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SOME REMARKS ON MORITA THEORY, AZUMAYA ALGEBRAS AND CENTER OF AN ALGEBRA IN BRAIDED MONOIDAL CATEGORIES

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Abstract. We study the left and the right center of an algebra in a braided monoidal category $\mathcal{C}$. We characterize them by specializing a general result and point out why in symmetric monoidal categories one does not distinguish the two centers. We review Morita theorems for a monoidal category $\mathcal{D}$ and analyze necessary conditions for the associativity of the tensor product over an algebra in $\mathcal{D}$. The central part of the Morita theorems in braided categories is also discussed. We review the notion of Azumaya algebra in $\mathcal{C}$ distinguishing left and right faithfully projective objects and study the relation between the two.

1. Introduction

The well known Morita Theorems address the question of the equivalence of categories of modules over algebras. The induced equivalence relation between algebras in the case of Azumaya algebras underlies the construction of the Brauer group. Pareigis generalized the Morita Theorems to arbitrary monoidal categories in [9] and the construction of the Brauer group to symmetric monoidal categories in [10]. The latter was a big step in the line of generalizations of the Brauer group. This group was initially considered for the category of vector spaces in order to classify finite dimensional central division algebras and its computation is related to number theory, algebraic geometry and K-Theory. This concept reaches its highest level of generalization in [11] – in the context of braided monoidal categories. For an overview of different constructions of Brauer groups see [1].

In the present paper we review (relying on [9] and [11]) the Morita Theorems in monoidal categories as well as the notion of Azumaya algebra analyzing some issues whose close study was omitted in the mentioned references. We prove that two Morita equivalent algebras in a braided monoidal category $\mathcal{C}$ have isomorphic centers and that Azumaya algebras are central, generalizing the results from [10].

The center of an algebra A in a symmetric monoidal category was introduced in [9, 10] as the inner hom-object ${}_A[A, A]_A$. We study this object in $\mathcal{C}$ and prove that it is a categorical limit, which embeds in two equalizers. Instead of applying Yoneda Lemma as in [10], we work with braided diagrams. We also study the left and the right center of an algebra in $\mathcal{C}$ and obtain their characterization as a particular instance of a more general fact (Proposition 3.3 with its symmetric version, Corollary 3.4). Our results generalize [11, Proposition 4.4], since we do not impose the separability condition on A (nor abelianity of $\mathcal{C}$), and we show that the isomorphism in question preserves the algebra structure. We explain why in symmetric categories one does not distinguish the left and the right center of an algebra, as one does in braided categories.

Developing Morita Theory in a monoidal category $\mathcal{C}$, Pareigis defined in [9] the concept of faithfully projective object in ${}_A\mathcal{C}$, where $A \in \mathcal{C}$ is an algebra. Symmetrically, in [11] the authors study the notion of faithfully projective object in $\mathcal{C}_A$ and work with the Morita Theorems from [9] together with their right hand-side counterparts. For $A{=}I$, where $I{\in}\,\mathcal{C}$ is the unit object, an object faithfully projective in ${}_I\mathcal{C}$ (respectively in $\mathcal{C}_I$) we call left (respectively right) faithfully projective object. In the construction of the Brauer group of a symmetric monoidal category in [10], an Azumaya algebra is an algebra A which is (left) faithfully projective and such that there is a certain algebra isomorphism $F\colon A \otimes A^{op} \longrightarrow [A, A]$. When extending the construction of the Brauer group from symmetric to general braided monoidal categories in [11], beside F there appears another algebra isomorphism $G\colon A^{op} \otimes A \longrightarrow [A, A]^{op}$ (in symmetric monoidal categories F and G coincide, as we show in Section 5). Though, it is not made explicit which kind of faithful projectiveness is used.

We verify in Theorem 5.1 that an Azumaya algebra A in a braided monoidal category is both left and right faithfully projective. More precisely, in [11, Theorem 3.1] the authors characterize Azumaya algebras in terms of two equivalence functors $\mathcal{F}$ and $\mathcal{G}$, without providing an explicit proof. We prove that the functor $\mathcal{F}$ gives rise to the algebra isomorphism F and it implies that A is right faithfully projective, whereas the algebra isomorphism G and the left faithful projectiveness of A come out from the functor $\mathcal{G}$. We also show that the two equivalence functors $\mathcal{F}$ and $\mathcal{G}$ coincide in a symmetric category $\mathcal{D}$ for the same reason for which the left and the right center of an algebra are not distinguished in $\mathcal{D}$.

Moreover, we address the issue of the relation between the notions of left and right faithfully projective objects in a braided monoidal category and prove the equivalence of the two notions. In view of this, the statement of [11, Theorem 3.1] remains valid.

For the Morita Theorems in [9] one needs the associativity of the tensor product over an algebra. We investigate necessary conditions for this associativity in Lemma 4.3. Our results lead to assumptions in Morita Theorems which slightly differ from those in [9].

The contents of the paper are organized as follows. In the second section we recall some definitions and basic facts for a braided monoidal category $\mathcal{C}$. In Section 3 we study and characterize the left and the right center of an algebra in $\mathcal{C}$. Section 4 is devoted to the study of necessary conditions for the associativity of the tensor product over algebras and how these affect the assumptions in Morita Theorems in monoidal categories. We prove that an object in $\mathcal{C}$ is left faithfully projective if and only if it is right faithfully projective and discus the central part of the Morita Theorems. In the last section we reconsider the definition of an Azumaya algebra in $\mathcal{C}$.

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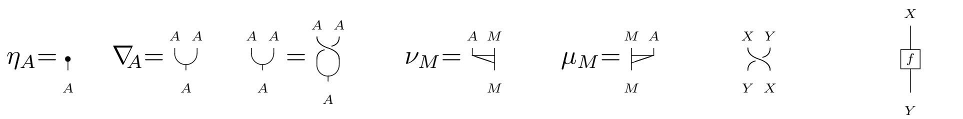
2. Preliminaries

We assume the reader is familiar with the general theory of (braided monoidal) categories. For reference we recommend [7, 4]. Throughout $\mathcal{C}$ will denote a braided monoidal category (with braiding Φ and unit I) unless otherwise specified. We set the notation for the structures of a (commutative) algebra A, an A-module M, a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ and for the braiding $\Phi_{X,Y}$ for $X, Y \in \mathcal{C}$:

Algebra — unit, multiplic., comm. alg.; Module — left, right; Braiding; Morphism

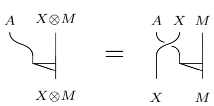
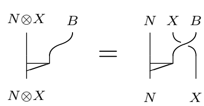


2.1. Observe that for algebras A, B in a monoidal category $\mathcal{C}$ the categories ${}_A\mathcal{C}$ of left A-modules and $\mathcal{C}_B$ of right B-modules are right and left module categories over $\mathcal{C}$ respectively. For the definition of a module category over a monoidal category and their functors we refer to [6]. For brevity, left and right module categories over $\mathcal{C}$ will be called *left and right $\mathcal{C}$-categories* and the corresponding strict module functors we will call shortly *$\mathcal{C}$-functors*. A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ between two right $\mathcal{C}$-functors $\mathcal{F}, \mathcal{G} : {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ is called a *right $\mathcal{C}$-natural transformation* if for all $M \in {}_A\mathcal{C}$ and $X \in \mathcal{C}$ the following diagram commutes

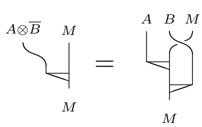
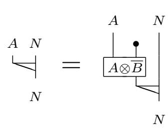
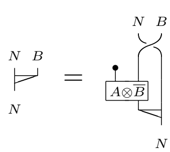
$$\begin{tikzcd}
\mathcal{F}(M \otimes X) \arrow[r, "\varphi(M \otimes X)"] \arrow[d, "\cong"'] & \mathcal{G}(M \otimes X) \arrow[d, "\cong"] \\
\mathcal{F}(M) \otimes X \arrow[r, "\varphi(M) \otimes X"] & \mathcal{G}(M) \otimes X.
\end{tikzcd}$$

Similarly, one defines a left $\mathcal{C}$-natural transformation.

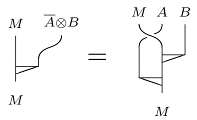
When $\mathcal{C}$ is braided we can consider ${}_A\mathcal{C}$ as a left and $\mathcal{C}_B$ as a right $\mathcal{C}$-category respectively with the corresponding structures:

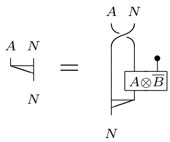

(1) and

(2)

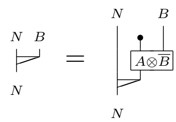
for $M \in {}_A\mathcal{C}, N \in \mathcal{C}_B$ and $X \in \mathcal{C}$. The category of A-B-bimodules ${}_A\mathcal{C}_B$ is isomorphic (as a right $\mathcal{C}$-category) to ${}_{A\otimes\overline{B}}\mathcal{C}$ through correspondences


; (3)

and


For $K \in {}_A\mathcal{C}_B$ and $X \in \mathcal{C}$ the object $K \otimes X$ inherits its left A-module structure from that on K, whereas its right B-module structure is given as in (2). Analogously, ${}_A\mathcal{C}_B$ is isomorphic (as a left $\mathcal{C}$-category) to $\mathcal{C}_{\overline{A} \otimes B}$ via



; (4)


and


where ${}_A\mathcal{C}_B$ is considered a left $\mathcal{C}$-category through (1). Hence, ${}_{A \otimes \overline{B}}\mathcal{C}$ is isomorphic to $\mathcal{C}_{\overline{A} \otimes B}$. However, this is neither left nor right $\mathcal{C}$-isomorphism of categories, *unless the category $\mathcal{C}$ is symmetric.*

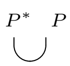
In Sections 4 and 5 both right $\mathcal{C}$-functors ${}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ and left $\mathcal{C}$-functors $\mathcal{C}_A \longrightarrow \mathcal{C}_B$ we will call shortly *$\mathcal{C}$-functors.*

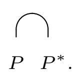
2.2. A monoidal category $\mathcal{C}$ is called *right closed* if the functor $- \otimes M : \mathcal{C} \to \mathcal{C}$ has a right adjoint, denoted by $[M, -]$ and called *inner hom-functor*, for all $M \in \mathcal{C}$. For $N \in \mathcal{C}$, the object $[M, N]$ is called *inner hom-object.* The counit of the adjunction evaluated at N is denoted by $ev_{M,N} : [M, N] \otimes M \to N$. The pair $([M, N], ev_{M,N})$ satisfies the following universal property: *for any morphism $f : T \otimes M \to N$ there is a unique morphism $g : T \to [M, N]$ such that $f = ev_{M,N}(g \otimes M)$.* If $f : N \longrightarrow N'$ is a morphism in $\mathcal{C}$, then $[M, f] : [M, N] \longrightarrow [M, N']$ is the unique morphism such that $ev_{M,N'}([M, f] \otimes M) = f \; ev_{M,N}$. The unit of the adjunction $\alpha : N \longrightarrow [M, N \otimes M]$ is induced by $ev_{M,N \otimes M}(\alpha \otimes M) = id_{N \otimes M}$.

A monoidal category $\mathcal{C}$ is called *left closed* if the functor $M \otimes - : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\{M, -\}$ for all $M \in \mathcal{C}$. The counit of this adjunction evaluated at $N \in \mathcal{C}$ is denoted by $\overline{ev}_{M,N} : M \otimes \{M, N\} \to N$. The properties analogous to those in the previous paragraph hold for $\{M, N\}$. We denote the unit of the adjunction $(M \otimes -, \{M, -\})$ by $\overline{\alpha} : N \longrightarrow \{M, M \otimes N\}$. The category $\mathcal{C}$ is called *closed* if it is both left and right closed. When there is no confusion, we will suppress the indexes for the counit ev.

When $\mathcal{C}$ is braided we have a natural isomorphism $\mathcal{C}(M \otimes P, N) \longrightarrow \mathcal{C}(P \otimes M, N)$, $f \mapsto f \circ (\Phi_{P,M})^{\pm 1}$. This clearly implies that $\mathcal{C}$ is right closed if and only if it is left closed. By uniqueness of the (right) adjoint functors, the functors $\{M, -\}$ and $[M, -]$ are naturally equivalent, then for all $N \in \mathcal{C}$ it is $\{M, N\} \cong [M, N]$. For the counits of the two adjunctions, putting $P = [M, N]$, we obtain $\overline{ev} = ev \circ (\Phi_{[M,N],M})^{\mp 1}$. Throughout we will write $[-, -]$ for both types of inner hom-bifunctors, unless otherwise specified. The difference will be clear from the context.

2.3. Let P be an object in a monoidal category $\mathcal{C}$. An object $P^* \in \mathcal{C}$ together with a morphism $e_P \colon P^* \otimes P \longrightarrow I$ is called a *left dual object* for P if there exists a morphism $d_P \colon I \longrightarrow P \otimes P^*$ in $\mathcal{C}$ such that $(P \otimes e_P)(d_P \otimes P) = id_P$ and $(e_P \otimes P^*)(P^* \otimes d_P) = id_{P^*}$. In braided diagrams the morphisms e_P and d_P are denoted by:

$e_P =$ 

and $d_P =$ 

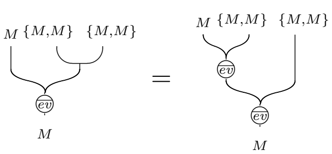
Symmetrically, one defines a right dual object. Dual objects are unique up to isomorphism. In a braided category left and right dual objects are isomorphic.

2.4. On inner hom-objects in a right closed monoidal category there is an associative pre-multiplication $\varphi_{M,Y,Z} : [Y,Z]\otimes[M,Y] \longrightarrow [M,Z]$ and a unital morphism $\eta_{[Y,Y]} : I \longrightarrow [Y,Y]$ such that

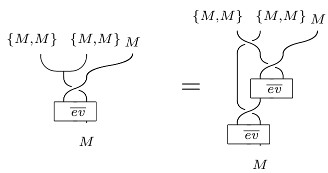
$$\varphi_{M,Y,Y}(\eta_{[Y,Y]} \otimes [M,Y]) = \lambda_{[M,Y]} \qquad \text{and} \qquad \varphi_{M,M,Y}([M,Y] \otimes \eta_{[M,M]}) = \rho_{[M,Y]}$$

where λ and ρ are unity constraints. The morphisms $\varphi_{M,Y,Z}$ and $\eta_{[Y,Y]}$ are given via the universal properties of $([M,Z], ev)$ and $([Y,Y], ev)$, respectively, by $ev_{M,Z}(\varphi_{M,Y,Z}\otimes M) = ev_{Y,Z}([Y,Z]\otimes ev_{M,Y})$ and $ev_{Y,Y}(\eta_{[Y,Y]}\otimes Y) = \lambda_Y$. Consequently, for each $M \in \mathcal{C}$, the object $[M,M]$ is equipped with an algebra structure via $\varphi_{M,M,M}$ and $\eta_{[M,M]}$. The analogous statement we have for the algebra $\{M,M\}$.

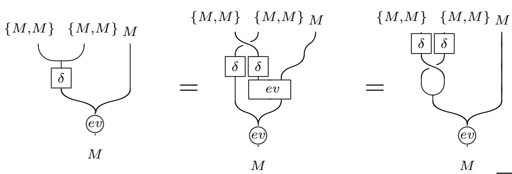
2.5. For any algebra A in a braided monoidal category $\mathcal{C}$ we adopt the convention to consider the multiplication on the opposite algebra $\overline{A}$ to be given by $\nabla_A\Phi_{A,A}$. This implies that from the two possibilities for the relation between the counits of the adjunctions (see 2.2) one has to consider $\overline{ev} = ev\Phi$. Indeed, observe that by the left version of the multiplication in 2.4 we have



which is equivalent to



Choosing $\overline{ev} = ev\Phi$ and using an isomorphism $\delta : \{M,M\} \longrightarrow [M,M]$ in order to pass from $\{M,M\}$ to $[M,M]$, we further obtain



which recovers the known algebra isomorphism $\{M,M\} \cong \overline{[M,M]}$ ([11, Corollary 2.2]). If we had chosen the other relation between $\overline{ev}$ and ev, we would have to impose symmetricity conditions on the braiding ($\Phi_{\{M,M\},\{M,M\}} = (\Phi_{\{M,M\},\{M,M\}})^{-1}$) in order to recover the mentioned isomorphism.

2.6. Let $\mathcal{C}$ be a monoidal category with coequalizers. A *tensor product over an algebra* A in $\mathcal{C}$ of a right A-module M and a left A-module N is the coequalizer

$$M\otimes A\otimes N \underset{M\otimes\nu_N}{\overset{\mu_M\otimes N}{\rightrightarrows}} M\otimes N \xrightarrow{\Pi_{M,N}} M\otimes_A N.$$

Consider a right A-linear morphism $f : M \longrightarrow M'$ and a left A-linear morphism $g : N \longrightarrow N'$. Then $f\otimes g : M\otimes N \longrightarrow M'\otimes N'$ induces a morphism $f\otimes_A g : M\otimes_A N \longrightarrow M'\otimes_A N'$.

3. The left and the right center of an algebra

The center of an algebra in a symmetric monoidal category was defined in [10]. If a category is braided, one distinguishes two counterparts of the center of an algebra. This was first recognized in [11] where the notion of the left and the right center was introduced for a separable algebra in an abelian braided monoidal category. In [6] this notion was studied for an algebra in a semisimple abelian ribbon category, while [3] treats it in terms of a maximality property for idempotents of symmetric special Frobenius algebras. In this section we study the left and the right center of an algebra A in a closed braided monoidal category $\mathcal{C}$. We work in a closed category in order to guarantee the existence of inner hom-objects. Before defining the left and the right center of A, we recall some concepts.

Take $M, N \in {}_A\mathcal{C}$ and let ${}_A[M,N]$ be the inner hom-object obtained from the adjunction $M \otimes - : \mathcal{C} \longrightarrow {}_A\mathcal{C}$. Pareigis proved in [8] that ${}_A[M,N]$ is the equalizer

$$ {}_A[M,N] \xrightarrow{\lambda} [M,N] \underset{l_2}{\overset{l_1}{\rightrightarrows}} [A\otimes M, N], \tag{5}$$

where $l_1, l_2 : [M,N] \longrightarrow [A\otimes M, N]$ are induced by $\overline{ev}(A\otimes M\otimes l_1) = \overline{ev}(\nu_M \otimes [M,N])$ and $\overline{ev}(A\otimes M \otimes l_2) = \nu_N(A\otimes \overline{ev}_{M,N})$. Then (5) yields:

$$ \overline{ev}(\nu_M \otimes \lambda) = \nu_N(A \otimes \overline{ev}(M\otimes \lambda)). \tag{6}$$

The functor ${}_A[M,-]$ acts on morphisms as follows. If $f : N \longrightarrow N'$ is a morphism in ${}_A\mathcal{C}$, then ${}_A[M,f]$: ${}_A[M,N] \longrightarrow {}_A[M,N']$ is induced by $[M,f]$: $[M,N] \longrightarrow [M,N']$ as a morphism on an equalizer via $\lambda_{N'} \circ {}_A[M,f] = [M,f]\lambda_N$. That $[M,f]$ induces ${}_A[M,f]$ is provided by A-linearity of f. Analogously, for $M, N \in \mathcal{C}_A$ considering the right adjoint functor for $- \otimes M : \mathcal{C} \longrightarrow \mathcal{C}_A$, one gets that

$$ [M,N]_A \xrightarrow{\rho} [M,N] \underset{r_2}{\overset{r_1}{\rightrightarrows}} [M\otimes A, N] \tag{7}$$

is the equalizer, with $r_1, r_2 : [M,N] \longrightarrow [M\otimes A, N]$ induced by $ev(r_1 \otimes M \otimes A) = ev([M,N]\otimes \mu_M)$ and $ev(r_2 \otimes M\otimes A) = \mu_N(ev_{M,N}\otimes A)$. A morphism $f : N \longrightarrow N'$ in $\mathcal{C}_A$ induces a morphism $[M,f]_A : [M,N]_A \longrightarrow [M,N']_A$.

For another algebra $B \in \mathcal{C}$ and $M \in {}_A\mathcal{C}_B$, consider the functors: $M\otimes -, - \otimes M : \mathcal{C} \longrightarrow {}_A\mathcal{C}_B$ with their right adjoint ${}_A[M,-]_B$. For $X \in \mathcal{C}$ we view $M\otimes X$ and $X \otimes M$ as A-B-bimodules as in 2.1. Pareigis defined in [10] the center of an algebra A in a *symmetric* monoidal category as the object ${}_A[A,A]_A$ (if it exists). We will study the properties of this object in $\mathcal{C}$ further below.

Definition 3.1. *The left (right) center of an algebra A in a closed braided monoidal category $\mathcal{C}$ is the object ${}_{A\otimes\overline{A}}[A,A]$ (resp. $[A,A]_{\overline{A}\otimes A}$) in $\mathcal{C}$.*

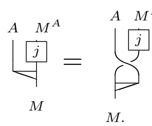
The algebra $A \otimes \overline{A}$ is called *the enveloping algebra of* A and it is denoted by A^e. The algebra $\overline{A} \otimes A$ we denote by $A^{\overline{e}}$. We regard A as a left A^e-module via (3) and as a right $A^{\overline{e}}$-module via (4). We give another interpretation of the left and the right center of an algebra A.

Definition 3.2. *For $M \in {}_{A^e}\mathcal{C}$, let M^A denote the equalizer:*

$$M^A \xrightarrow{j} M \xrightarrow{\overline{\alpha}_A} [A, A\otimes M] \underset{[A,\eta_A\otimes\overline{A}\otimes M]}{\overset{[A,A\otimes\eta_{\overline{A}}\otimes M]}{\rightrightarrows}} [A, A\otimes\overline{A}\otimes M] \xrightarrow{[A,\nu]} [A, M],$$

where $\overline{\alpha}_A$ is the unit of the adjunction $(A\otimes -, [A,-])$ and ν the structure morphism for M.

Note that $\nu(A\otimes\eta_{\overline{A}}\otimes M) = \mu_l$ and $\nu(\eta_A\otimes\overline{A}\otimes M) = \mu_r\Phi_{A,M}$ by (3), where μ_l and μ_r stand for the left- and the right A-action on M respectively. Using the identities from 2.2 we compute $\overline{ev}(A\otimes[A,\ \theta]\overline{\alpha}_A) = \theta\overline{ev}(A\otimes\overline{\alpha}_A) = \theta$ for any $\theta : A\otimes M \longrightarrow M$. Substituting θ by μ_l and $\mu_r\Phi_{A,M}$, we obtain a short description of the property satisfied by the equalizer M^A:



(8)

If $f : M \longrightarrow N$ is a morphism in ${}_{A^e}\mathcal{C}$, then $f^A : M^A \longrightarrow N^A$ is induced as a morphism on an equalizer via $j_N f^A = f j_M$.

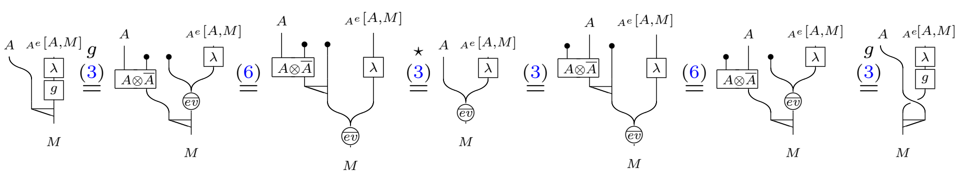
The following result is proved in author's Ph.D. thesis.

Proposition 3.3. *Let A be an algebra in $\mathcal{C}$. Then the functors ${}_{A^e}[A,-]$ and $(-)^A$ are isomorphic.*

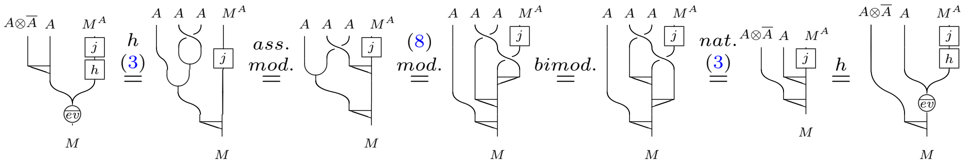
Proof. Take $M \in {}_{A^e}\mathcal{C}$. We define the morphisms $g : [A, M] \longrightarrow M$ and $h : M \longrightarrow [A, M]$ by $g := \overline{ev}(\eta_A\otimes[A, M])$ and $\overline{ev}(A\otimes h) = \mu_l$. We view A as a left A^e-module by (3). In the diagram

$$\begin{tikzcd}
{}_{A^e}[A,M] \arrow[r, "\lambda"] \arrow[d, shift right, "\overline{g}"'] \arrow[dr, phantom, "\boxed{1}"] & {[A,M]} \arrow[r, shift left, "l_1"] \arrow[r, shift right, "l_2"'] \arrow[d, shift right, "g"'] & {[(A\otimes\overline{A})\otimes A, M]} \\
M^A \arrow[r, "j"'] \arrow[u, shift right, "\overline{h}"'] & M \arrow[r, shift left, "\mu_l"] \arrow[r, shift right, "\mu_r\Phi_{A,M}"'] \arrow[u, shift right, "h"'] & {[A,M]}
\end{tikzcd}$$

we are going to prove that the composition $g\lambda$ induces $\overline{g}$ and hj induces $\overline{h}$ so that the square $\langle 1\rangle$ commutes both with left and right arrows. We proceed with $\overline{g}$:



By the equalizer property of M^A this means that $g\lambda$ induces $\overline{g}$ so that $j\overline{g} = g\lambda$. Let us now prove that $\overline{h}$ is well defined. We compute:



Thus hj induces $\overline{h}$ so that $\lambda\overline{h} = hj$. We next prove that $\overline{g}$ and $\overline{h}$ are inverse to each other. We have: $\overline{ev}(A \otimes hg\lambda) = \mu_l(A \otimes g\lambda) = \overline{ev}(A \otimes \lambda)$, where the latter equality holds by the identity $\star$. This yields $\lambda = hg\lambda = hj\overline{g} = \lambda\overline{h}\overline{g}$. Since λ is a monomorphism, we get $\overline{h}\overline{g} = id_{_{A^e}[A,M]}$. On the other hand, it is $gh = \overline{ev}(\eta_A \otimes h) = \mu_l(\eta_A \otimes M) = id_M$. Composing this from the right with j and using the fact that it is a monomorphism, similarly as above we obtain $\overline{g}\overline{h} = id_{M^A}$.

We finally prove that the isomorphism ${}_{A^e}[A, M] \cong M^A$ is natural. Let $N \in {}_{A^e}\mathcal{C}$ and let $f : M \longrightarrow N$ be a morphism in ${}_{A^e}\mathcal{C}$. Observe the following diagram

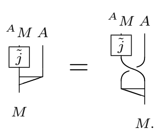
$$\begin{tikzcd}
{}_{A^e}[A,M] \arrow[rr, "\lambda_M"] \arrow[dr, "\overline{g}_M"] \arrow[dd, "{}_{A^e}[A,f]"'] & & {[A,M]} \arrow[dd, "g_M"] \\
& M^A \arrow[dr, "j_M"] \arrow[dd, "f^A"'] & \\
{}_{A^e}[A,N] \arrow[dr, "\overline{g}_N"'] \arrow[dd, "\lambda_N"'] & & M \arrow[dd, "f"] \\
& N^A \arrow[dr, "j_N"'] & \\
{[A,N]} \arrow[rr, "g_N"'] & & N.
\end{tikzcd}$$

The upper and the lower triangle in this picture commute by the definitions of $\overline{g}_M$ and $\overline{g}_N$, respectively. The right inner trapeze commutes by the definition of f^A. The outer diagram commutes as well, it can be seen as a juxtaposition

$$\begin{tikzcd}
{}_{A^e}[A,M] \arrow[r, "\lambda_M"] \arrow[d, "{}_{A^e}[A,f]"'] & {[A,M]} \arrow[r, "g_M"] \arrow[d, "{[A,f]}"'] & M \arrow[d, "f"] \\
{}_{A^e}[A,N] \arrow[r, "\lambda_N"] & {[A,N]} \arrow[r, "g_N"] & N
\end{tikzcd}$$

where the left inner parallelogram commutes by the definition of the morphism ${}_{A^e}[A, f]$, and the right one by the definitions of g and $[A, f]$. Now a diagram chasing argument applied to the previous diagram provides $j_N\overline{g}_N{}_{A^e}[A, f] = j_N f^A\overline{g}_M$, which, since j_N is a monomorphism, yields that $\overline{g}$ is a natural transformation. $\square$

Symmetrically to M^A one defines ${}^AM$ for $M \in \mathcal{C}_{A^{\overline{e}}}$ using the adjunction $(- \otimes A, [A, -])$. The short description of its equalizer property takes the form:


(9)

Similarly as in Proposition 3.3 one has that $[A,-]_{A^{\overline{e}}}$ and ${}^{A}(-)$ are isomorphic functors.

Corollary 3.4. *We have the following isomorphisms of commutative algebras:*

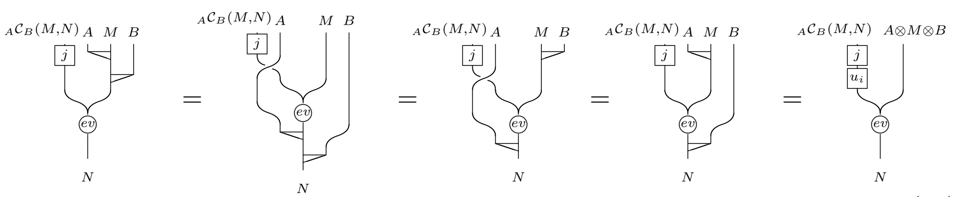
$$ {}_{A\otimes\overline{A}}[A,A] \cong A^A \qquad \text{and} \qquad [A,A]_{\overline{A}\otimes A} \cong {}^{A}A. $$

Proof. Using multiplication $\nabla_{[A,A]}$ in $[A,A]$ (see 2.4) and associativity of ∇_A, one proves that the morphism $h: A \longrightarrow [A,A]$ from the above proof is an algebra morphism. We define $\overline{\nabla}: {}_{A^e}[A,A] \otimes {}_{A^e}[A,A] \longrightarrow {}_{A^e}[A,A]$ by $\lambda\overline{\nabla}{=}\nabla_{[A,A]}(\lambda\otimes\lambda)$. Then ${}_{A^e}[A,A]$ is a subalgebra of $[A,A]$ and $\lambda: {}_{A^e}[A,A] \longrightarrow [A,A]$ is an algebra morphism. Similarly, $j: A^A \longrightarrow A$ is an algebra embedding. Then $\lambda\overline{\nabla}(\overline{h}\otimes\overline{h}) = \nabla_{[A,A]}(\lambda\overline{h}\otimes\lambda\overline{h}) \stackrel{\langle 1\rangle}{=} \nabla_{[A,A]}(hj\otimes hj) = h\nabla_A(j\otimes j) = hj\nabla_{A^A} \stackrel{\langle 1\rangle}{=} \lambda\overline{h}\nabla_{A^A}$, where $\langle 1\rangle$ denotes the square diagram in the proof of Proposition 3.3. Since λ is a monomorphism, the latter yields that $\overline{h}: A^A \longrightarrow {}_{A^e}[A,A]$ is an isomorphism of algebras. To prove commutativity, compose the identity (8) with $j\otimes A^A$ from above (in the braided diagram orientation), setting $M=A$. Since j is an algebra morphism and a monomorphism, we obtain that A^A is commutative. The other isomorphism is proved similarly. $\square$

The isomorphisms in the above Corollary were proved in [11, Proposition 4.4] on the level of objects for separable algebras (in abelian categories). We obtained our result as a special case of Proposition 3.3 and its right hand-side version. In view of (8) and (9) our left and right centers coincide with right and left centers of an algebra in a semisimple abelian ribbon category from [6].

3.1. The object ${}_A[A,A]_A$. Before we compare objects ${}_{A\otimes\overline{A}}[A,A]$, $[A,A]_{\overline{A}\otimes A}$ and ${}_A[A,A]_A$ in a symmetric category, let us study the latter object in a closed braided monoidal category $\mathcal{C}$. Being an inner hom-object, ${}_A[A,A]_A$ is the representing object of the two functors ${}_A\mathcal{C}_A(A\otimes -, A), {}_A\mathcal{C}_A(-\otimes A, A): \mathcal{C} \longrightarrow \mathcal{S}et$, where $\mathcal{S}et$ is the category of sets. Similarly as we saw that the inner hom-objects ${}_A[M,N]$ and $[M,N]_B$ are equalizers, we will prove that the inner hom-object ${}_A[M,N]_B$ is a (categorical) limit. This will provide us with some identities relating ${}_A[A,A]_A$ with objects ${}_A[A,A], [A,A]_A$ and $[A,A]$.

Let us first remark that for A-B-bimodules M and N the set of A-B-bimodule morphisms ${}_A\mathcal{C}_B(M,N)$ can be described using (braided) diagrams in the (braided monoidal and closed) category of sets as follows:



(10)

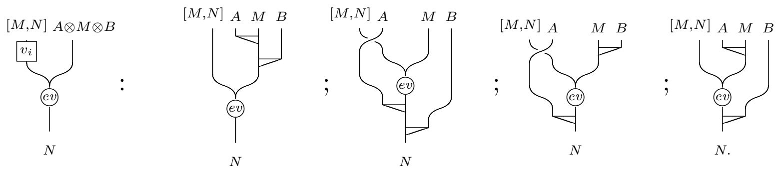
Here $j: {}_A\mathcal{C}_B(M,N) \longrightarrow \mathcal{C}(M,N)$ is an embedding and $u_i: \mathcal{C}(M,N) \longrightarrow \mathcal{C}(A\otimes M\otimes B, N)$ for $i{=}1,2,3,4$ are the morphisms defined via the universal property of

$(\mathcal{C}(A\otimes M\otimes B,N),ev)$ as the first, second, third and fourth diagram, respectively, in (10) show (omitting the morphism j). We use the same notation ev for both counit in $\mathcal{C}$ and in $\mathcal{S}et$. In $\mathcal{S}et$ this means that ${}_A\mathcal{C}_B(M,N)$ is the limit of the u_i's.

Lemma 3.5. *Let M and N be A-B-bimodules and suppose that $\mathcal{C}$ is complete. The object ${}_A[M,N]_B$ is the limit*

$$
{}_A[M,N]_B \xrightarrow{\iota} [M,N] \overset{v_1}{\underset{v_4}{\overset{v_2}{\underset{v_3}{\rightrightarrows\rightrightarrows}}}} [A\otimes M\otimes B,N], \tag{11}
$$

where v_1,v_2,v_3,v_4 are given by the universal property of $([A\otimes M\otimes B,N],ev)$ via the equality of the first diagram with the second, third, fourth and fifth diagram in the below picture, respectively:



(12)

Proof. We define morphisms $u'_i:\mathcal{C}(X\otimes M,N)\longrightarrow\mathcal{C}(A\otimes X\otimes M\otimes B,N), i=1,2,3,4,$ as the compositions

$$
u'_i:\quad \mathcal{C}(X\otimes M,N)\xrightarrow{u_i}\mathcal{C}(A\otimes X\otimes M\otimes B,N)\xrightarrow{\mathcal{C}(\Phi^{-1}_{A,X}\otimes M\otimes B,N)}\mathcal{C}(X\otimes A\otimes M\otimes B,N)
$$

being u_i's the morphisms we defined in (10), where M is substituted by $X\otimes M$. Let $L(M,N)$ be the limit of the morphisms v_1,v_2,v_3,v_4 from the Lemma. In the diagram:

$$
\begin{array}{ccccc}
{}_A\mathcal{C}_B(X\otimes M,N) & \longrightarrow & \mathcal{C}(X\otimes M,N) & \overset{u'_1,\,u'_2,\,u'_3,\,u'_4}{\rightrightarrows\rightrightarrows} & \mathcal{C}(X\otimes A\otimes M\otimes B,N)\\
 & & \cong\big\downarrow & & \cong\big\downarrow\\
\mathcal{C}(X,L(M,N)) & \longrightarrow & \mathcal{C}(X,[M,N]) & \overset{\mathcal{C}(X,v_1),\,\mathcal{C}(X,v_2),\,\mathcal{C}(X,v_3),\,\mathcal{C}(X,v_4)}{\rightrightarrows\rightrightarrows} & \mathcal{C}(X,[A\otimes M\otimes B,N])
\end{array}
$$

the first row is a limit, because $\mathcal{C}(\Phi^{-1}_{A,X}\otimes M\otimes B,N)$ is an isomorphism, and the second row is a limit too, because $\mathcal{C}(X,-)$ as a covariant representable functor preserves limits. Direct check shows that the right rectangle commutes taking into account the four pairs of arrows $(u'_i,\mathcal{C}(X,v_i))$. From the isomorphism of the two limits we obtain that $L(M,-)$ is a right adjoint for the functor $-\otimes M:\mathcal{C}\longrightarrow{}_A\mathcal{C}_B$, hence $L(M,N)\cong{}_A[M,N]_B$ and we have the claim. $\square$

Recall l_1 and l_2 from (5). From (12) is clear that $ev(v_1\otimes A\otimes M\otimes\eta_B)=ev(v_4\otimes A\otimes M\otimes\eta_B)=\overline{ev}(A\otimes M\otimes l_1)\Phi^{-1}_{A\otimes M,[M,N]}$ by naturality and $\overline{ev}=ev\Phi$, and similarly $ev(v_2\otimes A\otimes M\otimes\eta_B)=ev(v_3\otimes A\otimes M\otimes\eta_B)=\overline{ev}(A\otimes M\otimes l_2)\Phi^{-1}_{A\otimes M,[M,N]}$. This implies $\overline{ev}(A\otimes M\otimes l_1\iota) = \overline{ev}(A\otimes M\otimes l_2\iota)$ because of (11). The universal property of $([A\otimes M, N], \overline{ev})$ yields $l_1\iota = l_2\iota$, hence by the equalizer property of ${}_A[M,N]$, the morphism $\iota : {}_A[M,N]_B \longrightarrow [M,N]$ factors through a morphism $\iota_l : {}_A[M,N]_B \longrightarrow {}_A[M,N]$ making the triangle

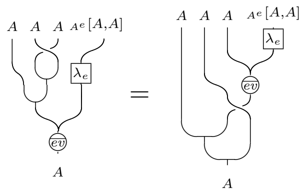
$$\begin{tikzcd} {}_A[M,N]_B \arrow[d, "\iota_l"'] \arrow[dr, "\iota"] & \\ {}_A[M,N] \arrow[r, "\lambda"] & {[M,N]} \end{tikzcd} \tag{13}$$

commutative. Similarly, we obtain that $\iota : {}_A[M,N]_B \longrightarrow [M,N]$ factors through a morphism $\iota_r : {}_A[M,N]_B \longrightarrow [M,N]_B$ so that the triangle

$$\begin{tikzcd} {}_A[M,N]_B \arrow[d, "\iota_r"'] \arrow[dr, "\iota"] & \\ {[M,N]_B} \arrow[r, "\rho"] & {[M,N]} \end{tikzcd} \tag{14}$$

commutes.

On the other hand, for the left center ${}_{A^e}[A,A]$ of A, from (6) and (3) one obtains the identity



After composing the latter equality from above with $A\otimes\eta_A\otimes A\otimes id$, one gets that $\lambda_e : {}_{A^e}[A,A] \longrightarrow [A,A]$ factors through a morphism $\lambda_l : {}_{A^e}[A,A] \longrightarrow {}_A[A,A]$ so that $\lambda\lambda_l = \lambda_e$. Similarly, one proves that $\rho_{\overline{e}} : [A,A]_{A^{\overline{e}}} \longrightarrow [A,A]$ factors through a morphism $\rho_r : [A,A]_{A^{\overline{e}}} \longrightarrow [A,A]_A$ so that $\rho\rho_r = \rho_{\overline{e}}$. Here $\lambda : {}_A[A,A] \longrightarrow [A,A]$ and $\rho : [A,A]_A \longrightarrow [A,A]$ denote the corresponding equalizer monomorphisms.

Lemma 3.6. *For $X \in {}_A\mathcal{C}$ it is ${}_A[A,X] \cong X$. Symmetrically, for $Y \in \mathcal{C}_A$ it is $[A,Y]_A \cong Y$.*

Proof. Define $f : X \longrightarrow [A,X]$ by $\overline{ev}(A\otimes f) = \nu_X$. It induces $\overline{f} : X \longrightarrow {}_A[A,X]$ such that $\lambda\overline{f} = f$, since: $\overline{ev}(\nabla_A\otimes f) = \nu_X(\nabla_A\otimes X) = \nu_X(A\otimes\nu_X) = \nu_X(A\otimes\overline{ev}(A\otimes f))$. Now the morphism $\zeta := \overline{ev}(\eta_A\otimes[A,X])\lambda : {}_A[A,X] \longrightarrow X$ is the inverse of $\overline{f}$. Indeed, $\zeta\overline{f} = \overline{ev}(\eta_A\otimes\lambda\overline{f}) = \overline{ev}(\eta_A\otimes f) = \nu_X(\eta_A\otimes X) = id_X$ and $\overline{ev}(A\otimes(\lambda\overline{f}\zeta))= \overline{ev}(A\otimes f\,\overline{ev}(\eta_A\otimes\lambda))= \nu_X(A\otimes\overline{ev}(\eta_A\otimes\lambda))= \overline{ev}(\nabla_A(A\otimes\eta_A)\otimes\lambda)= \overline{ev}(A\otimes\lambda)$. In the penultimate equality we applied (6). From the universal property of $([A,X],\overline{ev})$ it follows $\lambda\overline{f}\zeta = \lambda$. Since λ is a monomorphism, we get $\overline{f}\zeta = id_{{}_A[A,X]}$. For the second claim one proves that $g : Y \longrightarrow [A,Y]$ defined by $ev(g\otimes A) = \mu_Y$ induces $\overline{g} : Y \longrightarrow [A,Y]_A$ such that $\rho\overline{g} = g$. $\square$

Consider the diagrams

$$\begin{tikzcd}[column sep=small]
 & {}_{A^e}[A,A] \arrow[d, "\lambda_l"'] \arrow[dr, "\lambda_e"] & \\
{}_A[A,A]_A \arrow[r, "\iota_l"] \arrow[dr, "\iota_L"'] & {}_A[A,A] \arrow[r, "\lambda"] & {[A,A]} \\
 & A \arrow[u, "\overline{f}"] \arrow[ur, "f"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
 & {[A,A]_{A^{\overline{e}}}} \arrow[d, "\rho_r"'] \arrow[dr, "\rho_{\overline{e}}"] & \\
{}_A[A,A]_A \arrow[r, "\iota_r"] \arrow[dr, "\iota_R"'] & {[A,A]_A} \arrow[r, "\rho"] & {[A,A]} \\
 & A \arrow[u, "\overline{g}"] \arrow[ur, "g"'] &
\end{tikzcd}
\tag{15}$$

where $\overline{f}$ and $\overline{g}$ are the isomorphisms from the above Lemma with $X{=}Y{=}A$, so that the lower right triangles in the above diagrams commute. The morphisms ι_L and ι_R are defined so that the left triangles commute, and note that the upper right triangles commute, too. Observe that from the definitions of the morphisms f and g from Lemma 3.6 we obtain $\overline{ev}(A\otimes f)=\nabla_A$ and $ev(g\otimes A)=\nabla_A$.

Similarly as in Corollary 3.4, one proves that f and $\lambda: {}_A[A,A]\longrightarrow[A,A]$ are algebra morphisms so that ${}_A[A,A]$ is a subalgebra of $[A,A]$. Being both λ and $f=\lambda\overline{f}$ algebra morphisms and λ being a monomorphism, one obtains that $\overline{f}$ is an isomorphism of algebras. Similarly, so is $\overline{g}$. As for λ, one gets that $\iota_l: {}_A[A,A]_A \longrightarrow {}_A[A,A]$ and $\lambda_l: {}_{A^e}[A,A]\longrightarrow {}_A[A,A]$ are algebra embeddings, hence so are ι_L and λ_e. The same holds for ι_r,ι_R and $\rho_r,\rho_{\overline{e}}$.

In symmetric categories, like for instance in the category of modules over a commutative ring, one works only with one center, without distinguishing its left and right counterparts. The reason is the following.

Proposition 3.7. *Let A and B be algebras in a symmetric category $\mathcal{C}$ and let M and N be A-B-bimodules. The following objects (if they exist) are isomorphic:*

$${}_A[M,N]_B \cong {}_{A\otimes\overline{B}}[M,N] \cong [M,N]_{\overline{A}\otimes B}.$$

Proof. Take $X\in\mathcal{C}$. We consider that the left A- and $A\otimes\overline{B}$-module structures and the right B- and $\overline{A}\otimes B$-module structures on $M\otimes X$ are inherited from those on M via (2). Recall from 2.1 that we have left and right $\mathcal{C}$-isomorphisms of categories ${}_A\mathcal{C}_B\cong {}_{A\otimes\overline{B}}\mathcal{C}\cong\mathcal{C}_{\overline{A}\otimes B}$ since $\mathcal{C}$ is symmetric. This is why the sets in the first row of

$$\begin{array}{ccccc}
{}_A\mathcal{C}_B(M\otimes X,N) & \cong & {}_{A\otimes\overline{B}}\mathcal{C}(M\otimes X,N) & \cong & \mathcal{C}_{\overline{A}\otimes B}(M\otimes X,N)\\
\cong & & \cong & & \cong\\
\mathcal{C}(X,{}_A[M,N]_B) & & \mathcal{C}(X,{}_{A\otimes\overline{B}}[M,N]) & & \mathcal{C}(X,[M,N]_{\overline{A}\otimes B})
\end{array}$$

are isomorphic. The vertical isomorphisms come from the corresponding adjunctions. The claim now follows by uniqueness of right adjoint functors. □

4. Morita theory

In Morita theorems for monoidal categories the associativity of the tensor product of modules over an algebra is needed in the formation of a Morita context, but also for the consistence of certain equivalence functors. We study necessary conditions for this associativity and observe how these affect the definition of a

Morita context as well as the assumptions in Morita theorems. We also study faithfully projective objects making difference between left and right version of the notion and analyze how these are related. In this section $\mathcal{C}$ will denote a monoidal category (with unit I) unless otherwise specified.

4.1. Associativity of the tensor product of modules over an algebra. In the literature there have been discrepancies about necessary conditions for the associativity of the (co)tensor product over a (co)algebra. The issue for cotensor products in $\mathcal{C}$ was studied in the author's Ph.D. thesis. Even though the tensor product over an algebra is a dual concept, it gives rise to conditions which are not dual to those obtained in the cotensor product case. For the sake of Morita theorems we give here a complete proof of the associativity of the tensor product over an algebra in $\mathcal{C}$. We will first need:

Definition 4.1. *Let A and B be algebras in $\mathcal{C}$. An object $M \in {}_B\mathcal{C}_A$ is called A-*coflat *if for all algebras $R, S \in \mathcal{C}$ and objects $L \in {}_A\mathcal{C}_R$ the coequalizer $M \otimes_A L \in {}_B\mathcal{C}_R$ exists and if the natural morphism $M \otimes_A (L \otimes P) \longrightarrow (M \otimes_A L) \otimes P$ in ${}_B\mathcal{C}_S$, induced by the associativity of the tensor product, is an isomorphism for every $P \in \mathcal{C}_S$. Symmetrically we define that M is B-*coflat*. If it is both B- and A-coflat, we say that it is* bicoflat*.*

We have that left adjoint functors preserve coequalizers, [5]. In particular, in a closed category the tensor functor preserves coequalizers.

Remark 4.2. *If $\mathcal{C}$ is a category with coequalizers and it is left (right) closed, then any B-A-bimodule M is B-coflat (A-coflat). Indeed, for any $X \in {}_R\mathcal{C}, L \in \mathcal{C}_B, N \in {}_A\mathcal{C}$ and $Y \in \mathcal{C}_T$ it is $X \otimes (L \otimes_B M) \cong (X \otimes L) \otimes_B M$ (resp. $(M \otimes_A N) \otimes Y \cong M \otimes_A (N \otimes Y)$) in $\mathcal{C}$, because $X \otimes -$ (resp. $- \otimes Y$) preserves coequalizers. These isomorphisms are shown to be in ${}_R\mathcal{C}_A$ and ${}_B\mathcal{C}_T$, respectively. Clearly, if $\mathcal{C}$ is closed, then M is bicoflat. Recall that a braided category is closed if and only if it closed from at least one side.*

Lemma 4.3. *For every $M \in \mathcal{C}_A, N \in {}_A\mathcal{C}_B$ and $L \in {}_B\mathcal{C}$ the coequalizers $M \otimes_A (N \otimes_B L)$ and $(M \otimes_A N) \otimes_B L$ are isomorphic provided that the isomorphisms:*

$$X \otimes (N \otimes_B L) \cong (X \otimes N) \otimes_B L \; ; \quad (16) \qquad\qquad (M \otimes_A N) \otimes X \cong M \otimes_A (N \otimes X) \quad (17)$$

hold for any $X \in \mathcal{C}$. These conditions are fulfilled if any of the following holds:

(i) *M is A-coflat and L is B-coflat;*
(ii) *M is A-coflat and $\mathcal{C}$ is left closed;*
(iii) *L is B-coflat and $\mathcal{C}$ is right closed;*
(iv) *$\mathcal{C}$ is braided and M is A-coflat and N is B-coflat;*
(v) *$\mathcal{C}$ is braided and L is B-coflat and N is A-coflat;*
(vi) *$\mathcal{C}$ is closed.*

Proof. We view $N \otimes_B L$ as a left A-module and $M \otimes_A N$ as a right B-module via the structures inherited from N. Observe that in the diagram

$$
\begin{tikzcd}
(M\otimes_A N)\otimes_B L & (M\otimes_A N)\otimes L \arrow[l, "\Pi_{M\otimes_A N,L}"'] & (M\otimes_A N)\otimes B\otimes L \arrow[l, shift right, "\mu_{M\otimes_A N}\otimes L"'] \arrow[l, shift left, "(M\otimes_A N)\otimes\nu_L"] \\
(M\otimes N)\otimes_B L \arrow[u, "\Pi_{M,N}\otimes_B L"'] & (M\otimes N)\otimes L \arrow[l, "\Pi_{M\otimes N,L}"'] \arrow[u, "\Pi_{M,N}\otimes L"'] & (M\otimes N)\otimes B\otimes L \arrow[l, shift right, "\mu_{M\otimes N}\otimes L"'] \arrow[l, shift left, "(M\otimes N)\otimes\nu_L"] \arrow[u, "\Pi_{M,N}\otimes B\otimes L"'] \\
(M\otimes A\otimes N)\otimes_B L \arrow[u, shift left, "(\mu_M\otimes N)\otimes_B L"] \arrow[u, shift right, "(M\otimes\nu_N)\otimes_B L"'] & (M\otimes A\otimes N)\otimes L \arrow[l, "\Pi_{M\otimes A\otimes N,L}"'] \arrow[u, shift left, "(\mu_M\otimes N)\otimes L"] \arrow[u, shift right, "(M\otimes\nu_N)\otimes L"'] & (M\otimes A\otimes N)\otimes B\otimes L \arrow[l, shift right, "\mu_{M\otimes A\otimes N}\otimes L"'] \arrow[l, shift left, "(M\otimes A\otimes N)\otimes\nu_L"] \arrow[u, shift left, "(\mu_M\otimes N)\otimes B\otimes L"] \arrow[u, shift right, "(M\otimes\nu_N)\otimes B\otimes L"']
\end{tikzcd}
$$

the three rows are coequalizers. By (17) the second and the third column are coequalizers. Note that all inner rectangles commute. Then by the three by three Lemma we obtain that the first column is a coequalizer too.

In order to show that $(M\otimes_A(N\otimes_B L), \Pi_{M,N\otimes_B L})$ and $((M\otimes_A N)\otimes_B L, \Pi_{M,N}\otimes_B L)$ are isomorphic as coequalizers, observe the diagram

$$
\begin{tikzcd}
M\otimes_A(N\otimes_B L) & M\otimes(N\otimes_B L) \arrow[l, "\Pi_{M,N\otimes_B L}"'] & M\otimes A\otimes(N\otimes_B L) \arrow[l, shift right, "\mu_M\otimes(N\otimes_B L)"'] \arrow[l, shift left, "M\otimes\nu_{N\otimes_B L}"] \\
(M\otimes_A N)\otimes_B L \arrow[u, dashed, "\chi"] & (M\otimes N)\otimes_B L \arrow[l, "\Pi_{M,N}\otimes_B L"'] \arrow[u, "\beta_{M,N,L}"'] & (M\otimes A\otimes N)\otimes_B L, \arrow[l, shift right, "(\mu_M\otimes N)\otimes_B L"'] \arrow[l, shift left, "(M\otimes\nu_N)\otimes_B L"] \arrow[u, "\beta_{M\otimes A,N,L}"']
\end{tikzcd}
$$

where β's are the isomorphisms from (16). The right rectangle commutes obviously with upper lines, but also with lower ones, because $\nu_{N\otimes_B L}$ is induced by $\nu_N\otimes_B L$. Knowing that both rows are coequalizers, we get that $\Pi_{M,N\otimes_B L}\beta_{M,N,L}$ induces χ so that the left rectangle commutes. Similarly, $(\Pi_{M,N}\otimes_B L)\beta^{-1}_{M,N,L}$ induces the inverse of χ.

Note that for the second part of the Lemma by Remark 4.2 it suffices to discuss the conditions (i), (iv) and (v). Assume we are in the conditions of (iv), let Φ be the braiding of $\mathcal{C}$. Consider the diagram

$$
\begin{tikzcd}
N\otimes_B(L\otimes X) \arrow[d, dashed, "\theta"'] & N\otimes L\otimes X \arrow[l, "\Pi_{N,L\otimes X}"'] \arrow[d, "\Phi_{N\otimes L,X}"] & N\otimes B\otimes L\otimes X \arrow[l, shift right, "\mu_N\otimes(L\otimes X)"'] \arrow[l, shift left, "N\otimes(\nu_L\otimes X)"] \arrow[d, "\Phi_{N\otimes B\otimes L,X}"] \\
(X\otimes N)\otimes_B L & X\otimes N\otimes L \arrow[l, "\Pi_{X\otimes N,L}"'] & X\otimes N\otimes B\otimes L. \arrow[l, shift right, "(X\otimes\mu_N)\otimes L"'] \arrow[l, shift left, "(X\otimes N)\otimes\nu_L"]
\end{tikzcd}
$$

The right rectangle commutes obviously both with upper and lower lines. Knowing that both rows are coequalizers, we get that $\Pi_{X\otimes N,L}\Phi_{N\otimes L,X}$ induces θ so that the left rectangle commutes. Similarly, $\Pi_{N,L\otimes X}\Phi^{-1}_{N\otimes L,X}$ induces the inverse of θ. Now, since N is B-coflat, we have $(X\otimes N)\otimes_B L\cong N\otimes_B(L\otimes X)\cong(N\otimes_B L)\otimes X\cong X\otimes(N\otimes_B L)$, applying $\Phi^{-1}_{X,N\otimes_B L}$. Moreover, A-coflatness of M implies (17), and we are done with the condition (iv). One can argue symmetrically for (v), and (i) is evident. □

4.2. Morita theorems. Morita theorems for monoidal categories were developed by Pareigis in [9]. We define here the notion of Morita context that does not appear explicitly in [9].

Definition 4.4. *Let $A, B \in \mathcal{C}$ be algebras, $P \in {}_A\mathcal{C}_B$ and $Q \in {}_B\mathcal{C}_A$. Assume there exist isomorphisms:*

$$(P\otimes_B Q)\otimes_A P \cong P\otimes_B(Q\otimes_A P) \qquad \text{and} \qquad (Q\otimes_A P)\otimes_B Q \cong Q\otimes_A(P\otimes_B Q). \tag{18}$$

If the morphisms $f : P \otimes_B Q \longrightarrow A$ in ${}_A\mathcal{C}_A$ and $g : Q \otimes_A P \longrightarrow B$ in ${}_B\mathcal{C}_B$ are given so that the diagrams commute, we call the sextuple (A, B, P, Q, f, g) a Morita

$$\begin{tikzcd} (P\otimes_B Q)\otimes_A P \arrow[r, "f\otimes_A P"] \arrow[d, phantom, "\cong"] & A\otimes_A P \arrow[dd, "\cong"] \\ P\otimes_B (Q\otimes_A P) \arrow[d, "P\otimes_B g"'] & \\ P\otimes_B B \arrow[r, "\cong"'] & P \end{tikzcd} \tag{19}$$

$$\begin{tikzcd} (Q\otimes_A P)\otimes_B Q \arrow[r, "g\otimes_B Q"] \arrow[d, phantom, "\cong"] & B\otimes_B Q \arrow[dd, "\cong"] \\ Q\otimes_A (P\otimes_B Q) \arrow[d, "Q\otimes_A f"'] & \\ Q\otimes_A A \arrow[r, "\cong"'] & Q \end{tikzcd} \tag{20}$$

context *in $\mathcal{C}$.*

If moreover there exist morphisms $\zeta \in \mathcal{C}(I, P \otimes_B Q)$ and $\xi \in \mathcal{C}(I, Q \otimes_A P)$ so that $f \circ \zeta = id_{\mathcal{C}(I,A)} = \eta_A$ and $g \circ \xi = id_{\mathcal{C}(I,B)} = \eta_B$, we say that the context is strict*.*

Remark 4.5. *The isomorphisms in (18) hold if either of the conditions:*

(i) *P and Q are bicoflat;*
(ii) *P is B-coflat, Q is A-coflat and $\mathcal{C}$ is left closed;*
(iii) *P is A-coflat, Q is B-coflat and $\mathcal{C}$ is right closed;*
(iv) *P is B-coflat, Q is A-coflat and $\mathcal{C}$ is braided;*
(v) *P is A-coflat, Q is B-coflat and $\mathcal{C}$ is braided;*
(vi) *$\mathcal{C}$ is closed;*

is fulfilled.

Given a strict Morita context (A, B, P, Q, f, g), from the Morita theorems one can conclude that P and Q are faithfully projective, as we will see further below. Faithfully projective objects can be considered in ${}_A\mathcal{C}$ or in $\mathcal{C}_B$ for algebras $A, B \in \mathcal{C}$. For the sake of completeness, we give here the definitions of both notions which slightly differ from those in [9, 11].

Definition 4.6. *An object $P \in {}_A\mathcal{C}$ is called* faithfully projective*, if ${}_A[P, A]$ and ${}_A[P, P]$ exist, $P \in {}_A\mathcal{C}_{{}_A[P,P]}$ and ${}_A[P, A] \in {}_{{}_A[P,P]}\mathcal{C}_A$ are bicoflat, the dual basis morphism $\overline{db} : {}_A[P, A] \otimes_A P \longrightarrow {}_A[P, P]$ defined via the universal property of $({}_A[P, P], \overline{ev}_{P,P})$ by:*

$$\begin{tikzcd} P\otimes {}_A[P,A]\otimes_A P \arrow[d, "P\otimes \overline{db}"'] \arrow[dr, "\overline{ev}\otimes_A P"] & \\ P\otimes {}_A[P,P] \arrow[r, "\overline{ev}_{P,P}"'] & P\cong A\otimes_A P \end{tikzcd} \tag{21}$$

and the canonical morphism $\hat{ev} : P \otimes_{{}_A[P,P]} {}_A[P,A] \longrightarrow A$ *induced by* $\overline{ev} : P \otimes {}_A[P,A] \longrightarrow A$ *are isomorphisms.*

Definition 4.7. *An object* $P \in \mathcal{C}_B$ *is called* faithfully projective*, if* $[P,B]_B$ *and* $[P,P]_B$ *exist,* $P \in {}_{[P,P]_B}\mathcal{C}_B$ *and* $[P,B]_B \in {}_B\mathcal{C}_{[P,P]_B}$ *are bicoflat, the dual basis morphism* $db : P \otimes_B [P,B]_B \longrightarrow [P,P]_B$ *defined via the universal property of* $([P,P]_B, ev_{P,P})$ *by:*

$$
\begin{tikzcd}
P \otimes_B [P,B]_B \otimes P \arrow[d, "db \otimes P"'] \arrow[dr, "P \otimes_B ev"] & \\
{[P,P]_B \otimes P} \arrow[r, "ev_{P,P}"'] & P \cong P \otimes_B B
\end{tikzcd}
\tag{22}
$$

and the canonical morphism $\tilde{ev} : [P,B]_B \otimes_{[P,P]_B} P \longrightarrow B$ *induced by the morphism* $ev : [P,B]_B \otimes P \longrightarrow B$ *are isomorphisms.*

Faithfully projective objects in ${}_A\mathcal{C}$ and in $\mathcal{C}_B$ give rise to strict Morita contexts, as we will see later. In order for this to work, it is evident from Remark 4.5 that for faithfully projective objects we need to require complicated conditions (if $\mathcal{C}$ is not closed). To avoid this we decided to include bicoflatness in the above two definitions.

In the following we unify [9, Theorems 5.1 and 5.3] into an equivalence claim including the original results and their right hand-side counterparts. The assumptions that we find necessary slightly differ from the original ones.

Theorem 4.8. *Let* $\mathcal{C}$ *be a monoidal category.*

(i) *Assume that* $\mathcal{C}$ *is left closed (or braided). The functors* $\mathcal{F} : {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ *and* $\mathcal{G} : {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$ *are inverse* $\mathcal{C}$*-equivalences if and only if there exists a* B*-coflat object* $P \in {}_A\mathcal{C}_B$ *and an* A*-coflat object* $Q \in {}_B\mathcal{C}_A$ *such that* $\mathcal{F} \cong Q \otimes_A -$ *and* $\mathcal{G} \cong P \otimes_B -$ *and* (A,B,P,Q,f,g) *is a strict Morita context, where* $f : P \otimes_B Q \longrightarrow A$ *is an isomorphism in* ${}_A\mathcal{C}_A$ *and* $g : Q \otimes_A P \longrightarrow B$ *is an isomorphism in* ${}_B\mathcal{C}_B$*.*

(ii) *Assume that* $\mathcal{C}$ *is right closed (or braided). The functors* $\mathcal{F} : \mathcal{C}_A \longrightarrow \mathcal{C}_B$ *and* $\mathcal{G} : \mathcal{C}_B \longrightarrow \mathcal{C}_A$ *are inverse* $\mathcal{C}$*-equivalences if and only if there exists an* A*-coflat object* $P \in {}_A\mathcal{C}_B$ *and a* B*-coflat object* $Q \in {}_B\mathcal{C}_A$ *such that* $\mathcal{F} \cong - \otimes_A P$ *and* $\mathcal{G} \cong - \otimes_B Q$ *and* (A,B,P,Q,f,g) *is a strict Morita context, where* $f : P \otimes_B Q \longrightarrow A$ *is an isomorphism in* ${}_A\mathcal{C}_A$ *and* $g : Q \otimes_A P \longrightarrow B$ *is an isomorphism in* ${}_B\mathcal{C}_B$*.*

(iii) *Assume that* $\mathcal{C}$ *is closed. The two pairs of functors* $\mathcal{F} : {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ *and* $\mathcal{G} : {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$*; and* $\mathcal{F}' : \mathcal{C}_A \longrightarrow \mathcal{C}_B$ *and* $\mathcal{G}' : \mathcal{C}_B \longrightarrow \mathcal{C}_A$*, are inverse* $\mathcal{C}$*-equivalences if and only if there exist objects* $P \in {}_A\mathcal{C}_B$ *and* $Q \in {}_B\mathcal{C}_A$ *such that* $\mathcal{F} \cong Q \otimes_A -$ *and* $\mathcal{G} \cong P \otimes_B -$*, and* $\mathcal{F}' \cong - \otimes_A P$ *and* $\mathcal{G}' \cong - \otimes_B Q$*, and* (A,B,P,Q,f,g) *is a strict Morita context, where* $f : P \otimes_B Q \longrightarrow A$ *is an isomorphism in* ${}_A\mathcal{C}_A$ *and* $g : Q \otimes_A P \longrightarrow B$ *is an isomorphism in* ${}_B\mathcal{C}_B$*.*

Note that in the item (iii) P and Q are bicoflat in view of Remark 4.2. In part (i) (part (ii)) we require left (right) closedness of $\mathcal{C}$ or that it is braided in order that the functors $Q \otimes_A -$ and $P \otimes_B -$ $(- \otimes_A P$ and $- \otimes_B Q)$ give an equivalence of categories.

Namely, in the case of (i), for any $X \in {}_B\mathcal{C}$ we have $Q \otimes_A (P \otimes_B X) \cong (Q \otimes_A P) \otimes_B X$ since Q is A-coflat and $\mathcal{C}$ is left closed, or since Q is A-coflat, P is B-coflat and $\mathcal{C}$ is braided, due to Lemma 4.3. Similarly for $P \otimes_B (Q \otimes_A Y) \cong (P \otimes_B Q) \otimes_A Y$ for any $Y \in {}_A\mathcal{C}$. The reasoning for (ii) is similar.

Theorem 4.9. [11, Theorem 2.1] *Let (A, B, P, Q, f, g) be a strict Morita context in $\mathcal{C}$ and assume that P and Q are bicoflat. Then:*

(i) *There are isomorphisms $P \cong [Q, A]_A \cong {}_B[Q, B]$ in ${}_A\mathcal{C}_B$ and $Q \cong [P, B]_B \cong {}_A[P, A]$ in ${}_B\mathcal{C}_A$.*

(ii) *There are isomorphisms $A \cong [P, P]_B \cong {}_B[Q, Q]$ in ${}_A\mathcal{C}_A$ and $B \cong [Q, Q]_A \cong {}_A[P, P]$ in ${}_B\mathcal{C}_B$ that are also isomorphisms of algebras.*

(iii) *${}_AP$, P_B, ${}_BQ$ and Q_A are faithfully projective.*

Observe that in the proof of (iii) above, the diagrams (19) and (20) translate into the diagram (21) and the one determining morphism $\hat{ev} : P \otimes_{{}_A[P,P]} {}_A[P, A] \longrightarrow A$ (and (22) and the diagram determining $\tilde{ev} : [P, B]_B \otimes_{[P,P]_B} P \longrightarrow B$). Note that by assumption P and Q are bicoflat, as in our definition of faithful projectiveness. Translating the diagrams in the reversed order, one proves:

Proposition 4.10. *Let $\mathcal{C}$ be a monoidal category.*

(i) *If P_B is faithfully projective, then $([P, P]_B, B, P, [P, B]_B, db, \tilde{ev})$ is a strict Morita context, with db and $\tilde{ev}$ from Definition 4.7.*

(ii) *If ${}_AP$ is faithfully projective, then $(A, {}_A[P, P], P, {}_A[P, A], \hat{ev}, \overline{db})$ is a strict Morita context, with $\hat{ev}$ and $\overline{db}$ from Definition 4.6.*

From here and Theorem 4.8 we have:

Corollary 4.11. (i) *If P_B is faithfully projective and $\mathcal{C}$ is left closed (or braided), then the functor $P \otimes_B - : {}_B\mathcal{C} \longrightarrow {}_{[P,P]_B}\mathcal{C}$ is a $\mathcal{C}$-equivalence.*

(ii) *If ${}_AP$ is faithfully projective and $\mathcal{C}$ is right closed (or braided), then the functor $- \otimes_A P : \mathcal{C}_A \longrightarrow \mathcal{C}_{{}_A[P,P]}$ is a $\mathcal{C}$-equivalence.*

Let us now study faithfully projective objects with $A{=}B{=}I$. In order to distinguish two kinds of faithful projectiveness we will write $\{M, N\}$ and $[M, N]$ for the two inner-hom objects (recall 2.2).

An object faithfully projective in ${}_I\mathcal{C}$ we call *left faithfully projective* and an object faithfully projective in $\mathcal{C}_I$ we call *right faithfully projective*. Note that an object P is right faithfully projective if $P \in {}_{[P,P]}\mathcal{C}$ and $[P, I] \in \mathcal{C}_{[P,P]}$ are coflat, the dual basis morphism $db{:}P \otimes [P, I] \longrightarrow [P, P]$ defined via $ev_{P,P}(db \otimes P){=}P \otimes ev_{P,I}$ is an isomorphism (this defines the concept of *right finite object*) and $\tilde{ev}{:}[P, I] \otimes_{[P,P]I} P \longrightarrow I$ induced by $ev{:}[P, I] \otimes P \longrightarrow I$ is an isomorphism. We have a similar statement for a left faithfully projective object, and a left finite object is defined similarly as a right one.

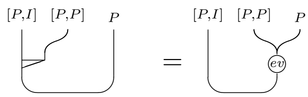
Remark 4.12. *One may easily prove that if P is right finite, then $([P, I], e_P{=}ev)$ is its left dual with $d_P = db^{-1}\eta_{[P,P]}$. A similar claim holds for a left finite object. In a braided monoidal category an object is left finite if and only if it is right finite.*

With A=B=I the requirement made in Corollary 4.11, that $\mathcal{C}$ be left (right) closed or braided, becomes superfluous in view of the comment after Theorem 4.8. Thus we may write:

Corollary 4.13. (i) *If P is a right faithfully projective object in a monoidal category $\mathcal{C}$, then $([P,P], I, P, [P,I], db, \tilde{ev})$ is a strict Morita context. Moreover, the functor $P\otimes - : \mathcal{C} \longrightarrow {}_{[P,P]}\mathcal{C}$ is a $\mathcal{C}$-equivalence.*

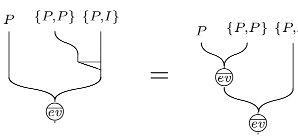
(ii) *If P is a left faithfully projective object in a monoidal category $\mathcal{C}$, then $(I, \{P,P\}, P, \{P,I\}, \hat{ev}, \overline{db})$ is a strict Morita context. Moreover, the functor $-\otimes P : \mathcal{C} \longrightarrow \mathcal{C}_{\{P,P\}}$ is a $\mathcal{C}$-equivalence.*

For the rest of this subsection assume that $\mathcal{C}$ is a braided monoidal category with braiding Φ. In the above Corollary P is a left $[P,P]$-module by $ev : [P,P]\otimes P \longrightarrow P$ and $P^* = [P,I]$ is a right $[P,P]$-module by

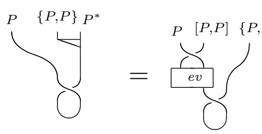


(23)

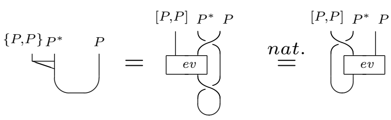
whereas P is a right $\{P,P\}$-module by $\overline{ev} : P\otimes\{P,P\} \longrightarrow P$, and $P^* = \{P,I\}$ is a left $\{P,P\}$-module by



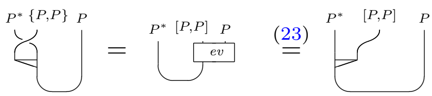
which is



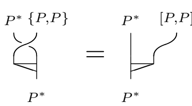
Composing the latter with $\Phi^{-1}_{P,\{P,P\}\otimes P^*}$ and then $\Phi^{-1}_{\{P,P\},P^*}\otimes P$ from above, we obtain:



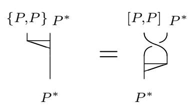
and



respectively. By the universal property of $([P,P], ev)$ this yields:



which implies

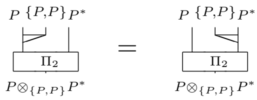
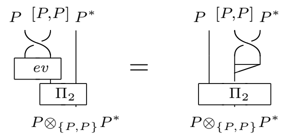


(24)

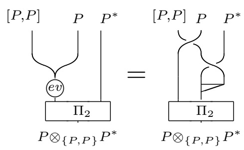
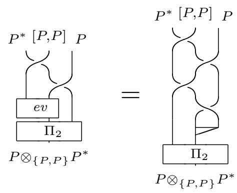
Now consider the diagram

$$\begin{tikzcd}
P^*\otimes[P,P]\otimes P \arrow[r, shift left, "\mu_{P^*}\otimes P"] \arrow[r, shift right, "P^*\otimes\nu_P"'] & P^*\otimes P \arrow[rr, "\Pi_1"] \arrow[dd, "\Phi^{-1}_{P,P^*}"'] \arrow[dr, "ev"'] & \arrow[d, phantom, "\boxed{1}"] & P^*\otimes_{[P,P]} P \arrow[dl, "\tilde{ev}"] \arrow[dd, dashed, shift right, "\theta"'] \\
& \arrow[r, phantom, "\boxed{2}"] & I & \\
P\otimes\{P,P\}\otimes P^* \arrow[r, shift left, "\mu_P\otimes P^*"] \arrow[r, shift right, "P\otimes\nu_{P^*}"'] & P\otimes P^* \arrow[rr, "\Pi_2"'] \arrow[ur, "\overline{ev}"] & \arrow[u, phantom, "\boxed{3}"] & P\otimes_{\{P,P\}} P^* \arrow[ul, "\tilde{\overline{ev}}"'] \arrow[uu, dashed, shift right, "\psi"']
\end{tikzcd} \tag{25}$$

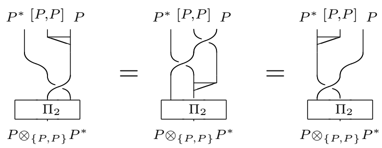
We want to define morphisms θ and ψ so that the right rectangle in the above picture commutes. The two rows are coequalizers. From the second one we have:


which by (24) is equivalent to


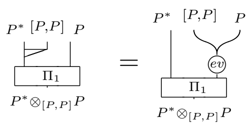
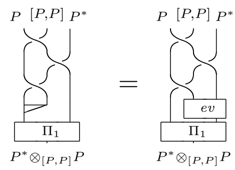
Composing to this identity $\Phi^{-1}_{P,[P,P]} \otimes P^*$ and then $\Phi^{-1}_{[P,P]\otimes P,P^*}$ from above, we obtain:


and


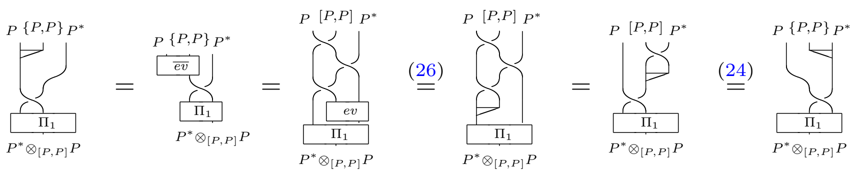
respectively. The latter is equivalent to



This proves that $\Pi_2\Phi^{-1}$ induces θ so that $\theta\Pi_1 = \Pi_2\Phi^{-1}$. On the other hand, by the first coequalizer in (25) we have:


and this implies

(26)

Then we find:



This proves that $\Pi_1\Phi$ induces ψ so that $\psi\Pi_2 = \Pi_1\Phi$. Now, $\psi\theta\Pi_1 = \psi\Pi_2\Phi^{-1} = \Pi_1\Phi\Phi^{-1} = \Pi_1$, which yields $\psi\theta = id$ since Π_1 is an epimorphism. Similarly, one proves $\theta\psi = id$.

From the commutativity of the inner diagrams $\langle 1\rangle, \langle 2\rangle$ and $\langle 3\rangle$ in (25) we obtain $\overline{\tilde{ev}}\theta\Pi_1 = \overline{\tilde{ev}}\Pi_2\Phi^{-1} = \tilde{ev}\Pi_1$, hence $\overline{\tilde{ev}}\theta = \tilde{ev}$. Since θ is an isomorphism, we have that $\overline{\tilde{ev}}$ is an isomorphism if and only if $\tilde{ev}$ is an isomorphism. We have proved:

Proposition 4.14. *In a braided monoidal category an object is left faithfully projective if and only if it is right faithfully projective.*

4.3. The central part of the Morita theorems. In [9] Pareigis proved that for two algebras A and B in a symmetric category $\mathcal{C}$ for which the categories of modules ${}_A\mathcal{C}$ and ${}_B\mathcal{C}$ are $\mathcal{C}$-equivalent the centers ${}_A[A,A]_A$ and ${}_B[B,B]_B$ are isomorphic as commutative algebras. This is proved using a more general result. Let $\mathcal{U},\mathcal{V}: {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ be two $\mathcal{C}$-functors such that $\mathcal{U} \cong P\otimes_A -$ and $\mathcal{V}\cong Q\otimes_A -$ are $\mathcal{C}$-isomorphisms for some B-A-bimodules P and Q. Then P_A and Q_A are coflat. For $Y\in\mathcal{C}$ the functor $\mathcal{U}\otimes Y: {}_A\mathcal{C}\longrightarrow {}_B\mathcal{C}$ is defined by $(\mathcal{U}\otimes Y)(M) := \mathcal{U}(M\otimes Y)$ for $M\in {}_A\mathcal{C}$. It is a $\mathcal{C}$-functor. Furthermore, one defines a (contravariant) functor $[\mathcal{U},\mathcal{V}]: \mathcal{C}\longrightarrow \mathcal{S}et$ by $[\mathcal{U},\mathcal{V}](Y) := \mathcal{C}\text{-}Nat(\mathcal{U}\otimes Y,\mathcal{V})$, the latter denoting the family of $\mathcal{C}$-natural transformations $\mathcal{U}\otimes Y\longrightarrow\mathcal{V}$, and for $h\in\mathcal{C}(Z,Y)$ one defines $[\mathcal{U},\mathcal{V}](h)$ by $[\mathcal{U},\mathcal{V}](h)(\varphi)(M) := \varphi(M)\circ\mathcal{U}(M\otimes h)$ where $\varphi\in\mathcal{C}\text{-}Nat(\mathcal{U}\otimes Y,\mathcal{V})$.

Proposition 4.15. [9, Theorem 6.1] *In a symmetric monoidal category $\mathcal{C}$ there is a natural isomorphism of functors $[\mathcal{U},\mathcal{V}]\cong {}_B\mathcal{C}_A(P\otimes -, Q)$. If ${}_B[P,Q]_A$ exists, then $[\mathcal{U},\mathcal{V}](Y)\cong\mathcal{C}(Y,{}_B[P,Q]_A)$.*

The Proposition remains valid also when $\mathcal{C}$ is a braided category. In its proof the braiding is not used, however in order to assure that $\mathcal{U}\otimes Y$ is a $\mathcal{C}$-functor in the non-symmetric case, we need to keep track on the braiding. Consider the diagram:

$$\begin{tikzcd}
P\otimes A\otimes M\otimes Y \arrow[r, shift left, "\mu_P\otimes M\otimes Y"] \arrow[r, shift right, "P\otimes\nu_{M\otimes Y}"'] \arrow[d, "P\otimes\Phi_{A\otimes M,Y}"] & P\otimes M\otimes Y \arrow[r, "\Pi_{P,M\otimes Y}"] \arrow[d, "P\otimes\Phi_{M,Y}"] & P\otimes_A(M\otimes Y) \arrow[d, dashed, "\beta"] \\
P\otimes Y\otimes A\otimes M \arrow[r, shift left, "\mu_{P\otimes Y}\otimes M"] \arrow[r, shift right, "P\otimes Y\otimes\nu_M"'] & P\otimes Y\otimes M \arrow[r, "\Pi_{P\otimes Y,M}"'] & (P\otimes Y)\otimes_A M.
\end{tikzcd}$$

The left rectangle commutes with lower lines by naturality, and also with upper lines, given that the right A-module structure of $P\otimes Y$ is given as in (2). Since both horizontal rows are coequalizers, we obtain the isomorphism β. Then $(\mathcal{U}\otimes Y)(M) = \mathcal{U}(M\otimes Y) = P\otimes_A(M\otimes Y)\cong(P\otimes Y)\otimes_A M$ and hence $(\mathcal{U}\otimes Y)(M\otimes X)\cong (P\otimes Y)\otimes_A(M\otimes X)\cong((P\otimes Y)\otimes_A M)\otimes X\cong(\mathcal{U}\otimes Y)(M)\otimes X$. The penultimate isomorphism is a consequence of the isomorphism β and of the A-coflatness of P. This proves that $\mathcal{U}\otimes Y$ is a $\mathcal{C}$-functor also when $\mathcal{C}$ is braided.

Applying Proposition 4.15 to the identity functor $\mathcal{U}=\mathcal{V}=\mathrm{Id}_A$, one obtains

$$[{}_A\mathrm{Id},{}_A\mathrm{Id}]\cong {}_A\mathcal{C}_A(A\otimes -, A). \tag{27}$$

Suppose that $\mathcal{F}: {}_A\mathcal{C}\longrightarrow {}_B\mathcal{C}$ is a $\mathcal{C}$-equivalence. In [9, Corollary 6.3] it is proved that $[{}_A\mathrm{Id},{}_A\mathrm{Id}](Y)$ and $[{}_B\mathrm{Id},{}_B\mathrm{Id}](Y)$ are isomorphic for $Y\in\mathcal{C}$. In the proof the braiding is used only once at a computation in order to intertwine two objects $X,Y\in\mathcal{C}$ when proving the $\mathcal{C}$-functor property, but the sign of the braiding is irrelevant. Now, if ${}_A[A,A]_A$ and ${}_B[B,B]_B$ exist, they both are representing objects for the same functor, due to (27), so they are isomorphic. Just before [9, Corollary 6.3] the multiplication and unit for the functors in (27) is described and it is concluded that this is an isomorphism of "monoids". On the object ${}_A[A,A]_A$ these structure morphisms translate via Yoneda Lemma into those in 2.4.

Observe that in the above construction we have worked with right $\mathcal{C}$-functors. In 2.1 we saw that the category ${}_B\mathcal{C}_A$ is isomorphic as a right $\mathcal{C}$-category to ${}_{B\otimes\overline{A}}\mathcal{C}$. Then the isomorphism in Proposition 4.15 can be rewritten as follows: $[\mathcal{U},\mathcal{V}] \cong {}_{B\otimes\overline{A}}\mathcal{C}(P\otimes -, Q)$, and the one in (27) as $[{}_A\mathrm{Id}, {}_A\mathrm{Id}] \cong {}_{A\otimes\overline{A}}\mathcal{C}(A\otimes -, A)$. The result of the above paragraph can then be restated as: ${}_{A\otimes\overline{A}}[A,A] \cong {}_{B\otimes\overline{B}}[B,B]$ as algebras.

By symmetry, starting with the functors $\mathcal{U}' \cong -\otimes_B P$, $\mathcal{V}' \cong -\otimes_B Q$ and $(Y\otimes\mathcal{U}')(M) := \mathcal{U}'(Y\otimes M)$ for $M \in \mathcal{C}_B$, one may prove the isomorphisms of functors $[\mathcal{U}',\mathcal{V}'] \cong \mathcal{C}_{\overline{B}\otimes A}(-\otimes P, Q)$ and $[\mathrm{Id}_B, \mathrm{Id}_B] \cong \mathcal{C}_{\overline{B}\otimes B}(-\otimes B, B)$. Given a $\mathcal{C}$-equivalence between $\mathcal{C}_A$ and $\mathcal{C}_B$, the latter yields, analogously as above, the algebra isomorphism $[A,A]_{\overline{A}\otimes A} \cong [B,B]_{\overline{B}\otimes B}$.

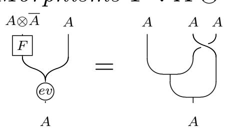
We now may claim:

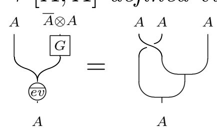
Theorem 4.16. *Let $\mathcal{C}$ be a braided monoidal category and $A, B \in \mathcal{C}$ algebras.*
- (i) *If ${}_A\mathcal{C}$ and ${}_B\mathcal{C}$ are $\mathcal{C}$-equivalent, then the left centers of A and B (if they exist) are isomorphic as commutative algebras.*
- (ii) *If $\mathcal{C}_A$ and $\mathcal{C}_B$ are $\mathcal{C}$-equivalent, then the right centers of A and B (if they exist) are isomorphic as commutative algebras.*

5. Azumaya algebras

In [2] we presented the construction of the Brauer group of a braided monoidal category $\mathcal{C}$ using the notation of braided diagrams. Azumaya algebras in $\mathcal{C}$, whose equivalence classes form the Brauer group of $\mathcal{C}$, were characterized in [11, Theorem 3.1]. However, an explicit proof is omitted and it is not stated which kind of faithful projectiveness is used. Observe that faithful projectiveness of P in $\mathcal{C}_B$ can not be deduced from the one in ${}_A\mathcal{C}$, and vice versa. Although in Proposition 4.14 we proved that ${}_IP$ is faithfully projective if and only if so is P_I, given that to the author's knowledge this was not discussed in the literature so far, we present below the full proof of a reformulated [11, Theorem 3.1] studying how both left and right faithful projectiveness of the (Azumaya) algebra in question are involved.

Theorem 5.1. *Let $\mathcal{C}$ be a closed braided monoidal category and $A \in \mathcal{C}$ an algebra. The following are equivalent:*
- (i) *The functors $A\otimes - : \mathcal{C} \longrightarrow {}_{A\otimes\overline{A}}\mathcal{C}$ and $-\otimes A : \mathcal{C} \longrightarrow \mathcal{C}_{\overline{A}\otimes A}$ establish $\mathcal{C}$-equivalences of categories;*
- (ii) *Morphisms $F : A\otimes\overline{A} \longrightarrow [A,A]$ and $G : \overline{A}\otimes A \longrightarrow \overline{[A,A]}$ defined via:*



(28) and


(29)

are isomorphisms and A is left and right faithfully projective.

Proof. From the $\mathcal{C}$-equivalence $A\otimes - : \mathcal{C} \longrightarrow {}_{A\otimes\overline{A}}\mathcal{C}$, by Theorem 4.8 part (i) and Theorem 4.9 part (i) we have a strict Morita context $(I, A\otimes\overline{A}, A^*, A, \tilde{ev}, db)$. By part (iii) it follows that A_I is faithfully projective, i.e. A is right faithfully projective. Similarly, the $\mathcal{C}$-equivalence $-\otimes A : \mathcal{C} \longrightarrow \mathcal{C}_{\overline{A}\otimes A}$, by Theorem 4.8 part (ii) and

Theorem 4.9 part (i), yields a strict Morita context $(I, \overline{A} \otimes A, A, A^*, \overline{ev}, \overline{d}b)$. From here we get that ${}_I A$ is faithfully projective, i.e. A is left faithfully projective.

The algebra isomorphism $\omega : B \longrightarrow [Q, Q]_{\mathbb{A}}$ from Theorem 4.9 part (ii) is given by $id_Q =: ev(\omega \otimes_B Q)\colon (Q \cong B \otimes_B Q \stackrel{\omega\otimes_B Q}{\longrightarrow} [Q, Q]_{\mathbb{A}} \otimes_B Q \stackrel{ev}{\longrightarrow} Q)$. The latter is induced as a morphism on the coequalizer by the commuting diagram:

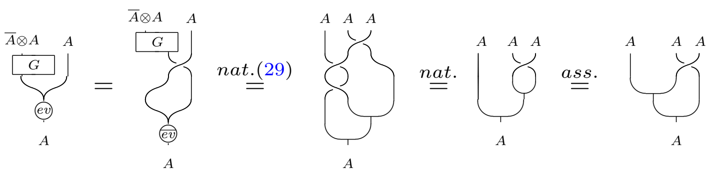
$$\begin{tikzcd}
B \otimes B \otimes Q \arrow[r, shift left, "\nabla_B \otimes Q"] \arrow[r, shift right, "B \otimes \nu"'] & B \otimes Q \arrow[r, "\Pi_{B,Q}"] \arrow[d, "\omega \otimes Q"'] \arrow[dr, "\nu"] & B \otimes_B Q \cong Q \arrow[d, "id_Q"] \\
& {[Q, Q]_{\mathbb{A}} \otimes Q} \arrow[r, "ev"'] & Q
\end{tikzcd}$$

Observe that ω is defined in terms of the action ν of B on Q, whereas on the other hand ν induces id_Q. Substituting in our situation using the first Morita context $B{=}A \otimes \overline{A}, Q{=}A$ and $\mathbb{A}{=}I$, we obtain that the algebra isomorphism $F\colon A \otimes \overline{A} \longrightarrow [A, A]$ is induced by the left action μ_l of $A{\otimes}\overline{A}$ on A, that is $ev(F{\otimes}A) = \mu_l$, which by (3) yields (28). Similarly, consider the algebra isomorphism $\theta : B \longrightarrow {}_{\mathbb{A}}[P, P]$ from Theorem 4.9 part (ii) defined via $\overline{ev}(P \otimes_B \theta) = id_P$ and substitute from our second Morita context $B = \overline{A} \otimes A, P = A$ and $\mathbb{A} = I$. Recalling from 2.5 that ${}_I[A, A] = \{A, A\} \cong \overline{[A, A]}$ as algebras, we obtain that the algebra isomorphism $G : \overline{A} \otimes A \longrightarrow \overline{[A, A]}$ is induced by the right action μ_r of $\overline{A} \otimes A$ on A. This means $\overline{ev}(A \otimes G) = \mu_r$, and by (4) we get (29).

Conversely, left and right faithful projectiveness of A, by Corollary 4.13 imply respectively that $- \otimes A : \mathcal{C} \longrightarrow \mathcal{C}_{\overline{[A,A]}}$ and $A \otimes - : \mathcal{C} \longrightarrow {}_{[A,A]}\mathcal{C}$ are $\mathcal{C}$-equivalences. Using the algebra isomorphisms $G\colon \overline{A} \otimes A \longrightarrow \overline{[A, A]}$ and $F\colon A \otimes \overline{A} \longrightarrow [A, A]$, we obtain the claim. □

An algebra satisfying one of the conditions from the above Theorem is called an *Azumaya algebra*. The right inverse functors for $A \otimes -$ and $- \otimes A$ are the functors ${}_{A^e}[A, -] \cong (-)^A$ and $[A, -]_{A^{\overline{e}}} \cong {}^A(-)$, respectively.

The morphisms $F : A \otimes \overline{A} \longrightarrow [A, A]$ and $G : \overline{A} \otimes A \longrightarrow \overline{[A, A]}$ are in general not equal. However, if the category is symmetric, they coincide:



which since $\mathcal{C}$ is symmetric equals (28). By the universal property of $([A, A], ev)$ it follows $F = G$. The structures chosen in (3) and (4) ($\Phi_{B,M}$ in (3) and $\Phi_{M,A}$ in (4)) assure that F and G are morphisms of algebras in any braided monoidal category.

From the above it is clear why in the construction in [10] only one of the two morphisms appears. Likewise, instead of two equivalence functors as in Theorem 5.1, in [10] occurs only the equivalence functor $A \otimes - : \mathcal{C} \longrightarrow {}_A\mathcal{C}_A$, whose right inverse is

${}_A[A,-]_A$. In Proposition 3.7 we showed that in a symmetric category all the three equivalence functors are mutually equivalent.

We close this section by proving that an Azumaya algebra is central.

Definition 5.2. *An algebra $A \in \mathcal{C}$ is called* left central *if its left center ${}_{A\otimes\overline{A}}[A,A]$ is trivial, that is, isomorphic to I. Similarly, one defines a right central algebra. A is called* central *if it is both left and right central.*

The following claim is implicitly contained in [11, Theorem 4.9, (3) $\Rightarrow$ (1)], where the category was assumed to be abelian. Our proof illustrates how right faithful projectiveness of A implies that A is left central, whereas left faithful projectiveness of A implies that it is right central.

Proposition 5.3. *An Azumaya algebra A in a braided monoidal category $\mathcal{C}$ is central.*

Proof. From the right faithful projectiveness of A we have the strict Morita context $([A,A], I, A, [A,I], db, \tilde{ev})$. Applying Theorem 4.9 part (ii), we get $I \cong {}_{[A,A]}[A,A]$. Taking into account the algebra isomorphism $F: A\otimes\overline{A} \longrightarrow [A,A]$, we obtain $I \cong {}_{A\otimes\overline{A}}[A,A]$. On the other hand, the left faithful projectiveness of A yields the strict Morita context $(I, \{A,A\}, A, \{A,I\}, \hat{ev}, \overline{db})$. From Theorem 4.9 part (ii) we get $I \cong [A,A]_{\overline{[A,A]}}$. This, together with the algebra isomorphism $G: \overline{A}\otimes A \longrightarrow \overline{[A,A]}$, implies $I \cong [A,A]_{\overline{A}\otimes A}$. □

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