a Euclidean space. Take a finite set $\mathcal{P} = \{P_1, \dots, P_q\}$ of real K-invariant polynomials on $\mathbb{R}^d$ generating the full algebra of K-invariant polynomials, and set

$$\rho_{\mathcal{P}}(\eta) = \big(P_1(\eta), \dots, P_q(\eta)\big)\ ,$$

called the *Hilbert map* associated to $\mathcal{P}$. The image $\Delta'_{\mathcal{P}}$ of $\rho_{\mathcal{P}}$ in $\mathbb{R}^q$ is homeomorphic to $\mathbb{R}^d/K$ and hence to Δ_K.

In analogy with what we have discussed for compact group actions on the Heisenberg group, $\Delta'_{\mathcal{P}}$ coincides with the joint L^2-spectrum of the self-adjoint differential operators

$$D_j = P_j(i^{-1}\nabla_x)\ .$$

The following statement, proved in [2], is an adaptation of a result of G. Schwarz [22].

Lemma 5.1. *Let K be a compact subgroup of $SO(d)$. Let $\mathcal{P} = \{P_1, \dots, P_q\}$ be a set of real generators of the algebra of K-invariant polynomials on $\mathbb{R}^d$, and let $\Delta'_{\mathcal{P}}$ be the image of the associated Hilbert map $\rho_{\mathcal{P}}$ in $\mathbb{R}^q$. Then the map $f \mapsto f \circ \rho_{\mathcal{P}}$ establishes a topological isomorphism between $\mathcal{S}(\Delta'_{\mathcal{P}})$ and $\mathcal{S}_K(\mathbb{R}^d)$.*

This lemma leads (in fact, it is equivalent) to the following analogue of our Main Theorem for the pairs $(K \ltimes \mathbb{R}^d, K)$. To see this, notice that the spherical transform $\mathcal{G}_{K,\mathbb{R}^d} f$ of $f \in L^1_K(\mathbb{R}^d)$ is related to its Fourier transform by

$$\hat{f} = \mathcal{G}_{K,\mathbb{R}^d} f \circ \rho_{\mathcal{P}}\ .$$

Proposition 5.2. *$\mathcal{G}_{K,\mathbb{R}^d}$ establishes a topological isomorphism between $\mathcal{S}_K(\mathbb{R}^d)$ and $\mathcal{S}(\Delta'_{\mathcal{P}})$.*

6. The Gelfand spectrum of $L^1_K(H_n)$ for general subgroups of $U(n)$

Going back to the pair $(K \ltimes H_n, K)$, we can get now a better understanding of its Gelfand spectrum. Look first at K acting on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, and let Δ_K be the Gelfand spectrum of $(K \ltimes \mathbb{R}^{2n}, K)$.

Take $\mathcal{P} = \{P_1, \dots, P_q\}$ a real generating set for the algebra of K-invariant polynomials on $\mathbb{R}^{2n}$. From each P_j we obtain by symmetrization a self-adjoint, left-invariant, K-invariant, differential operator D_j on H_n. If we add to $D_1,\dots,D_q$ the operator $D_{q+1}=i^{-1}T$, we obtain a full system $\mathcal{D}$ of self-adjoint generators of the algebra $\mathbb{D}_K(H_n)$.

We then have the corresponding immersions $\Sigma'_{\mathcal{D}}$ of Σ_K in $\mathbb{R}^{q+1}$, resp. $\Delta'_{\mathcal{P}}$ of Δ_K in $\mathbb{R}^q$. They are related as follows:

$$\Sigma'_{\mathcal{D}}\cap\{\xi:\xi_{q+1}=0\}=\Delta'_{\mathcal{P}}\times\{0\}\ .$$

Moreover, for $f\in L^1_K(H_n)$, one has the following identity:

$$\mathcal{G}_K f(\xi_1,\dots,\xi_q,0)=\mathcal{G}_{K,\mathbb{R}^{2n}}f^\flat(\xi_1,\dots,\xi_q)\ ,$$

with $f^\flat(z)=\int_{\mathbb{R}}f(z,t)\,dt$. Notice that Proposition 5.2 implies that, if $f\in\mathcal{S}_K(H_n)$, $\mathcal{G}_{K,\mathbb{R}^{2n}}f^\flat$ admits a Schwartz extension to $\mathbb{R}^q$.

The remaining part of $\Sigma'_{\mathcal{D}}$ consists of infinitely many unbounded curves clustering towards all of $\Delta'_{\mathcal{P}}\times\{0\}$. In the Heisenberg fan of Figure 1, the horizontal half-line represents $\Delta'_{\mathcal{P}}$ for $\mathcal{P}=\{P_1(z)=|z|^2\}$. Figure 2 below gives an idea of Σ_K for $K=\mathbb{T}\times SO(n)$. The immersion in $\mathbb{R}^3$ corresponds to the choice of the polynomials

$$P_1(z)=\left|z_1^2+\cdots+z_n^2\right|^2\ ,\qquad P_2(z)=|z|^2\ .$$

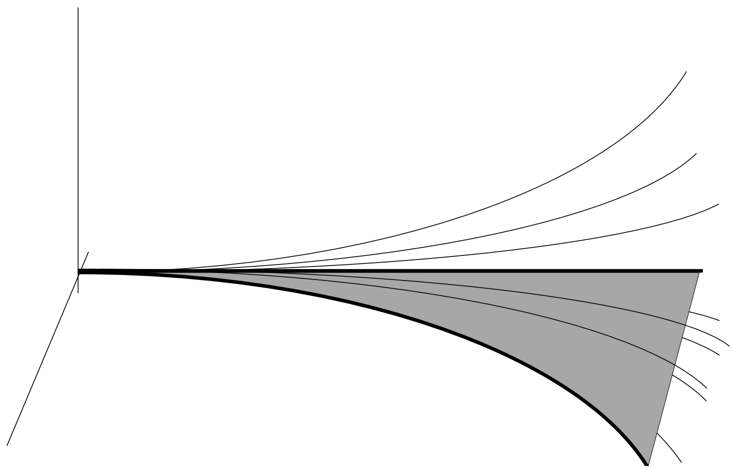

FIGURE 2. Σ_K for $K=\mathbb{T}\times SO(n)$

The completion of the general argument requires just one variant with respect to the case $K=U(n)$. In order to extend Lemma 4.1 as stated, we would need to know that, whenever $(\xi_1, \ldots, \xi_q, \xi_{q+1}) \in \Sigma'_{\mathcal{D}}$, then also $(\xi_1, \ldots, \xi_q) \in \Delta'_{\mathcal{P}}$. We do not know if this is true. However it is possible to modify the statement as follows.

Lemma 6.1. *Given $f \in \mathcal{S}_K(H_n)$, there exist K-invariant Schwartz functions f_j, $j \in \mathbb{N}$, with $f_0 = f$, and Schwartz extensions g_j of $\mathcal{G}_{K,\mathbb{R}^{2n}} f_j^\flat$ to $\mathbb{R}^q$ such that, for every $N \in \mathbb{N}$ and $(\xi_1, \ldots, \xi_q, \xi_{q+1}) \in \Sigma'_K$,*

$$\mathcal{G}_K f(\xi_1, \ldots, \xi_q, \xi_{q+1}) = \sum_{j=0}^{N} \frac{\lambda^j}{j!} g_j(\xi_1, \ldots, \xi_q, 0) + \frac{\lambda^{N+1}}{(N+1)!} \mathcal{G}_K f_{N+1}(\xi_1, \ldots, \xi_q, \xi_{q+1}) .$$

The rest of the proof goes as for $U(n)$.

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