REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 50, Número 2, 2009, Páginas 187–199

HANKEL OPERATORS BETWEEN HARDY-ORLICZ SPACES AND PRODUCTS OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. For $\mathbb{B}_n$ the unit ball of $\mathbb{C}^n$, we consider Hardy-Orlicz spaces of holomorphic functions $\mathcal{H}^\Phi$, which are preduals of spaces of *BMOA* type with weight. We characterize the symbols of Hankel operators that extend into bounded operators from the Hardy-Orlicz $\mathcal{H}^{\Phi_1}$ into $\mathcal{H}^{\Phi_2}$. We also consider the closely related question of integrability properties of the product of two functions, one in $\mathcal{H}^{\Phi_1}$ and the other one in the dual of $\mathcal{H}^{\Phi_2}$.

1. Introduction and statement of results

In the whole paper, Φ is a continuous and increasing function from $[0, \infty)$ onto itself, which is of strictly positive lower type, that is, there exists $p > 0$ and $c > 0$ such that, for $t > 0$ and $0 < s \leq 1$,

$$\Phi(st) \leq cs^p\Phi(t). \tag{1}$$

When this inequality is satisfied, we say also that Φ is of lower type p. Such a function Φ will be called a *growth function*.

We note σ the Lebesgue measure on the unit sphere $\mathbb{S}^n$. The Orlicz space $L^\Phi(\mathbb{S}^n)$ is the space of measurable functions f such that

$$\|f\|_{L^\Phi}^{\mathrm{lux}} := \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\sigma(x) \leq 1\right\} < \infty.$$

The quantity $\|f\|_{L^\Phi}^{\mathrm{lux}}$ is called the Luxembourg (quasi)-norm of f in the Orlicz space $L^\Phi(\mathbb{S}^n)$. It generalizes L^p norms, which correspond to the case $\Phi(t) = t^p$. Two growth functions Φ_1 and Φ_2 are said equivalent if there exists some constant c such that

$$c\Phi_1(ct) \leq \Phi_2(t) \leq c^{-1}\Phi_1(c^{-1}t).$$

Such equivalent growth functions define the same Orlicz space.

The following is proved in [VT]:

Proposition 1.1 (Volberg, Tolokonnikov)**.** *For Φ_1 and Φ_2 two growth functions, the bilinear map $(f, g) \mapsto fg$ sends $L^{\Phi_1} \times L^{\Phi_2}$ onto L^Φ, with the inverse mappings of Φ_1, Φ_2 and Φ related by*

$$\Phi^{-1} = \Phi_1^{-1} \times \Phi_2^{-1}. \tag{2}$$

2000 *Mathematics Subject Classification.* 32A37, 47B35, 47B10, 46E22.
Key words and phrases. Orlicz spaces, atomic decomposition, Hankel operators, *BMO*, Hardy spaces.

Moreover, there exists some constant c such that

$$\|fg\|^{\text{lux}}_{L^\Phi} \le c\|f\|^{\text{lux}}_{L^\Phi_1} \times \|g\|^{\text{lux}}_{L^\Phi_2}.$$

Next, we define the Hardy-Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$.

Definition 1.2. *$\mathcal{H}^\Phi(\mathbb{B}^n)$ is the space of holomorphic functions f in the unit ball $\mathbb{B}^n$ such that the functions f_r, defined by $f_r(w) = f(rw)$, are uniformly in $L^\Phi(\mathbb{S}^n)$. Moreover,*

$$\|f\|^{\text{lux}}_{\mathcal{H}^\Phi} := \sup_{r<1} \|f_r\|^{\text{lux}}_{L^\Phi}.$$

It follows from the last proposition that the product of two functions, one in $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and the other in $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$, belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$, with Φ given by (2). The converse, that is, the problem of factorizing a function of $\mathcal{H}^\Phi(\mathbb{B}^n)$ as a product $f_1 f_2$ with $f_i \in \mathcal{H}^{\Phi_i}(\mathbb{B}^n)$, is called the *Factorization Problem*. It is well known that it can be solved for $n = 1$ when the functions Φ are power functions. For $n \ge 2$, even in this case one only knows that there is weak factorization (see [CRW]) when $\Phi(t) = t^p$ with $p \le 1$.

We will consider a particular case of this problem. Moreover, instead of dealing with two Hardy-Orlicz spaces, we will replace one of them by a space of $BMOA$ type with weight. More precisely, we assume that ϱ is a continuous increasing function from $[0, \infty)$ onto itself, which is of *upper type α on* $[0, 1]$, that is,

$$\varrho(st) \le s^\alpha \varrho(t) \tag{3}$$

for $s > 1$, with $st \le 1$. In the next definition, B stands for a ball on $\mathbb{S}^n$ for the Koranyi metrics, given by $d(z, w) := |1 - z.\overline{w}|$ (see [R]). We first define

$$BMO(\varrho) = \left\{ f \in L^2(\mathbb{S}^n); \sup_B \inf_{R \in \mathcal{P}_N(B)} \frac{1}{(\varrho(\sigma(B)))^2 \sigma(B)} \int_B |f - R|^2 d\sigma < \infty \right\},$$

where, for B a ball centered at ζ_0, the space $\mathcal{P}_N(B)$ is the space of polynomials of order $\le N$ in the $(2n-1)$ last coordinates related to an orthonormal basis whose first element is ζ_0 and second element $\Im\zeta_0$. Here N is taken larger than $2n\alpha - 1$. We call $\|f\|_{BMO(\varrho)} := \|f\|_2 + C$, where C is the quantity appearing in the definition of $BMO(\varrho)$. Next, we define $BMOA(\varrho)$ as

$$BMOA(\varrho) = \left\{ f \in \mathcal{H}^2(\mathbb{B}^n); \sup_{r<1} \|f_r\|_{BMO(\varrho)} < \infty \right\}.$$

It follows from classical arguments that it coincides with the space of holomorphic functions in $\mathcal{H}^2(\mathbb{B}^n)$ such that their boundary values on $\mathbb{S}^n$ lie in $BMO(\varrho)$. The space $BMO(\varrho)$ is the dual of a so-called "real" Hardy-Orlicz space that we shall define later on (see Janson [J2] for the corresponding duality in the Euclidean setting, or Viviani [V] for spaces of homogeneous type). As a consequence, $BMOA(\varrho)$ spaces appear as duals of particular Hardy-Orlicz spaces, see also [BG] in this context.

Theorem 1.3. *The dual space of $\mathcal{H}^\Phi(\mathbb{B}^n)$ is the space $BMOA(\varrho)$ when Φ and ϱ are related by*

$$\varrho(t) := \varrho_\Phi(t) = \frac{1}{t\Phi^{-1}(1/t)} \tag{4}$$

The duality is given by the limit, as $r<1$ tends to 1, of scalar products on spheres of radius r.

Such Hardy-Orlicz spaces are defined by using growth functions Φ of lower type $(1+\alpha)^{-1}$. Functions Φ have moreover the additional property that the function $t\mapsto \dfrac{\Phi(t)}{t}$ is non-increasing. Without loss of generality, eventually replacing Φ by the equivalent growth function $\displaystyle\int_0^t \frac{\Phi(s)}{s}ds$ (which is well defined because of the positive lower type), we can assume that Φ is concave and Φ is a $\mathcal{C}^1$ function, with derivative $\Phi'(t)\simeq\frac{\Phi(t)}{t}$. This leads us to the following definition.

Definition 1.4. *We call $\mathfrak{T}_p$ the set of concave growth functions of lower type p, which are $\mathcal{C}^1$ functions and satisfy $\Phi'(t)\simeq\frac{\Phi(t)}{t}$. We assume also, without loss of generality, that $\Phi(1)=1$.*

Before stating our main result, let us define an auxiliary function.

Definition 1.5. *For $\Phi\in\mathfrak{T}_p$, we call λ_Φ the function defined for $t>0$ by*

$$\lambda_\Phi(t):=\int_0^t \min\left(1,\frac{\Phi(s)}{s^2}\right)ds. \tag{5}$$

When $\Phi(t)=t$, or $\varrho=1$, so that we are dealing with $\mathcal{H}^1$ and *BMOA*, this function is equivalent to a logarithm for $t>1$.

A variation of this function has been introduced in [BM] in relation to Hankel operators. It can be identified with the following function, introduced by [J1] in relation to multipliers of $BMO(\varrho)$,

$$\psi_\varrho(t):=\int_t^\infty \min\left(\frac{1}{s^2},\frac{\varrho(s)}{s}\right)ds.$$

The two following lemmas are elementary. Their proofs are given in the next section.

Lemma 1.6. *For $\Phi\in\mathfrak{T}_p$ and ϱ given by (4), then we have*

$$\lambda_\Phi(t)\simeq\psi_\varrho(1/t) \qquad \text{for } t>1.$$

Remark in particular that the Dini condition $\int_0^1\varrho(s)ds/s<\infty$ is equivalent to the condition $\int_1^\infty\frac{\Phi(s)}{s^2}ds<\infty$. When this condition holds, the function ψ_ϱ (resp. λ_Φ) is equivalent to a constant for $t<1$ (resp. $t>1$).

Lemma 1.7. *Let Φ_1 and Φ_2 in $\mathfrak{T}_p$, and let $\varrho_i(t)=\frac{1}{t\Phi_i^{-1}(1/t)}$. Then, if*

$$\varrho_\Phi:=\frac{\varrho_1}{\psi_{\varrho_2}},$$

we have also

$$\Phi(t)\simeq\Phi_1\left(\frac{t}{\lambda_{\Phi_2}(t)}\right) \qquad \text{for } t>1.$$

Theorem 1.8. *Let Φ_1 and Φ_2 in $\mathfrak{T}_p$, and $\varrho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$. Then the product of two functions, one in $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and the other one in BMOA(ϱ_2), is in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$, with Φ such that*

$$\varrho_\Phi := \frac{\varrho_1}{\psi_{\varrho_2}}, \tag{6}$$

or, equivalently,

$$\Phi(t) := \Phi_1\left(\frac{t}{\lambda_{\Phi_2}(t)}\right). \tag{7}$$

Moreover, functions in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ can be weakly factorized in terms of products of functions of $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and BMOA(ϱ_2).

We postpone to Section 4 the description of what we mean by a weak factorization.

Next, recall that, for $b \in \mathcal{H}^2(\mathbb{B}^n)$, the (small) Hankel operator h_b of symbol b is given, for functions $f \in \mathcal{H}^2(\mathbb{B}^n)$, by $h_b(f) = P(b\overline{f})$. Here P is the Szegö projection, that is the orthogonal projection from $L^2(\mathbb{S}^n)$ onto $\mathcal{H}^2(\mathbb{B}^n)$, considered as its closed subspace (see [R]). We can now state our second theorem.

Theorem 1.9. *Let Φ_1 and Φ_2 in $\mathfrak{T}_p$, and $\varrho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$. Assume that $b \in (\mathcal{H}^{\Phi}(\mathbb{B}^n))' =$ BMOA(ϱ_Φ), with*

$$\varrho_\Phi := \frac{\varrho_1}{\psi_{\varrho_2}}.$$

Then the Hankel operator h_b extends into a continuous operator from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ to $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$. Moreover the condition is necessary under one of the following conditions: Φ_2 is of lower type $\frac{n}{n+1}$, or Φ_2 satisfies the Dini condition $\int_1^\infty \frac{\Phi_2(t)}{t^2}dt < \infty$.

Let us comment these results. For the unit disc, Theorem 1.9 is due to Janson, Peetre and Semmes [JPS] when both Hardy spaces are $\mathcal{H}^1$. In whole generality (for the unit disc), it has been obtained by Bonami and Madan [BM] by a different method, based on balayage of generalized Carleson measures. See also Tolokonnikov [T] for $\mathcal{H}^p$ spaces. We have generalized the $\mathcal{H}^1$ case to the unit ball in $\mathbb{C}^n$ in [BGS]. The link between products and weak factorization for Hardy-Orlicz spaces has been considered by Bonami and Grellier in [BG], where both theorems are proved in the particular case $\Phi_2(t) = t$. Moreover, we will use the atomic decomposition of Hardy-Orlicz spaces given in [BG], so that this paper can be seen as a continuation of [BG]. Hardy-Orlicz spaces appear as a natural context for these questions of products of holomorphic functions and Hankel operators, which have been first considered in [BIJZ]. The main tool for weak factorization is the use of an explicit function in the weighted *BMO* space, which is due to Spanne [Sp] and replaces the logarithm in the usual case.

Aknowledgement. This paper is in the continuity of the course given by the first author at the CIMPA school in La Falda.

2. Technical lemmas on growth functions

We start by the following lemma.

Lemma 2.1. *Let $\Phi \in \mathfrak{T}_p$. The function λ_Φ satisfies the three following estimates. For $t > 1$,*

$$\lambda_\Phi(t) = 1 + \int_1^t \frac{\Phi(s)}{s^2} ds. \tag{8}$$

For $\varepsilon > 0$ there exists a constant C such that, for all $s, t > 1$,

$$\lambda_\Phi(st) \leq C s^\varepsilon \lambda_\Phi(t). \tag{9}$$

For $q > 1$ there exists a constant C such that, for $t > 1$,

$$\lambda_\Phi(t^q) \leq C \lambda_\Phi(t). \tag{10}$$

Proof. The first equality is direct (recall that $\Phi(1) = 1$). For the second inequality we use the elementary inequalities

$$\lambda_\Phi(s) \leq 1 + \log s,$$

$$\lambda_\Phi(st) \leq \lambda_\Phi(s) + \lambda_\Phi(t),$$

where, for the second one, we have cut the integral into the integral from 0 to t and the integral from t to st, changed of variable in the second one and used the decrease of $\Phi(u)/u$. To prove (10), it is sufficient to show that, for $t > 1$,

$$\int_1^{t^q} \frac{\Phi(s)}{s} \frac{ds}{s} \lesssim \int_1^t \frac{\Phi(s)}{s} \frac{ds}{s}.$$

By change of variables the left hand side is, up to a constant, equal to $\int_1^t \frac{\Phi(s^q)}{s^q} \frac{ds}{s}$. We conclude by using the monotonicity of $\Phi(s)/s$. □

Let us first now prove Lemmas 1.6 and 1.7.

Proof of Lemma 1.6.

We have

$$\int_{1/t}^1 \varrho(s) \frac{ds}{s} = \int_1^t \frac{ds}{\Phi^{-1}(s)} = \int_{\Phi^{-1}(1)}^{\Phi^{-1}(t)} \frac{\Phi'(s)}{s} ds.$$

We conclude by using the fact that $\Phi'(s) \simeq \Phi(s)/s$ and (10), since $\Phi(s)$ lies between two powers of s. □

Proof of Lemma 1.7. Let us pose $\lambda := \lambda_{\Phi_2}$ for simplification. By assumption, we have

$$\Phi^{-1}(t) = \Phi_1^{-1}(t) \psi_{\varrho_2}(1/t) \simeq \Phi_1^{-1}(t) \lambda(t).$$

Let us pose $\theta(t) = \frac{t}{\lambda(t)}$. We want to prove that $\Phi \simeq \Phi_1 \circ \theta$, or, equivalently, $\Phi^{-1} \simeq \theta^{-1} \circ \Phi_1^{-1}$. Remark that, since Φ_1 is bounded below and above by two power functions, the same is valid for Φ_1^{-1}, and by (10) we have the equivalence $\lambda \circ \Phi_1^{-1} \simeq \lambda$. So it is sufficient to prove that $\theta^{-1}(t) \simeq t\lambda(t)$ to conclude. Again, this last inequality is a consequence of the fact that $\lambda(t\lambda(t)) \simeq \lambda(t)$, since $t\lambda(t)$ lives between two powers of t. □

In the sequel, we will assume the following condition,

$$\int_1^\infty \frac{\Phi(t)}{t^2} dt = \infty, \tag{11}$$

so that λ_Φ defines an homemorphism of $[0, \infty)$. Under this condition, we define, for $\Phi \in \mathfrak{T}_p$, a pair of Young convex functions, namely

$$\Phi^*(t) = \int_0^t \lambda_\Phi(s) ds \tag{12}$$

$$\Phi^{**}(t) = \int_0^t \lambda_\Phi^{-1}(s) ds. \tag{13}$$

Moreover Φ^* is doubling (see also [BM]). So we have the following duality.

Lemma 2.2. *The space $L^{\Phi^{**}}$ is the dual of the space L^{Φ^*}.*

Remark 2.3. *When the condition (11) is not satisfied, or, equivalently, if the Dini condition $\int_0^1 \frac{\varrho(t)}{t} dt < \infty$ is satisfied, then $\Phi^*(t)$ is equivalent to t for $t > 1$, and the space L^{Φ^*} is the space L^1, with dual L^∞. Remark that, in this case, the space L^1_{weak} is contained in L^Φ: indeed, for f a positive measurable function,*

$$\int \Phi(f) d\sigma = \int \Phi'(t) \sigma(|f| > t) dt \leq \sup_t (t\sigma(|f| > t)) \int_0^\infty \frac{\Phi'(t)}{t} dt.$$

In particular, this means that PL^1 is contained in the space $\mathcal{H}^\Phi$, and, by duality, its dual $BMOA(\varrho_\Phi)$ is contained in the space $\mathcal{H}^\infty$ (in fact, in [Sp] it is proved that functions in $BMO(\varrho_\Phi)$ are continuous in this case). Then Theorem 1.8 follows at once, while Theorem 1.9 follows from Proposition 5.2 in [BG], where it is proved that h_b maps continuously $\mathcal{H}^{\Phi_1}$ into $\mathcal{H}^1_{\text{weak}}$ if and only if b belongs to the dual space $BMOA(\varrho_{\Phi_1})$. Indeed, we conclude for the sufficiency of the condition by using the fact that L^1_{weak} is contained in L^{Φ_2}. For the necessity, we conclude as in [BG], writing that, for f in a dense subset,

$$|\langle b, f\rangle| = |h_b(f)(0)| \lesssim \|h_b\| \|f\|_{\mathcal{H}^\Phi},$$

so that b defines a continuous linear form on the space $\mathcal{H}^\Phi$. We have used the fact that $\mathcal{H}^{\Phi_2}$ is continuously contained in $\mathcal{H}^p$, and the evaluation at 0 is bounded on this space.

In the next sections we will concentrate on growth functions that satisfy (11). We will need the following elementary property.

Lemma 2.4. *For $\Phi \in \mathfrak{T}_p$ satisfying the condition (11), we have the two estimates, for $t > 1$,*

$$\Phi^*(t) \simeq t\lambda_\Phi(t), \tag{14}$$

$$(\Phi^{**})^{-1}(t) \simeq \lambda_\Phi(t). \tag{15}$$

Proof. For the first one, a simple integration gives

$$\Phi^*(t) = t + \int_1^t (t-s)\frac{\Phi(s)}{s^2}ds \qquad \text{for } t > 1.$$

For proving that $\int_1^t \frac{\Phi(s)}{s^2}ds \lesssim \Phi^*(t)$, we first prove that the quantity is bounded by $2\Phi^*(2t)$, then use the doubling property of Φ^*. For the second one, we first remark that, from (10), it follows that, for $p > 1$ and $t > 1$,

$$\left(\lambda_\Phi^{-1}(t)\right)^p \leq \lambda_\Phi^{-1}(Ct).$$

Also, from the inequality $\Phi(t) \leq t$ it follows that $\lambda_\Phi(t) \leq 1 + \log t$ for $t > 1$, so that $\lambda_\Phi^{-1}(t) \geq e^{t-1}$. The required inequality on $(\Phi^{**})^{-1}$ is equivalent to the double inequality

$$\lambda_\Phi^{-1}(t/C) \leq \Phi^{**}(t) \leq \lambda_\Phi^{-1}(Ct).$$

The right hand side inequality follows from the fact that

$$\Phi^{**}(t) \leq t\lambda_\Phi^{-1}(t) \leq e^{t-1}\lambda_\Phi^{-1}(t) \leq \left(\lambda_\Phi^{-1}(t)\right)^2 \leq \lambda_\Phi^{-1}(Ct).$$

For the left hand side inequality, we use that, for $t > 2$, we have $\Phi^{**}(t) \geq \lambda_\Phi^{-1}(t/2)$. □

3. Sufficient conditions

We will use the following proposition (see [BM]).

Proposition 3.1. *The Hardy-Littlewood maximal operator $\mathcal{M}_{HL}$ maps continuously $L^{\Phi^*}(\mathbb{S}^n)$ into $L^\Phi(\mathbb{S}^n)$, while the Szegö projection P maps continuously $L^{\Phi^*}(\mathbb{S}^n)$ into $\mathcal{H}^\Phi(\mathbb{B}^n)$.*

Proof. For $f \in L^{\Phi^*}(\mathbb{S}^n)$ with norm ≤ 1, we use the maximal inequality and the equivalence $\Phi'(t) \simeq \frac{\Phi(t)}{t}$ to obtain

$$\begin{aligned}
\int_{\mathbb{S}^n} \Phi(\mathcal{M}_{HL}(f))d\sigma &= \int_0^\infty \Phi'(t)\sigma(\mathcal{M}_{HL}(f) > t)dt \\
&\lesssim 1 + \int_1^\infty \Phi'(t)\sigma(\mathcal{M}_{HL}(f) > t)dt \\
&\lesssim 1 + \int_1^\infty \frac{\Phi(t}{t^2} \int_{\{|f|>\frac{t}{2}\}} |f|d\sigma dt \\
&\lesssim 1 + \int_{\mathbb{S}^n} |f| \left(\int_1^{2|f|} \frac{\Phi(t)}{t^2}dt\right) d\sigma \lesssim 1.
\end{aligned}$$

We have used Lemma 2.4 for the last inequality. This allows to conclude for the maximal function. To conclude for the Szegö projection, we use the classical Calderón-Zygmund theory in view of the following inequality, since Φ is doubling,

$$\int \Phi(|Pf|)d\sigma \lesssim \int \Phi(|\mathcal{M}_{HL}f|)d\sigma.$$ □

As a consequence, we have the following, which could as well be obtained directly, by the classical method of John and Nirenberg, see [Sp].

Proposition 3.2. *The space $BMO(\varrho_\Phi)$ is contained in $L^{\Phi^{**}}(\mathbb{S}^n)$.*

Proof. For f a distribution on $\mathbb{S}^n$, its Poisson-Szegö maximal function is defined by

$$\mathcal{M}f(\xi) = \sup_{r<1}\left|\int_{\mathbb{S}^n} P(r\xi, w)f(w)d\sigma(w)\right|,$$

where $P(z,w)$ is the Poison-Szegö kernel. Then, by analogy with the spaces $\mathcal{H}^p(\mathbb{S}^n)$, let us define the Hardy-Orlicz "real" space $\mathcal{H}^\Phi(\mathbb{S}^n)$ as the space of distributions f on $\mathbb{S}^n$ such that

$$\int_{\mathbb{S}^n} \Phi(\mathcal{M}f)d\sigma < \infty.$$

A direct generalization of the methods of Janson in [J2] allows to prove that the dual of $H^\Phi(\mathbb{S}^n)$ is $BMO(\varrho_\Phi)$. It follows from the previous lemma and the well-known inequality

$$\mathcal{M}f \lesssim \mathcal{M}_{HL}f$$

that L^{Φ^*} is continuously contained in $\mathcal{H}^\Phi(\mathbb{S}^n)$. We conclude for the proposition by duality. □

Proof of Theorem 1.8, sufficient condition. By Proposition 3.2, it is sufficient to prove that the product of two functions, one in L^{Φ_1} and one in $L^{\Phi_2^{**}}$, is in L^Φ. One verifies that the link between the three growth functions coincides with the one of Theorem 1.1. □

The sufficient condition in Theorem 1.9 is a consequence of the following proposition, which is stronger, since the Szegö projection maps $L^{\Phi_2^*}$ into $\mathcal{H}^\Phi$.

Proposition 3.3. *With the notations of Theorem 1.9, for $b \in BMOA(\varrho_\Phi)$ the operator h_b maps $\mathcal{H}^{\Phi_1}$ into $P(L^{\Phi_2^*}(\mathbb{S}^n))$.*

Proof. We use the fact that $P(L^{\Phi_2^*}(\mathbb{S}^n))$ is a Banach space, with dual $\mathcal{H}^{\Phi_2^{**}}(\mathbb{B}^n)$. The norm of $h_b(f)$ is obtained by duality, testing on $g \in \mathcal{H}^{\Phi_2^{**}}(\mathbb{B}^n)$. So,

$$\|h_b(f)\|_{P(L^{\Phi_2^*})} = \sup|\langle h_b(f), g\rangle| = \sup|\langle b, fg\rangle| \leq \sup\|b\|_{BMO(\varrho_\Phi)}\|fg\|^{\text{lux}}_{\mathcal{H}^\Phi},$$

where the sup is taken on all functions $g \in \mathcal{H}^{\Phi_2^{**}}$ with (Luxembourg) norm less than 1. We use again Proposition 1.1 for fg to conclude. □

4. Weak factorization and necessary conditions

We start from the following definition (see [BG] and the references therein).

Definition 4.1. *A holomorphic function $A \in \mathcal{H}^2(\mathbb{B}^n)$ is called a molecule of order L, associated to the ball $B := B(z_0, r_0) \subset \mathbb{S}^n$, if it satisfies*

$$\|A\|_{\text{mol}(B,L)} := \left(\sup_{r<1}\int_{\mathbb{S}^n}\left(1 + \frac{d(z_0, r\xi)^{L+n}}{r_0^{L+n}}\right)|A(r\xi)|^2\frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2} < \infty. \quad (16)$$

We have used the notation $d(z,w) := |1 - z.\overline{w}|$ for $z, w \in \overline{\mathbb{B}^n}$.

Every molecule A of order L so that $L > L_p := 2n(1/p - 1)$ belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$ with

$$\|A\|_{\mathcal{H}^\Phi} \lesssim \Phi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B),$$

see [BG]. Every function has a molecular decomposition.

Theorem 4.2 ([BG])**.** *For any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$, there exists molecules A_j of order $L > L_p$, associated to the balls B_j, so that f may be written as*

$$f = \sum_j A_j$$

with $\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)} \simeq \sum_j \Phi(\|A_j\|_{\mathrm{mol}(B_j,L)})\sigma(B_j)$.

What we describe now is a factorization of each molecule. More precisely, we have the following theorem.

Theorem 4.3. *Let Φ_1 and Φ_2 in $\mathfrak{T}_p$, and $\varrho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$. Let $\Phi(t) := \Phi_1\left(\frac{t}{\lambda_{\Phi_2}(t)}\right)$. We assume moreover that ϱ_2 is of upper type $1/n$ on $[0,1]$ (or that $p \geq \frac{n}{n+1}$). Then a molecule A associated to the ball B may be written as fg, where f is a molecule and g is in BMOA(ϱ_2). Moreover, for L, L' given with $L' < L$, then f and g may be chosen such that*

$$\|g\|_{BMOA(\varrho_2)} \leq 1, \qquad \|f\|_{\mathrm{mol}(B,L')} \lesssim \frac{\|A\|_{\mathrm{mol}(B,L)}}{\psi_{\varrho_2}(\sigma(B))}, \tag{17}$$

or such that

$$\|g\|_{BMOA(\varrho_2)} \leq 1, \qquad \|f\|^{\mathrm{lux}}_{\mathcal{H}^{\Phi_1}} \lesssim \|A\|_{\mathrm{mol}(B,L)}\sigma(B)\varrho(\sigma(B)). \tag{18}$$

Proof. The main point is the following proposition.

Proposition 4.4. *Let ϱ be a continuous increasing function of upper type $1/n$ on $[0,1]$. Then, for each ball $B := B(z_0, r)$, there exists a function g such that $\|g\|_{BMOA(\varrho)} \leq 1$ and $|g(z)| \gtrsim \psi_\varrho(r + d(z, z_0))$ for $z \in \overline{\mathbb{B}^n}$.*

Let us take for granted this proposition and prove the theorem. For simplification we use the notation $\psi := \psi_{\varrho_2}$. We pose $f = A/g$, where g is given by the proposition, and prove that

$$\|f\|_{\mathrm{mol}(B,L')} \lesssim \frac{\|A\|_{\mathrm{mol}(B,L)}}{\psi(\sigma(B))}.$$

We cut the integral into two parts, one in $B(z_0, 2r)$, the other one outside. For the first one, we prove that

$$\|f\|_{\mathcal{H}^2} \lesssim \frac{\|A\|_{\mathrm{mol}(B,L)}}{\psi(\sigma(B))}.$$

The $\mathcal{H}^2$ norm is obtained as the L^2 norm at the boundary $\mathbb{S}^n$, which we cut again into two parts: the integral inside $B(z_0, 2r)$ and the integral outside. Inside the ball $B(z_0, 2r)$, we use the fact that $|g(z)| \gtrsim \psi(3r) \simeq \psi(\sigma(B))$. The remaining integral can be treated as the second term appearing in the computation of $\|f\|_{\mathrm{mol}(B,L')}$. One sees easily that it is sufficient to prove that, for $\varepsilon > 0$

$$\frac{\psi(r)}{\psi(r+d(z_0,w))} \lesssim \left(\frac{r}{d(z_0,w)}\right)^{\varepsilon}.$$

This is a consequence of (3), using lemma 1.6. We conclude from this for (17).

It remains to prove (18). By homogeneity, it is sufficient to prove that, for $\|A\|_{\mathrm{mol}(B,L)}\sigma(B)\varrho(\sigma(B)) = 1$, the function f that has been chosen is such that $\int \Phi(|f|)d\sigma \lesssim 1$, or, which is stronger, $\Phi(\|f\|_{\mathrm{mol}(B,L')})\sigma(B) \lesssim 1$, which in turn is equivalent to $\|f\|_{\mathrm{mol}(B,L')} \lesssim \Phi^{-1}(\sigma(B))$. We use the definition of ϱ to conclude. $\square$

It remains to prove Proposition 4.4. Without loss of generality we may assume that the ball $B(\zeta_0, r)$ is such that $\zeta_0 = (1, 0, \cdots, 0)$. The general case is then obtained by the action of $\mathrm{SU}(n)$.

We start with the dimension 1 with the following lemma, which has been essentially proved by Spanne in [Sp], Theorem 2.

Lemma 4.5. *For $n = 1$, the function $\phi(e^{iu}) := \psi(|1-re^{iu}|)$ belongs to $BMO(\varrho)(\mathbb{T})$, with norm bounded independently of $r < 1$.*

Next we choose for function g the holomorphic function γ that has ϕ as real part on the boundary of $\mathbb{S}^1 = \mathbb{T}$. The fact that g is in $BMOA(\varrho)$ is a consequence of the fact that the Hilbert transform maps continuously $BMO(\varrho)$ into itself, which has been proved by Janson in [J1]. Finally, to conclude for Proposition 4.4 in dimension 1, we use the following lemma.

Lemma 4.6. *Let $\phi_s := \phi * P_s$, with P_s the Poisson kernel in the unit disc. Then $\phi_s(e^{iu}) \gtrsim \psi(r + |1 - se^{iu}|)$.*

Proof. We shall prove that $\phi_s(e^{iu}) \gtrsim \psi(r + 3|1 - se^{iu}|)$, which is equivalent by (3). Coming back to the definition of ϕ and changing the order of integration, we have to prove that

$$1 + \int_r^1 \left[\int_{|1-e^{i(u-\theta)}|\leq v-r} P_s(e^{i\theta})d\theta\right] \frac{\varrho(v)}{v}dv \geq c\int_{r+3|1-se^{iu}|}^1 \frac{\varrho(v)}{v}dv.$$

It is sufficient to prove that, for $v \geq r + 3|1 - se^{iu}|$, the quantity in the bracket is bounded below independently of s. Since the integral of the Poisson kernel P_s on the interval $|e^{i\theta} - 1| \leq 1 - s$ is uniformly bounded below, it is sufficient to prove that this interval is contained in the set of integration when $v \geq r + 3|1 - se^{iu}|$. In other words, we use the fact that the two inequalities $v \geq r + 3|1 - se^{iu}|$ and $|e^{i\theta} - 1| \leq 1 - s$ imply the third one, $|1 - e^{i(u-\theta)}| \leq v - r$. This is an easy consequence of the triangular inequality, since $|e^{i\theta} - e^{iu}| \leq |e^{i\theta} - 1| + |1 - s| + |s - e^{iu}| \leq 3|1 - se^{iu}|$. This allows to conclude for the lemma. $\square$

This concludes for the proof of Proposition 4.4 for $n = 1$. We consider now the general case. Assuming again that $\zeta_0 = (1, 0, \cdots, 0)$, we choose as function g the function $g(z) := \gamma(z_1)$, with γ the function used in dimension 1 when ϱ is replaced by $\tilde{\varrho}(t) := \varrho(t^n)$ (so that $\tilde{\varrho}$ has upper type 1 on $[0,1]$). More precisely, γ is the function that has $\psi_{\tilde{\varrho}}(|1-re^{iu}|)$ as real part on the boundary of $\mathbb{S}^1 = \mathbb{T}$. The bound below of g is straightforward, using (10) to see that $\psi_\varrho \simeq \psi_{\tilde{\varrho}}$. The only point that we have to prove is the fact that g is in $BMOA(\varrho)$. This is a consequence of the Carleson type characterization of $BMOA(\varrho)$, see [Sm].

Proposition 4.7. *The function $h \in \mathcal{H}^2(\mathbb{B}^n)$ is in $BMOA(\varrho)$ if and only if the condition*

$$\int_{T(Q)} |\nabla h(z)|^2(1-|z|)dV_n(z) \leq C\sigma(Q)(\varrho(\sigma(Q)))^2,$$

for all balls Q on $\mathbb{S}^n$.

Here $T(Q)$ denotes the tent above Q, and dV_n the volume measure inside $\mathbb{C}^n$. Moreover, there is an equivalence between the norm in $BMOA$ and $\|h\|_{\mathcal{H}^2} + \inf C$, where the infimum is taken on all possible constants.

Then, it is sufficient to prove the following, which we will use with $\widetilde{\varrho}$ in place of ϱ to conclude for the proof of Proposition 4.4.

Lemma 4.8. *Let ϱ of upper type 1 on $[0,1]$. Assume that the holomorphic function in one variable γ satisfies in*

$$\int_{T(I)} |\gamma'|^2(1-|z|)dV_1(z) \leq C\sigma(I)(\varrho(\sigma(I)))^2$$

for each interval of the unit circle. Then $g(z) = \gamma(z_1)$ satisfies the condition

$$\int_{T(Q)} |\nabla g(z)|^2(1-|z|)dV_n(z) \leq C\sigma(Q)\varrho((\sigma(Q)^{\frac{1}{n}}))^2,$$

for all balls Q on $\mathbb{S}^n$.

Proof. Let Q be the ball $B(w,s)$. We call $z' := (z_2, \cdots, z_n)$. Recall that we can take $T(Q) := \{z \in \mathbb{B}^n \; ; \frac{z}{|z|} \in Q \; , 1-|z| > s\}$.

Let us first consider balls Q such that $|w_1| = 1$, and, so $w' = 0$. Then $T(Q)$ is contained in the set defined by $|1 - z_1\overline{w_1}| < Cs$ and $|z'|^2 < Cs$ for some constant C that depends only on the geometry. Integrating first in z', we find the required property.

Next, let us consider balls Q such that $T(Q)$ contains a point z such that $|1 - |z_1| \leq s$. Then the same is valid for Q. This means that, for some $z \in Q$, we have $d(z, (\frac{z_1}{|z_1|}, 0)) \leq s$. So the distance $d(w, (\frac{z_1}{|z_1|}, 0))$ is bounded by Cs, for some geometric constant C, and there is a ball Q', centered at the point $(\frac{z_1}{|z_1|}, 0)$, which contains Q and has same measure, up to a geometric constant. So we can rely on the first case to conclude.

Finally, in the remaining cases, the projection on the complex line $z' = 0$ of $T(Q)$ is contained in the disc $|1 - |z_1| \leq s$. We will use the following lemma.

Lemma 4.9. *Assume that the holomorphic function in one variable γ satisfies in*

$$\int_{T(I)} ||\gamma'|^2(1-|z|)dV_1(z) \leq C\sigma(I)(\varrho(\sigma(I)))^2$$

for each interval of the unit circle. Then there exists some C' such that

$$|\gamma'(z)| \leq C' \frac{\varrho(1-|z|)}{1-|z|}.$$

Let us take this for granted and finish our proof in this last case. We have ∇g bounded by $C\varrho(s)/s$ on $T(Q)$, while

$$\int_{T(Q)} (1-|z|)dV \simeq s^2\sigma(s^n).$$

Finally, the proof of Lemma 4.9 is classical. We use the mean value property to write that

$$|\gamma'(z)|^2 \leq \frac{C}{(1-|z|)^3} \int_{|\zeta-z|\leq(1-|z|)/2} |\gamma'(\zeta)|^2(1-|\zeta|)dV_1(\zeta).$$

We conclude by using the assumption and the fact that the disc $|\zeta-z| \leq (1-|z|)/2$ is contained in $T(I)$, where I is centered at $z/|z|$ and has radius $2(1-|z|)$.

This finishes the proofs of Lemma 4.9 and 4.8. □

The proof of Proposition 4.4 in dimension $n > 1$ follows at once.

Proof of Theorem 1.8, necessary condition. We have already considered the case of growth functions satisfying the Dini condition in Remark 2.3. We assume now that ϱ is of upper type $1/n$, so that we have the factorization of molecules. We adapt the proof given in [BGS]. Let $b \in \mathcal{H}^2(\mathbb{B}^n)$. We assume that h_b is bounded from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ to $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ and prove that b belongs to $BMOA(\varrho_\Phi)$, with $\varrho_\Phi = \frac{\varrho_1}{\psi_{\varrho_2}}$. For this, it is sufficient to prove that there exists a positive constant C such that, for each ball B we can find a polynomial $R \in \mathcal{P}_N(B)$ such that

$$\int_B |b-R|^2 d\sigma \leq C(\varrho(\sigma(B)))^2\sigma(B).$$

We take for R the orthogonal projection of b onto $\mathcal{P}_N(B)$, so that the function $a := \chi_B(b-R)$ on $\mathbb{S}^n$ is an atom (see [BG] for a precise definition). One knows that $A := Pa$ is a molecule associated to B, with $\|A\|_{\mathrm{mol}(B,L)} \lesssim \|a\|_2\sigma(B)^{-1/2}$. From Theorem 4.3, we know that A may be written as fg, with

$$\|g\|_{BMOA(\varrho_2)} \leq 1, \qquad \|f\|^{\mathrm{lux}}_{\mathcal{H}^{\Phi_1}} \lesssim \|A\|_{\mathrm{mol}(B,L)}\sigma(B)\varrho(\sigma(B)).$$

Next, we write that

$$\int_B |b-R|^2 d\sigma = \langle b, a\rangle = \langle b, Pa\rangle = \langle h_b(f), g\rangle,$$

so that

$$\int_B |b-R|^2 d\sigma \leq \|h_b\| \|f\|^{\mathrm{lux}}_{\mathcal{H}^{\Phi_1}} \|g\|_{BMOA(\varrho_2)} \lesssim \|h_b\| \|a\|_2 \sigma(B)^{1/2}\varrho(\sigma(B)).$$

We divided both sides by $\|a\|_2 = (\int_B |b-R|^2 d\sigma)^{1/2}$ to conclude.

This ends the proof.

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Recibido: 9 de marzo de 2009
Aceptado: 18 de marzo de 2009