

REMARKS ON SPECTRAL MULTIPLIER THEOREMS
 ON HARDY SPACES
 ASSOCIATED WITH SEMIGROUPS OF OPERATORS

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ABSTRACT. Let L be a non-negative, self-adjoint operator on $L^2(\Omega)$, where (Ω, d, μ) is a space of homogeneous type. Assume that the semigroup $\{T_t\}_{t>0}$ generated by $-L$ satisfies Gaussian bounds, or more generally Davies-Gaffney estimates. We say that f belongs to the Hardy space H_L^1 if the square function

$$S_h f(x) = \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{\mu(B_d(x, t))} \frac{dt}{t} \right)^{1/2}$$

belongs to $L^1(\Omega, d\mu)$, where $\Gamma(x) = \{(y, t) \in \Omega \times (0, \infty) : d(x, y) < t\}$. We prove spectral multiplier theorems for L on H_L^1 .

1. INTRODUCTION.

A classical Hörmander multiplier theorem [37] asserts that if m is a bounded function on \mathbb{R}^d such that for some $\beta > d/2$ and any radial function $\eta \in C_c^\infty$, $\text{supp } \eta \subset \{\xi \in \mathbb{R}^d : 2^{-1} \leq |\xi| \leq 2\}$, one has

$$\sup_{t>0} \|\eta(\cdot) m(t \cdot)\|_{W^{2, \beta}(\mathbb{R}^d)} \leq C_\eta,$$

where $\|\cdot\|_{W^{2, \beta}(\mathbb{R}^d)}$ is the standard Sobolev norm on \mathbb{R}^d , then the multiplier operator $f \mapsto \mathcal{F}^{-1}(m\mathcal{F}f)$, initially defined on $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and is of weak-type (1,1). Here \mathcal{F} denotes the Fourier transform.

Let $(\Omega, d(x, y))$ be a metric space equipped with a positive measure μ . We assume that (Ω, d, μ) is a space of homogeneous type in the sense of Coifman-Weiss [9], that is, there exists a constant $C > 0$ such that

$$\mu(B_d(x, 2t)) \leq C\mu(B_d(x, t)) \quad \text{for every } x \in \Omega, t > 0, \quad (1.1)$$

where $B_d(x, t) = \{y \in \Omega : d(x, y) < t\}$. The condition (1.1) implies that there exist constants $C > 0$ and $q > 0$ such that

$$\mu(B_d(x, st)) \leq C_0 s^q \mu(B_d(x, t)) \quad \text{for every } x \in \Omega, t > 0, s > 1. \quad (1.2)$$

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Of course we wish to get q as small as possible even at the expense of large C_0 .

Let $\{T_t\}_{t>0}$ be a semigroup of linear operators on $L^2(\Omega, d\mu)$ generated by $-L$, where L is a non-negative, self-adjoint operator which is injective on its domain. Assume the operators T_t have the following form

$$T_t f(x) = \int_{\Omega} T_t(x, y) f(y) d\mu(y), \tag{1.3}$$

where the kernels $T_t(x, y)$ satisfy Gaussian bounds, that is, there exist constants $C_0, c_0 > 0$ such that for every $x, y \in \Omega, t > 0$, we have

$$|T_t(x, y)| \leq \frac{C_0}{V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)^2}{c_0 t}\right), \tag{1.4}$$

where here and subsequently $V(x, t) = \mu(B_d(x, t))$. The estimate (1.4) implies that for every $k \in \mathbb{N}$ there exist constants $C_k, c_k > 0$ such that

$$\left| \frac{\partial^k}{\partial t^k} T_t(x, y) \right| \leq \frac{C_k}{t^k V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)^2}{c_k t}\right) \quad \text{for } x, y \in \Omega, t > 0. \tag{1.5}$$

The constants C_k, c_k in (1.5) depend only on k and the constants C, C_0, q, c in (1.2) and (1.4).

For a suitable function f (e.g., from $L^2(\Omega)$) we consider the square function $S_h f$ associated with L defined by

$$S_h f(x) = \left(\iint_{\Gamma(x)} |t^2 L T_{t^2} f(y)|^2 \frac{d\mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}, \tag{1.6}$$

where $\Gamma(x) = \{(y, t) \in \Omega \times (0, \infty) : d(x, y) \leq t\}$.

Following [2], [3], [36] (see also [4], [23]) we define the Hardy space $H_L^1 = H_{L, S_h}^1(\Omega)$ as the completion of $\{f \in L^2(\Omega) : \|S_h f\|_{L^1(\Omega)} < \infty\}$ in the norm $\|f\|_{H_L^1} = \|S_h f\|_{L^1(\Omega)}$.

It was proved in Hofmann, Lu, Mitrea, Mitrea, Yan [36] that the space H_L^1 , where $-L$ generates a semigroup having Gaussian bounds, admits the following atomic decomposition.

Let $M \geq 1, M \in \mathbb{N}$. A function a is a $(1, 2, M)$ -atom for H_L^1 if there exist a ball $B = B_d(y_0, r) = \{y \in \Omega : d(y, y_0) < r\}$ and a function $b \in \mathcal{D}(L^M)$ such that

$$a = L^M b; \tag{1.7}$$

$$\text{supp } L^k b \subset B, \quad k = 0, 1, \dots, M; \tag{1.8}$$

$$\|(r^2 L)^k b\|_{L^2(\Omega)} \leq r^{2M} \mu(B)^{-1/2}, \quad k = 0, 1, \dots, M. \tag{1.9}$$

The atomic norm $\|f\|_{H_L^1\text{-atom}}$ is defined by

$$\|f\|_{H_L^1\text{-atom}} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all representation $f = \sum_j \lambda_j a_j$, where a_j are $(1, 2, M)$ -atoms for $H_L^1, \lambda_j \in \mathbb{C}$. Theorem 7.1 of [36] asserts that there exists a

constant $C > 0$ such that

$$C^{-1}\|f\|_{H_L^1} \leq \|f\|_{H_L^1\text{-atom}} \leq C\|f\|_{H_L^1}. \tag{1.10}$$

Let

$$Lf = \int_0^\infty \lambda dE_L(\lambda)f \tag{1.11}$$

be the spectral resolution of L .

Our first goal in this paper is to present a simple proof of the following spectral multiplier theorem.

Theorem 1.12. *Let m be a bounded function defined on $(0, \infty)$ such that for some real number $\alpha > q/2$ and any nonzero function $\eta \in C_c^\infty(2^{-1}, 2)$ there exists a constant C_η such that*

$$\sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{W^{\infty,\alpha}(\mathbb{R})} \leq C_\eta, \tag{1.13}$$

where $\|F\|_{W^{p,\alpha}(\mathbb{R})} = \|(I-d^2/dx^2)^{\alpha/2}F\|_{L^p(\mathbb{R})}$. Then the spectral multiplier operator

$$m(L) = \int_0^\infty m(\lambda)dE_L(\lambda), \tag{1.14}$$

maps $(1, 2, 1)$ -atoms for H_L^1 into H_L^1 . Moreover, there exists a constant $C > 0$ such that

$$\|m(L)a\|_{H_L^1} \leq C \quad \text{for every } (1, 2, 1)\text{-atom}. \tag{1.15}$$

Remark 1.16. If we additionally assume that for every $y \in \Omega$ there exist constants $\kappa > 0$ and $c > 0$ such that $\mu(B_d(y, s)) \geq cs^\kappa$ for $s > 1$, then the operator $m(L)$ extends uniquely to a bounded operator on H_L^1 (see Section 5 for details).

Remark 1.17. It turns out that if we replace (1.13) by the stronger condition

$$\sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{W^{2,\alpha}(\mathbb{R})} \leq C_\eta, \tag{1.18}$$

with some $\alpha > (q + 1)/2$, then the multiplier theorem holds for Hardy spaces associated with more general semigroups, that is, semigroups satisfying Davies-Gaffney estimates. This will be discussed in Section 4. We present two seemingly similar theorems with two different proofs. The first proof, thanks to [34], could be also adapted to cover the case of a broader class of semigroups with integral kernels of a very mild decay. The other one does not even require the existence of integral kernels of the semigroups under consideration, however depends very much on the finite speed propagation of the wave equation associated with generators which is in fact equivalent to Davies-Gaffney estimates, see, e.g., [45], [10].

Spectral multiplier theorems on various spaces attracted attention of many authors (see, for example, [1], [6], [7], [11], [12], [15], [18], [19], [30], [32], [33], [35], [40], [43], [44], [46], and references there). E. M. Stein [46] proved that if $-A$ is the infinitesimal generator of a symmetric diffusion semigroup and m is of Laplace transform type, then $m(A)$ is bounded on L^p , $1 < p < \infty$. E. Stein and A. Hulanicki (see [30]) noticed that if $-A$ is a sublaplacian on a stratified Lie group G , then

the convolution kernel of the operator $m(A)$ satisfies Calderón-Zygmund type estimates. This fact together with atomic decompositions of the Hardy spaces $H^p(G)$ leads to a spectral multiplier theorem on these spaces (see [30, Theorem 6.25]). The finite speed propagation of the wave equation was used by Sikora [44] and [45] for proving L^p bounds for certain operators. Actually, the technique of the proof of Lemma 4.8 is taken from [44].

The development of the theory of real Hardy spaces in \mathbb{R}^d had its origin in works Stein-Weiss [47] and Fefferman-Stein [29]. An important contribution to the theory were atomic decompositions proved by Coifman [8] for $d = 1$ and Latter [38] for $d > 1$. The extension of H^p on the spaces of homogenous type is due to Coifman-Weiss [9] (see also [43]). Hardy spaces associated with various semigroups of linear operators were considered by many authors. For their properties and equivalent characterizations we refer the reader to [2]-[5], [13]-[28], [31], [36], [41], [42].

2. FUNCTIONAL CALCULI

For $\beta \geq 0$ let $\omega_\beta(x, y) = (1 + d(x, y))^\beta$. The function is submultiplicative, that is, $\omega_\beta(x, y) \leq \omega_\beta(x, z)\omega_\beta(z, y)$.

For an integral kernel $k(x, y)$ and $\beta > 0$ we define

$$\begin{aligned} \|k(x, y)\|_{\omega(\beta)} &= \sup_{x \in \Omega} \int |k(x, y)|(1 + d(x, y))^\beta d\mu(y) \\ &\quad + \sup_{y \in \Omega} \int |k(x, y)|(1 + d(x, y))^\beta d\mu(x). \end{aligned}$$

The following theorem is a consequence of (1.4) and results of W. Hebisch [34, Theorem 2.10].

Theorem 2.1. *Let (Ω, d, μ) and $\{T_t\}_{t>0}$ satisfy (1.2) and (1.4) respectively. For $\alpha, \beta > 0$ with $\alpha > \beta + q/2$ there exists a constant $C' > 0$ such that for every function $\eta \in W^{\infty, \alpha}(\mathbb{R})$ with $\text{supp } \eta \subset (1/4, 4)$ the multiplier operator*

$$\eta(L)f = \int_0^\infty \eta(\lambda) dE_L(\lambda)f$$

is of the form

$$\eta(L)f(x) = \int_\Omega \eta(L)(x, y)f(y) d\mu(y)$$

with

$$\|\eta(L)(x, y)\|_{\omega(\beta)} \leq C' \|\eta\|_{W^{\infty, \alpha}(\mathbb{R})}. \quad (2.2)$$

The constant C' in (2.2) depends only on α, β and the constants C, q from (1.2) and constants C_0, c_0 from (1.4).

In this paper we shall use the following scaling argument. For $\tau > 0$, let $d^{\{\tau\}}(x, y) = \tau^{-1/2}d(x, y)$. Then the space $(\Omega, d^{\{\tau\}}(x, y), \mu)$ is the space of homogeneous type such that

$$V_\tau(x, st) = \mu(B_{d^{\{\tau\}}}(x, st)) \leq Cs^q \mu(B_{d^{\{\tau\}}}(x, t)), \quad s > 1, \quad (2.3)$$

with the same constants C, q as in (1.2). Similarly, let $L^{\{\tau\}} = \tau L$ and $\{T_t^{\{\tau\}}\}_{t>0}$ be the semigroup generated by $-L^{\{\tau\}}$. Clearly, $T_t^{\{\tau\}}(x, y) = T_{\tau t}(x, y)$ are the integral kernels of $T_t^{\{\tau\}}$. Hence, for $k = 0, 1, 2, \dots$, we have

$$\left| \frac{\partial^k}{\partial t^k} T_t^{\{\tau\}}(x, y) \right| \leq \frac{C_k}{t^k V_\tau(x, \sqrt{t})} \exp\left(-\frac{d^{\{\tau\}}(x, y)^2}{c_k t}\right) \quad \text{for } x, y \in \Omega, t > 0, \tag{2.4}$$

with the same constants C_k, c_k as in (1.4) and (1.5) independent of τ .

Therefore, from Theorem 2.1 we conclude that

$$\begin{aligned} & \int_\Omega |\eta(\tau L)(x, y)| \left(1 + \frac{d(x, y)}{\sqrt{\tau}}\right)^\beta d\mu(y) + \int_\Omega |\eta(\tau L)(x, y)| \left(1 + \frac{d(x, y)}{\sqrt{\tau}}\right)^\beta d\mu(x) \\ & \leq C \|\eta\|_{W^{\infty, \alpha}(\mathbb{R})}, \end{aligned} \tag{2.5}$$

provided $\text{supp } \eta \subset (4^{-1}, 4)$, $\alpha > \beta + q/2$.

Proposition 2.6. *Assume that m satisfies the assumptions of Theorem 1.12. For $N = 1, 2$, we set*

$$\Phi_t^{\langle N \rangle}(\lambda) = (t^2 \lambda)^N e^{-t^2 \lambda} m(\lambda).$$

Then there exist $\beta > 0$ and $C'' > 0$ such that

$$\int_\Omega |\Phi_t^{\langle N \rangle}(L)(x, y)| \left(1 + \frac{d(x, y)}{t}\right)^\beta d\mu(x) \leq C'', \tag{2.7}$$

$$\int_\Omega |\Phi_t^{\langle N \rangle}(L)(x, y)| \left(1 + \frac{d(x, y)}{t}\right)^\beta d\mu(y) \leq C''. \tag{2.8}$$

Proof. It suffices to prove (2.7) for $t = 1$ and then use the scaling argument. Fix a $C_c^\infty(\frac{1}{2}, 2)$ function ψ such that

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j} \lambda) = 1 \quad \text{for } \lambda > 0. \tag{2.9}$$

Denote $n_j(\lambda) = \psi(2^{-j} \lambda) \lambda^N e^{-\lambda} m(\lambda)$, $\tilde{n}_j(\lambda) = n_j(2^j \lambda) = \psi(\lambda) (2^j \lambda)^N e^{-2^j \lambda} m(2^j \lambda)$. Clearly, $\text{supp } \tilde{n}_j \subset (2^{-1}, 2)$ and

$$\|\tilde{n}_j\|_{W^{\infty, \alpha}(\mathbb{R})} \leq \begin{cases} C 2^{-j} & \text{for } j \geq 0; \\ C 2^{jN} & \text{for } j < 0. \end{cases} \tag{2.10}$$

Let $0 < \beta \leq 1/2$ be such that $\alpha > \beta + q/2$. Applying (2.5) combined with (2.10) we obtain

$$\int_\Omega |n_j(L)(x, y)| \left(1 + 2^{j/2} d(x, y)\right)^\beta d\mu(x) \leq \begin{cases} C 2^{-j} & \text{for } j \geq 0; \\ C 2^{jN} & \text{for } j < 0, \end{cases} \tag{2.11}$$

with the same bounds when integrating with respect to $d\mu(y)$. Obviously,

$$\int_\Omega |\Phi_1^{\langle N \rangle}(L)(x, y)| d\mu(x) \leq \sum_{j \in \mathbb{Z}} \int_\Omega |n_j(L)(x, y)| d\mu(x) \leq C'. \tag{2.12}$$

Moreover, from (2.11) we also deduce that

$$\begin{aligned} \int_{\Omega} |\Phi_1^{(N)}(L)(x, y)| d(x, y)^\beta d\mu(x) &\leq \sum_{j \in \mathbb{Z}} \int_{\Omega} |n_j(L)(x, y)| d(x, y)^\beta d\mu(x) \\ &\leq \sum_{j \geq 0} C 2^{-j-\beta/2} + \sum_{j < 0} C' 2^{jN-j\beta/2} \leq C', \end{aligned} \tag{2.13}$$

which implies (2.7) for $t = 1$. To prove (2.8) we proceed in the same way. □

Lemma 2.14. *For $N = 1$ or $N = 2$, let*

$$\Theta_j^{(N)}(x, y) = \sup_{2^j \leq t < 2^{j+1}} \sup_{d(x, x') < t} |\Phi_t^{(N)}(x', y)|. \tag{2.15}$$

Then there exist constants $C' > 0$ and $\beta > 0$ such that

$$\int_{\Omega} \Theta_j^{(N)}(x, y) \left(1 + \frac{d(x, y)}{2^j}\right)^\beta d\mu(x) \leq C'. \tag{2.16}$$

Proof. Fix $2^j \leq t < 2^{j+1}$ and let $d(x, x') < t$. Since

$$\Phi_t^{(N)}(\lambda) = (2^{1-j}t)^{2N} \exp(-(t^2 - 2^{2(j-1)})\lambda) \Phi_{2^{j-1}}^{(N)}(\lambda)$$

and $V(x', (t^2 - 2^{2(j-1)})^{1/2}) \sim V(x, 2^j)$ for $d(x, x') < t$, we have

$$\begin{aligned} |\Phi_t^{(N)}(x', y)| &= (2^{1-j}t)^{2N} \left| \int_{\Omega} T_{t^2 - 2^{2(j-1)}}(x', z) \Phi_{2^{j-1}}^{(N)}(z, y) d\mu(z) \right| \\ &\leq C'' \int_{\Omega} \frac{\exp(-d(x', z)^2/c_0(t^2 - 2^{2(j-1)}))}{V(x', (t^2 - 2^{2(j-1)})^{1/2})} |\Phi_{2^{j-1}}^{(N)}(z, y)| d\mu(z) \\ &\leq C \int_{\Omega} \frac{\exp(-d(x, z)^2/c'2^{2j})}{V(x, 2^j)} |\Phi_{2^{j-1}}^{(N)}(z, y)| d\mu(z). \end{aligned} \tag{2.17}$$

Using (2.17) and Proposition 2.6 we obtain

$$\begin{aligned} &\int_{\Omega} \Theta_j^{(N)}(x, y) \left(1 + \frac{d(x, y)}{2^j}\right)^\beta d\mu(x) \\ &\leq C \int_{\Omega} \int_{\Omega} \frac{\exp(-\frac{d(x, z)^2}{c'2^{2j}})}{V(x, 2^j)} |\Phi_{2^{j-1}}^{(N)}(z, y)| \left(1 + \frac{d(x, z)}{2^j}\right)^\beta \left(1 + \frac{d(z, y)}{2^j}\right)^\beta d\mu(z) d\mu(x) \\ &\leq C'. \end{aligned}$$

□

3. PROOF OF THEOREM 1.12

It suffices to establish that there exists a constant C such that for every $(1, 2, 1)$ -atom a for H_L^1 we have

$$\|S_h(m(L)a)\|_{L^1(\Omega)} \leq C. \tag{3.1}$$

Our proof of (3.1) borrows ideas from [17]. Let a be a $(1, 2, 1)$ -atom for H_L^1 and let b and $B = B_d(y_0, r)$ be as in (1.7)–(1.9). Since S_h is bounded on $L^2(\Omega)$, we have

$$\|S_h m(L)a\|_{L^1(B_d(y_0, 2r), d\mu)} \leq C' \|m(L)a\|_{L^2(\Omega)} \mu(B)^{1/2} \leq C \|a\|_{L^2(\Omega)} \mu(B)^{1/2} \leq C. \tag{3.2}$$

It suffices to estimate $S_h m(L)a$ on $(2B)^c$, where $2B = B_d(y_0, 2r)$. Clearly, $\Phi_t^{(1)}(L)a = t^{-2}\Phi_t^{(2)}(L)b$. Set $j_0 = \log_2 r$. Then

$$\begin{aligned} (S_h m(L)a(x))^2 &= \iint_{\Gamma(x)} |t^2(LT_{t^2}m(L)a)(x')|^2 \frac{d\mu(x')}{V(x', t)} \frac{dt}{t} \\ &= \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \int_{d(x, x') < t} \left| \Phi_t^{(1)}(L)a(x') \right|^2 \frac{d\mu(x')}{V(x', t)} \frac{dt}{t} \\ &= \sum_{j \leq j_0} \int_{2^j}^{2^{j+1}} \int_{d(x, x') < t} \left| \Phi_t^{(1)}(L)a(x') \right|^2 \frac{d\mu(x')}{V(x', t)} \frac{dt}{t} \\ &\quad + \sum_{j > j_0} \int_{2^j}^{2^{j+1}} \int_{d(x, x') < t} \left| t^{-2}\Phi_t^{(2)}(L)b(x') \right|^2 \frac{d\mu(x')}{V(x', t)} \frac{dt}{t}. \end{aligned}$$

Using (2.15) we have

$$\begin{aligned} (S_h m(L)a(x))^2 &\leq C \sum_{j \leq j_0} \int_{2^j}^{2^{j+1}} \int_{d(x, x') < t} \left(\int_{\Omega} \Theta_j^{(1)}(x, y) |a(y)| d\mu(y) \right)^2 \frac{d\mu(x')}{V(x', t)} \frac{dt}{t} \\ &\quad + C \sum_{j > j_0} \int_{2^j}^{2^{j+1}} \int_{d(x, x') < t} \left(\int_{\Omega} t^{-2}\Theta_j^{(2)}(x, y) |b(y)| d\mu(y) \right)^2 \frac{d\mu(x')}{V(x', t)} \frac{dt}{t} \\ &\leq C \sum_{j \leq j_0} \left(\int_{\Omega} \Theta_j^{(1)}(x, y) |a(y)| d\mu(y) \right)^2 \\ &\quad + C \sum_{j > j_0} \left(\int_{\Omega} 2^{-2j}\Theta_j^{(2)}(x, y) |b(y)| d\mu(y) \right)^2, \end{aligned} \tag{3.3}$$

because

$$\int_{2^j}^{2^{j+1}} \int_{d(x, x') < t} \frac{d\mu(x')}{V(x', t)} \frac{dt}{t} \leq C.$$

From (3.3) we trivially get

$$\begin{aligned} (S_h m(L)a)(x) &\leq C \left(\sum_{j \leq j_0} \int_{\Omega} \Theta_j^{(1)}(x, y) |a(y)| d\mu(y) + \sum_{j > j_0} \int_{\Omega} 2^{-2j}\Theta_j^{(2)}(x, y) |b(y)| d\mu(y) \right). \end{aligned}$$

Applying Lemma 2.14 we obtain

$$\begin{aligned}
 & \int_{(2B)^c} (S_h m(L)a)(x) \, d\mu(x) \\
 & \leq C \sum_{j \leq j_0} \int_{(2B)^c} \int_{d(y, y_0) < r} \Theta_j^{(1)}(x, y) \left(\frac{d(x, y)}{2^j} \right)^\beta \left(\frac{2^j}{r} \right)^\beta |a(y)| \, d\mu(y) \, d\mu(x) \\
 & \quad + C \sum_{j > j_0} \int_{(2B)^c} \int_{d(y, y_0) < r} 2^{-2j} \Theta_j^{(2)}(x, y) |b(y)| \, d\mu(y) \, d\mu(x) \\
 & \leq C \sum_{j \leq j_0} \int_{d(y, y_0) < r} \left(\frac{2^j}{r} \right)^\beta |a(y)| \, d\mu(y) + C \sum_{j > j_0} 2^{-2j} \int_{d(y, y_0) < r} |b(y)| \, d\mu(y) \, d\mu(x).
 \end{aligned} \tag{3.4}$$

By the Cauchy-Schwarz inequality $\|a\|_{L^1(\Omega)} \leq 1$ and $\|b\|_{L^1(\Omega)} \leq r^2$. Since $2^{j_0} \sim r$, we easily conclude from (3.4) that

$$\int_{d(x, y_0) > 2r} (S_h m(L)a)(x) \, d\mu(x) \leq C,$$

which together with (3.2) completes the proof of (3.1)

4. SPECTRAL MULTIPLIER THEOREM FOR SEMIGROUPS SATISFYING DAVIES-GAFFNEY ESTIMATES.

Let $\{\mathcal{T}_t\}_{t>0}$ be a semigroup of linear operators on $L^2(\Omega)$ generated by $-\mathcal{L}$, where \mathcal{L} is a non-negative, self-adjoint operator which is injective on its domain. We assume that $\{\mathcal{T}_t\}_{t>0}$ satisfies Davies-Gaffney estimates, which briefly speaking means that

$$|\langle \mathcal{T}_t f_1, f_2 \rangle| \leq C \exp\left(-\frac{\text{dist}(U_1, U_2)^2}{ct}\right) \|f_1\|_{L^2(\Omega)} \|f_2\|_{L^2(\Omega)} \tag{4.1}$$

for every $f_i \in L^2(\Omega)$, $\text{supp } f_i \subset U_i$, $i = 1, 2$, U_i are open subsets of Ω (see e.g., [10], [36] for details).

The Hardy space $H_{\mathcal{L}}^1$, defined as in Section 1 by means of $L^1(\Omega)$ bounds of the square function (1.6), were considered by Auscher, McIntosh, Russ [3] and Hofmann, Lu, Mitrea, Mitrea, Yan [36]. It was proved in [36] that the space $H_{\mathcal{L}}^1$ admits atomic decompositions into $(1, 2, M)$ -atoms associated with \mathcal{L} , provided $M > q/4$, $M \in \mathbb{N}$ (see [36]). Clearly, L is replaced by \mathcal{L} in the definition (1.7)–(1.9) of $(1, 2, M)$ -atoms for $H_{\mathcal{L}}^1$.

In this section we show that the following spectral multiplier theorem holds for Hardy spaces associated with semigroups satisfying the Davies-Gaffney estimates.

Theorem 4.2. *Let $M > q/4$, $M \in \mathbb{N}$. Assume m be a bounded function defined on $(0, \infty)$ such that for some real number $\alpha > (q + 1)/2$ and any nonzero function $\eta \in C_c^\infty(2^{-1}, 2)$ the condition (1.18) holds. Then there exists a constant $C > 0$*

such that

$$\|m(\mathcal{L})a\|_{H^1_{\mathcal{L}}} \leq C \quad \text{for every } (1, 2, 2M)\text{-atom } a \text{ for the space } H^1_{\mathcal{L}}. \quad (4.3)$$

Fix $\varepsilon > 0$ and $M > q/4$, $M \in \mathbb{N}$. We say that a function \tilde{a} is a $(1, 2, M, \varepsilon)$ -molecule associated to \mathcal{L} if there exist a function $\tilde{b} \in \mathcal{D}(\mathcal{L}^M)$ and a ball $B = B_d(y_0, r)$ such that

$$\tilde{a} = \mathcal{L}^M \tilde{b}; \quad (4.4)$$

$$\|(r^2 \mathcal{L})^k \tilde{b}\|_{L^2(U_j B)} \leq r^{2M} 2^{-j\varepsilon} V(y_0, 2^j r)^{-1/2} \quad (4.5)$$

for $k = 0, 1, \dots, M$, $j = 0, 1, 2, \dots$, where $U_0 = B$, $U_j(B) = B_d(y_0, 2^j r) \setminus B_d(y_0, 2^{j-1} r)$ for $j \geq 1$.

It was proved in [36, Corollary 5.2] that every $(1, 2, M, \varepsilon)$ -molecule \tilde{a} belongs to $H^1_{\mathcal{L}}$ and

$$\|\tilde{a}\|_{H^1_{\mathcal{L}}} \leq C_{\varepsilon, M}. \quad (4.6)$$

Of course the condition (1.18) is invariant under the change of variable $\lambda \mapsto \lambda^s$ in multipliers. Hence (4.3) will be established if we have proved the following proposition for $\sqrt{\mathcal{L}}$.

Proposition 4.7. *Assume that m satisfies (1.18). Fix $M > q/4$, $M \in \mathbb{N}$. Then there exists $\varepsilon > 0$ such that for every $(1, 2, 2M)$ -atom a for $H^1_{\mathcal{L}}$ the function*

$$\tilde{a}(x) = m(\sqrt{\mathcal{L}})a(x)$$

is a multiple of $(1, 2, M, \varepsilon)$ -molecule. The multiple constant is independent of a .

Proof. Let a be a $(1, 2, 2M)$ -atom for $H^1_{\mathcal{L}}$ and let b and $B = B_d(y_0, r)$ be as in (1.7)–(1.9). Set $\tilde{b} = m(\sqrt{\mathcal{L}})\mathcal{L}^M b$. Clearly, $\tilde{a} = \mathcal{L}^M \tilde{b}$. In order to complete the proof of the proposition it suffices to verify (4.5). To do this we need the following lemma.

Lemma 4.8. *Let $\gamma > 1/2$, $\beta > 0$. Then there exists a constant $C > 0$ such that for every even function $F \in W^{2, \gamma + \beta/2}(\mathbb{R})$ and every $g \in L^2(\Omega)$, $\text{supp } g \subset B_d(y_0, r)$, we have*

$$\int_{d(x, y_0) > 2r} |F(2^{-j} \sqrt{\mathcal{L}})g(x)|^2 \left(\frac{d(x, y_0)}{r} \right)^\beta d\mu(x) \leq C (r 2^j)^{-\beta} \|F\|_{W^{2, \gamma + \beta/2}}^2 \|g\|_{L^2(\Omega)}^2$$

for $j \in \mathbb{Z}$.

Proof of the lemma. The lemma seems to be well-known. For the convenience of the reader we provide a proof. To this end we borrow methods from [44]. Since F is even,

$$F(2^{-j} \sqrt{\mathcal{L}})g = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(\xi) \cos(2^{-j} \xi \sqrt{\mathcal{L}})g \, d\xi,$$

where $\hat{F} = \mathcal{F}F$ is the Fourier transform of F . The Davies-Gaffney estimates (4.1) imply the finite speed propagation of the wave equation $\mathcal{L}u + u_{tt} = 0$ (see, e.g., [45], [10]), which means that there exists a constant $C' > 0$ that

$$\text{supp } \cos(2^{-j} \xi \sqrt{\mathcal{L}})g \subset B_d(y_0, r + C' 2^{-j} |\xi|).$$

Hence,

$$\begin{aligned}
 & \left(\int_{d(x,y_0) > 2r} |F(2^{-j}\sqrt{\mathcal{L}})g(x)|^2 \left(\frac{d(x,y_0)}{r} \right)^\beta d\mu(x) \right)^{1/2} \\
 &= \left(\int_{d(x,y_0) > 2r} \left| \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(\xi) \cos(2^{-j}\xi\sqrt{\mathcal{L}})g(x) d\xi \right|^2 \left(\frac{d(x,y_0)}{r} \right)^\beta d\mu(x) \right)^{1/2} \\
 &\leq C \int_{C2^{-j}|\xi| > 2r} \left(\int_{2r < d(x,y_0) < C2^{-j}|\xi|} |\hat{F}(\xi)|^2 |\cos(2^{-j}\xi\sqrt{\mathcal{L}})g(x)|^2 \frac{d(x,y_0)^\beta}{r^\beta} d\mu(x) \right)^{1/2} d\xi \\
 &\leq C \int_{\mathbb{R}} |\hat{F}(\xi)| \left(\frac{2^{-j}|\xi|}{r} \right)^{\beta/2} \|g\|_{L^2(\Omega)} d\xi \\
 &\leq C'' (r2^j)^{-\beta/2} \|F\|_{W^{2,\gamma+\beta/2}} \|g\|_{L^2(\Omega)}.
 \end{aligned}$$

□

We are now in a position to complete the proof of Proposition 4.7. Fix $\varepsilon > 0$ and $\gamma > 1/2$ such that $\gamma + \varepsilon + q/2 = \alpha$. Set $\beta = q + 2\varepsilon$. Then $\gamma + \beta/2 = \alpha$. Let $j_0 = -\log_2 r$. For an integer number k , $0 \leq k \leq M$, write

$$\begin{aligned}
 (r^2\mathcal{L})^k \tilde{b} &= r^{2k} \sum_{j \geq j_0} \psi(2^{-j}\sqrt{\mathcal{L}})m(\sqrt{\mathcal{L}})\mathcal{L}^{k+M}b + r^{2k} \sum_{j < j_0} \psi(2^{-j}\sqrt{\mathcal{L}})\mathcal{L}^M m(\sqrt{\mathcal{L}})\mathcal{L}^k b \\
 &= r^{2k} \sum_{j \geq j_0} \psi(2^{-j}\sqrt{\mathcal{L}})m(\sqrt{\mathcal{L}})g_1 + r^{2k} \sum_{j < j_0} \psi(2^{-j}\sqrt{\mathcal{L}})\mathcal{L}^M m(\sqrt{\mathcal{L}})g_2,
 \end{aligned} \tag{4.9}$$

where $g_1 = \mathcal{L}^{k+M}b$, $g_2 = \mathcal{L}^k b$. Since a is a $(1, 2, 2M)$ -atom for \mathcal{L} associated with $B = B_d(y_0, r)$ and b (see (1.7)-(1.9)), we have

$$\|g_1\|_{L^2(\Omega)} \leq r^{2M-2k} \mu(B)^{-1/2}, \quad \|g_2\|_{L^2(\Omega)} \leq r^{4M-2k} \mu(B)^{-1/2}. \tag{4.10}$$

Put

$$F_j(\lambda) = \begin{cases} m(2^j\lambda)\psi(\lambda) & \text{for } j \geq j_0; \\ 2^{2Mj}m(2^j\lambda)\lambda^{2M}\psi(\lambda) & \text{for } j < j_0; \end{cases} \tag{4.11}$$

and extend each F_j to the even function. Clearly,

$$\|F_j\|_{W^{2,\alpha}(\mathbb{R})} \leq \begin{cases} C & \text{for } j \geq j_0; \\ C2^{2Mj} & \text{for } j < j_0. \end{cases} \tag{4.12}$$

Using Lemma 4.8 combined with (4.9) – (4.12), we get

$$\begin{aligned} & \left(\int_{d(x,y_0) > 2r} |(r^2 \mathcal{L})^k \tilde{b}(x)|^2 \frac{d(x,y_0)^\beta}{r^\beta} d\mu(x) \right)^{1/2} \\ & \leq Cr^{2k} \sum_{j \geq j_0} (r2^j)^{-\beta/2} \|F_j\|_{W^{2,\alpha}(\mathbb{R})} \|g_1\|_{L^2(\Omega)} \\ & \quad + Cr^{2k} \sum_{j < j_0} (r2^j)^{-\beta/2} \|F_j\|_{W^{2,\alpha}(\mathbb{R})} \|g_2\|_{L^2(\Omega)} \\ & \leq Cr^{2M} \mu(B)^{-1/2}. \end{aligned} \tag{4.13}$$

Moreover,

$$\begin{aligned} \|(r^2 \mathcal{L})^k \tilde{b}\|_{L^2(\Omega)} &= \|r^{2k} m(\sqrt{L}) \mathcal{L}^{k+M} b\|_{L^2(\Omega)} \\ &\leq Cr^{2k} \|m\|_{L^\infty(\mathbb{R})} \|g_1\|_{L^2(\Omega)} \leq Cr^{2M} \mu(B)^{-1/2}. \end{aligned} \tag{4.14}$$

Let $j \in \mathbb{Z}, j \geq 0$. Applying (4.13) and (4.14) we obtain

$$\begin{aligned} \|(r^2 \mathcal{L})^k \tilde{b}\|_{L^2(U_j(B))}^2 &\leq C \int_{U_j(B)} |(r^2 \mathcal{L})^k \tilde{b}(x)|^2 \left(1 + \frac{d(x,y_0)}{r} \right)^\beta 2^{-j\beta} d\mu(x) \\ &\leq Cr^{4M} 2^{-j\beta} \mu(B)^{-1} \\ &\leq C'' r^{4M} 2^{-2j\epsilon} V(y_0, 2^j r)^{-1}, \end{aligned} \tag{4.15}$$

where in the last inequality we have used (1.2). □

5. REMARKS

1. Assume that $-L$ generates a semigroup with Gaussian bounds. If we additionally assume that the space (Ω, d, μ) is such that for every $y \in \Omega$ there exists $\kappa = \kappa(y)$ and $c = c(y) > 0$ such that

$$\mu(B_d(y, s)) \geq cs^\kappa \text{ for } s > 1, \tag{5.1}$$

then the multiplier operator $m(L)$ (see (1.14)) extends uniquely to a bounded operator on H_L^1 . To see this we define the space \mathfrak{T} of test functions in the following way: a function g belongs to \mathfrak{T} if there exist $t > 0$, a ball $B_d(y, r)$, and a function $\zeta \in L^\infty(\Omega)$ such that $\text{supp } \zeta \subset B_d(y, r)$ and $g = T_t \zeta$. We say that g_n converge to g_0 in \mathfrak{T} if there exist $t > 0$, a ball $B = B_d(y, r)$, and functions ζ_n such that $\text{supp } \zeta_n \subset B, \sup \|\zeta_n\|_\infty < \infty, g_n = T_t \zeta_n$, and $\zeta_n(x) \rightarrow \zeta_0(x)$ a.e. Clearly, $\mathfrak{T} \subset L^p(\Omega)$ for every $1 \leq p \leq \infty$. One can easily prove that if $f \in L^1(\Omega)$ is such that $\int_\Omega fg d\mu = 0$ for every $g \in \mathfrak{T}$, then $f = 0$.

Lemma 5.2. *Assume that m satisfies (1.13). Then $\bar{m}(L)$ maps continuously \mathfrak{T} into $L^\infty(\Omega)$.*

Proof. Recall that $\alpha > q/2$. Observe that there exists a constant $C > 0$ such that for every function $n(\lambda)$ such that $n \in W^{\infty,\alpha}(\mathbb{R}), \text{supp } n \subset (2^{j-1}, 2^{j+1})$, one has

$$|n(L)(x, y)| \leq C \mu(B_d(y, 2^{-j/2}))^{-1} \|n(2^j \cdot)\|_{W^{\infty,\alpha}(\mathbb{R})}. \tag{5.3}$$

It suffices to prove (5.3) for $j = 0$ and then use the scaling argument. Set $\xi(\lambda) = e^{\lambda}n(\lambda)$. Then $\|\xi\|_{W^{\infty,\alpha}(\mathbb{R})} \sim \|n\|_{W^{\infty,\alpha}(\mathbb{R})}$. Hence, by Theorem 2.1,

$$|n(L)(x, y)| \leq \int |\xi(L)(x, z)T_1(z, y)| d\mu(z) \leq C\mu(B_d(y, 1))^{-1}\|n\|_{W^{\infty,\alpha}(\mathbb{R})}.$$

Assume that $g \in \mathfrak{F}$. Then there are $t > 0$, $B = B_d(y_0, r)$, and a bounded function ζ such that $g = T_t\zeta$, $\text{supp } \zeta \subset B$. Of course we can assume that $r > 1$. Let ψ be as in (2.9). Let $n_j(\lambda) = m(\lambda)\psi(2^{-j}\lambda)e^{-t\lambda}$. Then

$$\bar{m}(L)g(x) = \sum_j \bar{n}_j(L)\zeta(x) = \sum_j \int \bar{n}_j(L)(x, y)\zeta(y) d\mu(y).$$

Set $j_0 = -2\log_2 r$. Obviously $\|\bar{n}_j(2^j \cdot)\|_{W^{\infty,\alpha}(\mathbb{R})} \leq Ce^{-c2^{jt}}$. Thus

$$\sum_j |\bar{n}_j(L)(x, y)| \leq C \sum_{j \leq j_0} \mu(B_d(y, 2^{-j/2}))^{-1} + C \sum_{j > j_0} e^{-ct2^j} \mu(B_d(y, 2^{-j/2}))^{-1}. \tag{5.4}$$

By (5.1) there exist $c(y_0)$ and $\kappa = \kappa(y_0) > 0$ such that for $y \in B_d(y_0, r)$ and $j \leq j_0$ we have

$$\mu(B_d(y, 2^{-j/2})) \sim \mu(B_d(y_0, 2^{-j/2})) \geq c(y_0)2^{-j\kappa/2}. \tag{5.5}$$

On the other hand, by (1.2), for $y \in B_d(y_0, r)$ and $j > j_0$, we have

$$\mu(B_d(y_0, r)) \sim \mu(B_d(y, r)) \leq C(2^{j/2}r)^q \mu(B_d(y, 2^{-j/2})). \tag{5.6}$$

From (5.4)-(5.6) we conclude that there exists a constant $C(y_0, r)$ such that

$$\sum_j |\bar{n}_j(L)(x, y)| \leq C(y_0, r) \quad \text{for } x \in \Omega \text{ and } y \in B_d(y_0, r).$$

□

We are now in a position to define the action of $m(L)$ on the space $L^1(\Omega)$ in the weak (distributional) sense by putting

$$\langle m(L)f, g \rangle = \int_{\Omega} f(x)\overline{\bar{m}(L)g(x)} d\mu(x).$$

Let us observe that $m(L)$ is uniquely defined on H_L^1 . Indeed, if $f = \sum_j \lambda_j a_j$, where a_j are $(1, 2, 1)$ -atoms, $\lambda_j \in \mathbb{C}$, $\sum |\lambda_j| \sim \|f\|_{H_L^1} < \infty$ then, by Theorem 1.12 and Lemma 5.2, for every $g \in \mathfrak{F}$ we have

$$\begin{aligned} \langle m(L)f, g \rangle &= \int_{\Omega} \left(\sum_j \lambda_j a_j(x) \right) \overline{\bar{m}(L)g(x)} d\mu(x) \\ &= \sum_j \lambda_j \int_{\Omega} a_j(x) \overline{\bar{m}(L)g(x)} d\mu(x) \\ &= \sum_j \lambda_j \int_{\Omega} m(L)a_j(x) \overline{g(x)} d\mu(x) \\ &= \int_{\Omega} \left(\sum_j \lambda_j m(L)a_j(x) \right) \overline{g(x)} d\mu(x). \end{aligned}$$

Since $\sum_j \lambda_j m(L)a_j$ belongs to $L^1(\Omega)$, we obtain that $m(L)f = \sum_j \lambda_j m(L)a_j$, which gives the required uniqueness. Obviously, $\|m(L)f\|_{H^1_L} \leq C\|f\|_{H^1_L}$.

2. One of distinguished examples of semigroups of linear operators with Gaussian bounds is that generated by a Schrödinger operator $-A = \Delta - V$ on \mathbb{R}^d , where V is a nonnegative potential such that $V \in L^1_{loc}(\mathbb{R}^d)$. By the Feynman-Kac formula the integral kernels $p_t(x, y)$ of the semigroup e^{-tA} satisfy

$$0 \leq p_t(x, y) \leq (4\pi t)^{-d/2} \exp(-|x - y|^2/4t).$$

Clearly, considering $(\mathbb{R}^d, d(x, y) = |x - y|, dx)$ as a space of homogeneous type, we have that (1.2) and (5.1) hold with $q = d$. Thus, as a corollary of Theorem 1.12, we obtain that any bounded function $m : (0, \infty) \rightarrow \mathbb{C}$ which satisfies (1.13) with $\alpha > d/2$ is an H^1_A spectral multiplier for A .

We would like to remark that the space H^1_A admits also characterization by means of maximal function from the semigroup e^{-tA} (see [36]). Using arguments similar to those of [19] one can prove the spectral multiplier theorem on Hardy spaces associate with the Schrödinger operators by applying both atomic and maximal function characterizations.

Another molecule decomposition of Hardy space H^1 associated with semigroups generated by Schrödinger operators was communicated to us by Jacek Zienkiewicz [48]. These decompositions also lead to multiplier theorems.

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