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BALANCED BILINEAR FORMS AND FINITENESS PROPERTIES FOR INCIDENCE COALGEBRAS OVER A FIELD

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Abstract. Let $C = IC(X)$ be the incidence coalgebra of an intervally finite partially ordered set X over a field. We investigate finiteness properties of C. We determine all C^*-balanced bilinear forms on C, and we deduce that C is left (or right) quasi-co-Frobenius if and only if C is left (or right) co-Frobenius, and this is equivalent to the order relation on X being the equality.

1. Introduction and preliminaries

Let k be a field. The aim of this note is to study finiteness conditions for a well-known class of k-coalgebras, namely for incidence k-coalgebras of intervally finite partially ordered sets. Incidence coalgebras were defined in Sweedler's book [6] in 1969. It was explained by Joni and Rota [4] in 1979 how incidence coalgebras provide a good framework for interpreting several combinatorial problems in terms of coalgebras. The aim of this note is to study finiteness conditions for incidence coalgebras.

Let $(X, \leq)$ be an intervally finite partially ordered set, i.e. the interval $[x, y] = \{z| \ x \leq z \leq y\}$ is finite for any $x \leq y$. Let $C = IC(X)$ be the incidence k-coalgebra of X over the field k. This is the k-linear space with basis $\{e_{x,y}|x, y \in X, x \leq y\}$, and comultiplication Δ and counit ε defined by

$$\Delta(e_{x,y}) = \sum_{x\leq z\leq y} e_{x,z} \otimes e_{z,y}$$

$$\varepsilon(e_{x,y}) = \delta_{x,y}$$

for any $x, y \in X$ with $x \leq y$, where by $\delta_{x,y}$ we denote Kronecker's delta.
For any $x \in X$ we denote by $x^+ = \{y| \ x \leq y\}$, the right cone of x, and by $x^- = \{y| \ y \leq x\}$, the left cone of x.

We recall that if C is a coalgebra, then C is a left C, right C-bicomodule with coactions defined by the comultiplication, and this makes C a left C^*, right C^*-bimodule. Denote by $c^* \cdot c$ and $c \cdot c^*$ the left and the right actions of $c^* \in C^*$ on $c \in C$.

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We consider some finiteness properties for coalgebras. A coalgebra C is called

• Right semiperfect if the category of right C-comodules has enough projectives.
• Right quasi-co-Frobenius if C can be embedded as a right C^*-module in a free right C^*-module.
• Right co-Frobenius if C can be embedded as a right C^*-module in C^*.

It is known that the following implications hold

$$C \text{ is right co-Frobenius } \Rightarrow C \text{ is right quasi-co-Frobenius } \Rightarrow C \text{ is right semiperfect,}$$

but the converse implications are not true in general. Left side versions of these three concepts are similarly defined, and in general they are not left-right symmetric. We are interested in studying when an incidence coalgebra satisfies one of these finiteness properties. In order to do it, we study the coradical filtration, the injective envelopes of the simple comodules, and the balanced bilinear forms on incidence coalgebras. The description of the injective envelopes of the simple comodules was done by Simson [5], who used it to show that $IC(X)$ is left (respectively right) semiperfect if and only if x^+ (respectively x^-) is finite for any $x \in X$. We include this result in Section 2, for completeness. In the same section we give a description of the subcoalgebras of an incidence coalgebra, and we show that the wedge of two finite dimensional subcoalgebras of an incidence coalgebra is also finite dimensional, thus showing that the incidence coalgebra is locally finite. In Section 3 we look at the quasi-co-Frobenius and co-Frobenius properties. We use equivalent characterizations of these properties by using balanced bilinear forms. More precisely, a coalgebra C is right quasi-co-Frobenius if and only if there exists a family $(B_i)_{i\in I}$ of C^*-balanced bilinear forms on C such that for any non-zero $c \in C$ there is some $i \in I$ with $B_i(c, C) \neq 0$. A coalgebra C is right co-Frobenius if and only if there exists a C^*-balanced bilinear form on C which is right nondegenerate. We determine all C^*-balanced bilinear forms on an incidence coalgebra $C = IC(X)$, and as an application we find all situations when C is right (or left) quasi-co-Frobenius or co-Frobenius. More precisely we show that C is right (or left) quasi-co-Frobenius if and only if C is left (or right) co-Frobenius, and this happens if and only if the order relation on X is just the equality, in which case C is just the grouplike coalgebra on the set X.

For basic definitions and notations on coalgebras we refer the reader to [6] and [2].

2. Some finiteness conditions on incidence coalgebras

Let $(X, \leq)$ be an intervally finite partially ordered set and let $C = IC(X)$ be the incidence k-coalgebra of X. For any $x, y \in X$ such that $x \leq y$ we denote by $p_{x,y}$ the element of C^* such that $p_{x,y}(e_{u,v}) = \delta_{x,u}\delta_{y,v}$ for any $u, v \in X$ with $u \leq v$. Using the comultiplication formula of C, we obtain that

$$p_{x,y} \cdot e_{u,v} = \begin{cases} 0, \text{ if } y \neq v \\ 0, \text{ if } y = v \text{ and } x \notin [u, v] \\ e_{u,x}, \text{ if } y = v \text{ and } u \leq x \leq v \end{cases} \tag{1}$$

and

$$e_{u,v} \cdot p_{x,y} = \begin{cases} 0, \text{ if } x \neq u \\ 0, \text{ if } x = u \text{ and } y \notin [u,v] \\ e_{y,v}, \text{ if } x = u \text{ and } u \leq y \leq v \end{cases} \tag{2}$$

We use these formulas to describe all subcoalgebras of C.

Proposition 2.1. *Any non-zero subcoalgebra D of C has a basis consisting of elements of the form $e_{x,y}$. Moreover, there is a bijective correspondence between the non-zero subcoalgebras D of C and the non-empty subsets $\mathcal{D}$ of $\{(x,y)|\ x,y \in X, x \leq y\}$ with the property that for any $(x,y) \in \mathcal{D}$ and any $x \leq u \leq v \leq y$ we also have that $(u,v) \in \mathcal{D}$. This correspondence associates to a subcoalgebra D the set $\mathcal{D} = \{(x,y)|\ e_{x,y} \in D\}$, and to the set $\mathcal{D}$ the subcoalgebra $D =< e_{x,y}|\ (x,y) \in \mathcal{D} >$.*

Proof: Let D be a non-zero subcoalgebra of C, and let $c = \sum_{1\leq i\leq n} \alpha_i e_{x_i,y_i}$ be an element of D, represented with all α_i's non-zero. Then by using equations (1) and (2) we have that $p_{y_j,y_j} \cdot c \cdot p_{x_j,x_j} = \alpha_j e_{x_j,y_j}$ for any $1 \leq j \leq n$, so $e_{x_j,y_j} \in D$ for any $1 \leq j \leq n$. Hence D is spanned by elements of the form $e_{x,y}$. Moreover, if we consider the set $\mathcal{D} = \{(x,y)|\ e_{x,y} \in D\}$, which is a basis of D, let $(x,y) \in \mathcal{D}$, and let u,v such that $x \leq u \leq v \leq y$. Then $p_{v,y} \cdot e_{x,y} \cdot p_{x,u} = e_{u,v}$, so we must have $(u,v) \in \mathcal{D}$.
Conversely, it is clear that for any non-empty set $\mathcal{D}$ with the property from the statement, the subspace $D =< e_{x,y}|\ (x,y) \in \mathcal{D} >$ is a non-zero subcoalgebra of C. It is easy to see that the mappings $D \mapsto \mathcal{D}$ and $\mathcal{D} \mapsto D$ are inverse to each other. ∎

Remark 2.2. *If we make the convention that the empty set is a basis of a zero vector space, then Proposition 2.1 can be regarded as a bijective correspondence between the subcoalgebras of C and the subsets of $\{(x,y)|\ x,y \in X, x \leq y\}$ with the indicated property. The empty set is associated to the zero subcoalgebra through this correspondence. In the sequel we regard the correspondence of Proposition 2.1 extended in this way.*

Corollary 2.3. *The simple subcoalgebras of C are the 1-dimensional subspaces of the form $< e_{x,x} >$ with $x \in X$. In particular C is a pointed coalgebra and the coradical C_0 of C is the subspace spanned by all $e_{x,x}, x \in X$.*

We recall that if V and W are two linear subspaces of a coalgebra C, then the wedge $V \wedge W$ is the linear subspace of C defined by $V \wedge W = \Delta^{-1}(V \otimes C + C \otimes W)$. The following result is useful for determining the coradical filtration of C.

Proposition 2.4. *Let A and B be subcoalgebras of C, and let $\mathcal{A}$ and $\mathcal{B}$ the associated subsets of $\{(x,y)|\ x,y \in X, x \leq y\}$ as in Proposition 2.1. Then $A \wedge B$ is the subcoalgebra associated to the set $\{(x,y)|\ \forall z \in [x,y]$ either $(x,z) \in \mathcal{A}$ or $(z,y) \in \mathcal{B}\}$.*

Proof: It is known that $A \wedge B$ is a subcoalgebra of C. Let $\mathcal{C}$ be its associated subset of $\{(x,y)|\ x,y \in X, x \leq y\}$. Then for $x \leq y$ we have that $(x,y) \in \mathcal{C}$ if and only if $\Delta(e_{x,y}) = \sum_{x\leq z\leq y} e_{x,z} \otimes e_{z,y} \in A \otimes C + C \otimes B$, and this is equivalent to the fact that for any $z \in [x,y]$ we have either $e_{x,z} \in A$ or $e_{z,y} \in B$, which is the same to $(x,z) \in \mathcal{A}$ or $(z,y) \in \mathcal{B}$. ∎

We recall from [3] that a coalgebra C is called locally finite if for any finite dimensional subspaces V and W of C, their wedge $V \wedge W$ is finite dimensional. Since by the Fundamental Theorem of Coalgebras any finite dimensional subspace is contained in a finite dimensional subcoalgebra, we have that C is locally finite if and only if for any finite dimensional subcoalgebras A and B, their wedge $A \wedge B$ is finite dimensional.

Corollary 2.5. *C is a locally finite coalgebra.*

Proof: Let A and B be finite dimensional subcoalgebras of C. Let $\mathcal{A}$ and $\mathcal{B}$ be the subsets of $\{(x,y)|\ x,y \in X, x \leq y\}$ associated to the subcoalgebras A and B. Then by Proposition 2.4 the subcoalgebra $A \wedge B$ is associated to the subset $\mathcal{C} = \{(x,y)|\ \forall z \in [x,y] \text{ either } (x,z) \in \mathcal{A} \text{ or } (z,y) \in \mathcal{B}\}$. We show that $\mathcal{C} \subseteq \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}'$, where

$$\mathcal{C}' = \{(x,y)|\ x \leq y,\ (x,x) \in \mathcal{A} \text{ and } (y,y) \in \mathcal{B}\}$$

Indeed, let $(x,y) \in \mathcal{C}$. If $(x,y) \in \mathcal{A} \cup \mathcal{B}$, then clearly $(x,y) \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}'$. Assume that $(x,y) \notin \mathcal{A} \cup \mathcal{B}$. Then since $(x,y) \in \mathcal{C}$ and $(x,y) \notin \mathcal{B}$, we must have $(x,x) \in \mathcal{A}$. Similarly, since $(x,y) \in \mathcal{C}$ and $(x,y) \notin \mathcal{A}$, we must have $(y,y) \in \mathcal{B}$. Thus $(x,y) \in \mathcal{C}'$ and $\mathcal{C} \subseteq \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}'$. Since $\mathcal{A}$ and B are finite, we also have that $\mathcal{C}'$ is finite, and we obtain that $\mathcal{C}$ is also finite, so $A \wedge B$ is finite dimensional. ∎

If $x, y \in X$ and $x < y$, we say that the interval $[x,y]$ has length n if for any sequence $x = u_0 < u_1 < \ldots < u_m = y$ we have $m \leq n$, and there exists such a sequence for $m = n$. We write $\text{length}([x,y]) = n$. We make the convention that $\text{length}([x,x]) = 0$ for any $x \in X$. Now we are able to describe the coradical filtration of C.

Corollary 2.6. *The $(n+1)$-th term C_n of the coradical filtration of C is*

$$C_n =< e_{x,y}|\ \text{length}([x,y]) \leq n >$$

Proof: It follows immediately by induction using Corollary 2.3 and Proposition 2.4. ∎

The following result and its corollary appear in [5, Lemma 5.1]. We include them for completeness.

Proposition 2.7. *The injective envelope of the simple right C-comodule $S_x =< e_{x,x} >$ is $E_x =< e_{x,y}|\ y \in x^+ >$.*

Proof: It is clear that E_x is a right C-subcomodule of C, and that $C = \oplus_{x \in X} E_x$. This shows that each E_x is an injective right C-comodule. Since $S_x \subseteq E_x$, we have that an injective envelope $E(S_x)$ is a direct summand of E_x, so $E_x = E(S_x) \oplus H$ for some right C-comodule H. If $H \neq 0$, then it contains a simple C-comodule different from S_x, and then so does E_x, a contradiction. Hence $E_x = E(S_x)$. ∎

Corollary 2.8. *C is left semiperfect if and only if x^+ is finite for any $x \in X$. Similarly, C is right semiperfect if and only if x^- is finite for any $x \in X$.*

3. Balanced bilinear forms on incidence coalgebras

If C is a coalgebra, a bilinear form $B : C \times C \to k$ is called C^*-balanced if $B(c \cdot c^*, d) = B(c, c^* \cdot d)$ for any $c, d \in C$, $c^* \in C^*$. This is equivalent to

$$\sum B(c_2, d)c_1 = \sum B(c, d_1)d_2 \tag{3}$$

for any $c, d \in C$.

The aim of this section is to determine all C^*-balanced bilinear forms on C, where $C = IC(X)$ is the incidence coalgebra of an intervally finite partially ordered set X over the field k.

Theorem 3.1. *Let $(X, \leq)$ be an intervally finite partially ordered set and let $C = IC(X)$ be the incidence coalgebra of X over the field k. Let $\mathcal{P}_X$ be the set of all pairs (x, y) such that $x \leq y$, x is a minimal element of X, y is a maximal element of X, and $x^+ = y^- = [x, y]$. Then the following assertions hold.*
(i) A bilinear form $B : C \times C \to k$ is C^*-balanced if and only if there exists a family $(\alpha_{x,y})_{(x,y)\in\mathcal{P}_X}$ of scalars such that

$$B(e_{x,y}, e_{z,t}) = \begin{cases} \alpha_{z,y}, & \text{if } (z, y) \in \mathcal{P}_X \text{ and } x = t, \\ 0, & \text{otherwise} \end{cases}$$

for any $x, y, z, t \in X$ with $x \leq y$ and $z \leq t$.
(ii) The C^*-balanced bilinear forms on C are in a bijective correspondence to $k^{\mathcal{P}_X}$.

Proof: (i) Let B be a bilinear form on C. Using equation (3) we have that B is C^*-balanced if and only if

$$\sum_{x\leq u\leq y} B(e_{u,y}, e_{z,t})e_{x,u} = \sum_{z\leq v\leq t} B(e_{x,y}, e_{z,v})e_{v,t} \tag{4}$$

for any $x \leq y$ and $z \leq t$. We see that given some $x \leq y$ and $z \leq t$, there exist elements of the standard basis of C appearing in both the left hand side and the right hand side of equation (4) if and only if $z \leq x \leq t$ and $x \leq t \leq y$, and in this case the only common basis element is $e_{x,t}$. Therefore B is C^*-balanced if and only if

- For any $z \leq x \leq t \leq y$ we have that

$$B(e_{t,y}, e_{z,t}) = B(e_{x,y}, e_{z,x}) \tag{5}$$

$$B(e_{u,y}, e_{z,t}) = 0 \text{ for any } x \leq u \leq y, u \neq t \tag{6}$$

$$B(e_{x,y}, e_{z,v}) = 0 \text{ for any } z \leq v \leq t, v \neq x \tag{7}$$

- If $x \leq y$ and $z \leq t$, but it is not true that $z \leq x \leq t \leq y$, then

$$B(e_{u,y}, e_{z,t}) = 0 \text{ for any } x \leq u \leq y \tag{8}$$

$$B(e_{x,y}, e_{z,v}) = 0 \text{ for any } z \leq v \leq t \tag{9}$$

Assume now that B is a C^*-balanced bilinear form on C. Let $z, y \in X$ such that it is not true that $z \leq y$. Then for any x, t such that $x \leq y$ and $z \leq t$ we have

that $B(e_{x,y}, e_{z,t}) = 0$ (we get this by taking $u = x$ in equation (8)).
Let now $z, y \in X$ such that $z \le y$. By equation (5) we see that $B(e_{x,y}, e_{z,x})$ takes is the same for any x such that $z \le x \le y$. Hence there exists $\alpha_{z,y} \in k$ with $B(e_{x,y}, e_{z,x}) = \alpha_{z,y}$ for any $x \in [z, y]$. By equation (6) for $u = x$ we get that for any $x \le y$ and $z \le t$ with $t \ne x$ we have that $B(e_{x,y}, e_{z,t}) = 0$.
Therefore there exists a family $(\alpha_{x,y})_{(x,y)\in\mathcal{P}_X}$ of scalars such that

$$B(e_{x,y}, e_{z,t}) = \begin{cases} \alpha_{z,y}, & \text{if } z \le y \text{ and } x = t, \\ 0, & \text{otherwise} \end{cases}$$

Now let $z, y \in X$ be such that $z \le y$.
If z is not a minimal element, i.e. there exists $x \in X$ with $x < z$, then take $t = y$ and apply equation (8) for $u = y$. We obtain that $B(e_{y,y}, e_{z,y}) = 0$, which shows that $\alpha_{z,y} = 0$. Thus $\alpha_{z,y} = 0$ whenever z is not minimal.
On the other hand, if y is not a maximal element of X, i.e. there is $t \in X$ with $y < t$, take $x = z$ and apply equation (9) for $v = z$. We obtain that $B(e_{z,y}, e_{z,z}) = 0$, showing that $\alpha_{z,y} = 0$. Thus $\alpha_{z,y} = 0$ whenever y is not maximal.

Assume now that $z \le y$, z is minimal and y is maximal.
If there exists $x \le y$ such that x is not comparable to z, then take $t = y$ and apply equation (8) for $u = y$. We get that $B(e_{y,y}, e_{z,y}) = 0$, which means that $\alpha_{z,y} = 0$. Thus $\alpha_{z,y} = 0$ whenever $y^- \ne [z, y]$.
On the other hand, if there exists $z \le t$ such that t is not comparable to y, then take $x = z$ and apply equation (9) for $v = z$. We get that $B(e_{z,y}, e_{z,z}) = 0$, so $\alpha_{z,y} = 0$. Thus $\alpha_{z,y} = 0$ whenever $z^+ \ne [z, y]$.

We conclude that $\alpha_{z,y}$ may be non-zero only when z is minimal, y is maximal, and $y^- = z^+ = [z, y]$, i.e. when $(z, y) \in \mathcal{P}_X$.

Conversely, assume that for a bilinear form B there is a family $(\alpha_{x,y})_{(x,y)\in\mathcal{P}_X}$ of scalars such that

$$B(e_{x,y}, e_{z,t}) = \begin{cases} \alpha_{z,y}, & \text{if } (z, y) \in \mathcal{P}_X \text{ and } x = t, \\ 0, & \text{otherwise} \end{cases}$$

for any $x, y, z, t \in X$ with $x \le y$ and $z \le t$. Then (5), (6) and (7) are obviously satisfied. In order to check (8) and (9), let $x \le y$ and $z \le t$ such that it is not true that $z \le x \le t \le y$.
If $(z, y) \notin \mathcal{P}_X$, then it is clear that (8) and (9) are satisfied (both sides are zero).
If $(z, y) \in \mathcal{P}_X$, then we must have $z \le t < x \le y$, and then (8) is satisfied since $u \ne t$ whenever $x \le u \le y$, while (9) is satisfied since $v \ne x$ whenever $z \le v \le t$.

(ii) It follows directly from the description given by (i). ∎

Corollary 3.2. *Let $C = IC(X)$ be the incidence coalgebra of an intervally finite partially ordered set X. The following assertions are equivalent.*
(1) C is right quasi-co-Frobenius.
(2) C is left quasi-co-Frobenius.

(3) C is right co-Frobenius.
(4) C is left co-Frobenius.
(5) C is cosemisimple.
(6) The order relation on X is the equality.

Proof: (5) $\Rightarrow$ (3) $\Rightarrow$ (1) are clear.
(6) $\Rightarrow$ (5) C is isomorphic to the group-like coalgebra of the set X, so it is cosemisimple.
(1) $\Rightarrow$ (6) If C is right quasi-co-Frobenius then there exists a family $(B_i)_{i\in I}$ of C^*-balanced bilinear forms on C such that for any non-zero $c \in C$ there is some $i \in I$ with $B_i(c, C) \neq 0$. If the order relation on X were not equality, then let $z, y \in X$ such that $z < y$. Then for any C^*-balanced bilinear form B on C, $B(e_{z,z}, e_{u,v})$ may be non-zero only if $v = z$ and $(u, z) \in \mathcal{P}_X$. But z is not maximal, so (u, z) can not lie in $\mathcal{P}_X$. Hence $B(e_{z,z}, C) = 0$, a contradiction. We conclude that the order relation on X is the equality.
(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) are similar. ∎

Remark 3.3. *Let n be a positive integer and let $X = \{1, \ldots, n\}$, with the order relation inherited from the usual order of integers. Then $IC(X)$ is the matrix-like coalgebra $L(n, k)$ considered by Beattie and Rose in* [1, Section 3]*, and* [1, Proposition 3.2] *follows from our Theorem 3.1. As showed in* [1]*, it is interesting that the Taft n^2-dimensional Hopf algebra T_{n^2} is a union of subcoalgebras isomorphic to $L(n, k)$.*

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