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DRINFEL'D DOUBLES AND SHAPOVALOV DETERMINANTS

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Abstract. The Shapovalov determinant for a class of pointed Hopf algebras is calculated, including quantized enveloping algebras, Lusztig's small quantum groups, and quantized Lie superalgebras. Our main tools are root systems, Weyl groupoids, and Lusztig type isomorphisms. We elaborate powerful novel techniques for the algebras at roots of unity, and pass to the general case using a density argument.

This work is dedicated to Hans-Jürgen Schneider
on the occasion of his 65th birthday

1. Introduction

We study finite-dimensional representations of a large class of Hopf algebras $U(\chi)$, where χ is a bicharacter on $\mathbb{Z}^I$ for some finite index set I. These algebras emerged from a program of Andruskiewitsch and Schneider to classify pointed Hopf algebras [AS98], [Hec10]. Prominent examples are quantized enveloping algebras of semisimple Lie algebras, where the deformation parameter is not a root of 1, and Lusztig's (finite-dimensional) small quantum groups, see Sect. 8. Other relevant examples are quantized enveloping algebras of Lie superalgebras, see [KT91] and [Yam99, Yam01], and Drinfeld doubles of bosonizations of Nichols algebras of diagonal type classified in [Hec09].

Our main combinatorial tools towards the study of representations are the root system and the Weyl groupoid associated to χ. For quantized enveloping algebras of semisimple Lie algebras the Weyl groupoid is nothing but the Weyl group of the Lie algebra. The main concern of this paper is the determination of the Shapovalov determinants for all algebras $U(\chi)$ with finite root system. We obtain a natural analog of Shapovalov's original formula (for complex semisimple Lie algebras) as a product of linear factors. For our approach we need that $\chi(\beta,\beta)\neq 1$ for all positive roots β. This assumption is fulfilled for the special cases mentioned above.

The generality of our setting forces us to understand the representation theory of algebras $U(\chi)$, where many values of χ are roots of 1. We turn this bondage into a promising leading principle of our approach. We concentrate first on those bicharacters, which take values in the set of roots of 1. In this case the positive and negative parts of $U(\chi)$ are finite-dimensional algebras. For these algebras

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we develop special techniques, which are very different from the usual ones for semisimple Lie algebras, based on reflections in the Weyl groupoid. With these techniques we are able to characterize easily the irreducibility of Verma modules. The characterization leads quickly to a formula for the Shapovalov determinants of Verma modules by a variant of the usual density argument. In a next step we extend our results to more general bicharacters by a new density argument. This is possible because of our good knowledge of the root system of bicharacters.

The history of Shapovalov determinants started with Shapovalov's work [Sha72], where he defined a bilinear form on Verma modules and determinants on homogeneous subspaces to obtain information on the reducibility of Verma modules. These structures have been generalized by Kac and Kazhdan [KK79] to symmetrizable Kac-Moody algebras and by Kac [Kac79] [Kac86] to Lie superalgebras with symmetrizable Cartan matrix. Shapovalov determinants have been calculated for quantized enveloping algebras by de Concini and Kac [CK90] and for quantized Kac-Moody algebras by Joseph [Jos95], see also [JL96]. For Lusztig's small quantum groups Kumar and Letzter [KL97] factorize the Shapovalov determinants under the assumption that the deformation parameter q has prime order and the base field is a cyclotomic field. Shapovalov determinants have been considered recently in various contexts, see for example [BK02], [GS05], [Gor06], [AL05], [Hil08]. Our approach yields in particular an entirely new proof of the formula of de Concini and Kac. In Sect. 8 we improve Kumar's and Letzter's result by allowing arbitrary base fields and arbitrary orders of q.

The paper is organized as follows. In Sect. 2 the axioms of Cartan schemes, Weyl groupoids, and root systems are recalled. The Weyl groupoid of a bicharacter fits into this framework. Besides collecting the most important facts we introduce a character ρ^χ on $\mathbb{Z}^I$ which will play a similar role as the linear form 2ρ on the root lattice. In Sect. 3 the definition and properties of the Drinfeld doubles $U(\chi)$ are recalled. Sect. 4 deals with Lusztig type isomorphisms between two (usually different) Drinfeld doubles. With Thm. 4.9 we establish a Lusztig type PBW basis of these algebras. Moreover, in Thm. 4.8 we develop important properties for q-commutators of root vectors. In Sect. 5 we start to study Verma modules and special maps between them. Prop. 5.11 gives a criterion for bijectivity of such maps, and Prop. 5.16 identifies irreducible Verma modules. In Sect. 6 we study Shapovalov determinants following the approach in [Jos95]. Here our main result is Thm 6.8, which gives a formula for the Shapovalov determinant of $U(\chi)$, where all values of χ are roots of 1, and the root system of χ is finite. Then we pass to more general bicharacters: Thm. 7.3 states a similar result for bicharacters with finite root system. We conclude the paper with the adaptation of our formulas to quantized enveloping algebras and Lusztig's small quantum groups in Sect. 8, and with some commutative algebra in the Appendix.

We write $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of non-negative integers.

2. Preliminaries

Let $\Bbbk$ be a field and $\Bbbk^\times = \Bbbk \setminus \{0\}$. For all $n \in \mathbb{N}_0$ and $q \in \Bbbk^\times$ let

$$(n)_q = \sum_{j=0}^{n-1} q^j, \qquad (n)_q^! = \prod_{j=1}^{n} (j)_q,$$

where $(0)_q^! = 1$. For any finite set I let $\{\alpha_i \,|\, i \in I\}$ be the standard basis of the free $\mathbb{Z}$-module $\mathbb{Z}^I$.

2.1. **Cartan schemes, Weyl groupoids, and root systems.** The combinatorics of a Drinfel'd double of a Nichols algebra of diagonal type is controlled to a large extent by its Weyl groupoid. We use the language developed in [CH09]. Substantial part of the theory was obtained first in [HY08]. We recall the most important definitions and facts.

Let I be a non-empty finite set. By [Kac90, §1.1] a generalized Cartan matrix $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \le 0$ for all $i, j, k \in I$ with $j \ne k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

Definition 2.1. Let I be a non-empty finite set, A a non-empty set, $r_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $r_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{r_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

Example 2.2. Let $A = \{a\}$ be a set with a single element, and let C be a generalized Cartan matrix. Then $r_i = \mathrm{id}$ for all $i \in I$, and $\mathcal{C}$ becomes a Cartan scheme.

A Cartan scheme $\mathcal{C}$ is *connected*, if the subgroup $\langle r_i \,|\, i \in I\rangle$ of $\mathrm{Aut}(A)$ acts transitively on A, that is, if for all $a, b \in A$ with $a \ne b$ there exist $n \in \mathbb{N}$ and $i_1, i_2, \dots, i_n \in I$ such that $b = r_{i_n} \cdots r_{i_2} r_{i_1}(a)$. Two Cartan schemes $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (r'_i)_{i\in I'}, (C'^a)_{a\in A'})$ are *equivalent*, if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(r_i(a)) = r'_{\varphi_0(i)}(\varphi_1(a)), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij} \tag{2.1}$$

for all $i, j \in I$ and $a \in A$.

Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{2.2}$$

This map is a reflection. The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are generated by the maps $\sigma^a_i \in \mathrm{Hom}(a, r_i(a))$

with $i \in I$, $a \in A$. Formally, for $a, b \in A$ the set $\mathrm{Hom}(a,b)$ consists of the triples (b, f, a), where

$$f = \sigma_{i_n}^{r_{i_{n-1}} \cdots r_{i_1}(a)} \cdots \sigma_{i_2}^{r_{i_1}(a)} \sigma_{i_1}^a$$

and $b = r_{i_n} \cdots r_{i_2} r_{i_1}(a)$ for some $n \in \mathbb{N}_0$ and $i_1, \dots, i_n \in I$. The composition is induced by the group structure of $\mathrm{Aut}(\mathbb{Z}^I)$:

$$(a_3, f_2, a_2) \circ (a_2, f_1, a_1) = (a_3, f_2 f_1, a_1)$$

for all morphisms (a_3, f_2, a_2) and (a_2, f_1, a_1) of $\mathcal{W}(\mathcal{C})$. By abuse of notation we will write $f \in \mathrm{Hom}(a,b)$ instead of $(b, f, a) \in \mathrm{Hom}(a,b)$. The class of morphisms of $\mathcal{W}(\mathcal{C})$ will be denoted by $\mathcal{W}(\mathcal{C})$.

The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$.

Recall that a groupoid is a category such that all morphisms are isomorphisms. The Weyl groupoid $\mathcal{W}(\mathcal{C})$ of a Cartan scheme $\mathcal{C}$ is a groupoid, see [CH09]. For all $i \in I$ and $a \in A$ the inverse of σ_i^a is $\sigma_i^{r_i(a)}$. If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes, then $\mathcal{W}(\mathcal{C})$ and $\mathcal{W}(\mathcal{C}')$ are isomorphic groupoids.

A groupoid G is called *connected*, if for each $a, b \in \mathrm{Ob}(G)$ the class $\mathrm{Hom}(a,b)$ is non-empty. Hence $\mathcal{W}(\mathcal{C})$ is a connected groupoid if and only if $\mathcal{C}$ is a connected Cartan scheme.

Definition 2.3. Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subset \mathbb{Z}^I$, and define $m_{i,j}^a = |R^a \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a \in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma_i^a(R^a) = R^{r_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m_{i,j}^a$ is finite, then $(r_i r_j)^{m_{i,j}^a}(a) = a$.

If $\mathcal{R}$ is a root system of type $\mathcal{C}$, then $\mathcal{W}(\mathcal{R}) = \mathcal{W}(\mathcal{C})$ is the *Weyl groupoid of* $\mathcal{R}$. Further, $\mathcal{R}$ is called *connected*, if $\mathcal{C}$ is a connected Cartan scheme. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a \in A})$ is a root system of type $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R}'(\mathcal{C}', ({R'}^a_{a \in A'}))$ is a root system of type $\mathcal{C}'$, then we say that $\mathcal{R}$ and $\mathcal{R}'$ are *equivalent*, if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes given by maps $\varphi_0 : I \to I'$, $\varphi_1 : A \to A'$ as in Def. 2.1, and if the map $\varphi_0^* : \mathbb{Z}^I \to \mathbb{Z}^{I'}$ given by $\varphi_0^*(\alpha_i) = \alpha_{\varphi_0(i)}$ satisfies $\varphi_0^*(R^a) = {R'}^{\varphi_1(a)}$ for all $a \in A$.

There exist many interesting examples of root systems of type $\mathcal{C}$ related to semisimple Lie algebras, Lie superalgebras and Nichols algebras of diagonal type, respectively. For further details and results we refer to [HY08] and [CH09].

Remark 2.4. Let $\mathfrak{g}$ be a finite-dimensional contragredient Lie superalgebra with a root system containing a non-isotropic odd root α. Then 2α is an even root and hence the root system of $\mathfrak{g}$ does not satisfy the axioms in Def. 2.3. However, if for all such α one removes 2α from the root system, then one obtains a root system in the sense of Def. 2.3. Observe that for the construction of a PBW basis of the universal enveloping algebra of $\mathfrak{g}$ the root 2α is not necessary since $x_\alpha^n \neq 0$ for a root vector x_α of weight α and for all $n > 0$.

The appearance of various different sets R^a corresponds to the observation that $\mathfrak{g}$ has non-isomorphic Borel subalgebras and the root systems corresponding to these Borel subalgebras may take different forms.

Convention 2.5. In connection with Cartan schemes $\mathcal{C}$, upper indices usually refer to elements of A. Often, these indices will be omitted if they are uniquely determined by the context. In particular, for any $w, w' \in \mathcal{W}(\mathcal{C})$ and $a \in A$, the notation $1^a w$ and $w' 1^a$ means that $w \in \mathrm{Hom}(\_, a)$ and $w' \in \mathrm{Hom}(a, \_)$, respectively.

A fundamental result about Weyl groupoids is the following theorem.

Theorem 2.6. [HY08, Thm. 1] *Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Let $\mathcal{W}$ be the abstract groupoid with $\mathrm{Ob}(\mathcal{W}) = A$ such that the class of morphisms of $\mathcal{W}$ is generated by abstract morphisms $s_i^a \in \mathrm{Hom}(a, r_i(a))$, where $i \in I$ and $a \in A$, satisfying the relations*

$$s_i s_i 1^a = 1^a, \quad (s_j s_k)^{m^a_{j,k}} 1^a = 1^a, \qquad a \in A,\, i, j, k \in I,\, j \neq k,$$

see Conv. 2.5. Here 1^a is the identity of the object a, and $(s_j s_k)^\infty 1^a$ is understood to be 1^a. The functor $\mathcal{W} \to \mathcal{W}(\mathcal{R})$, which is the identity on the objects, and on the class of morphisms is given by $s_i^a \mapsto \sigma_i^a$ for all $i \in I$, $a \in A$, is an isomorphism of groupoids.

If $\mathcal{C}$ is a Cartan scheme, then the Weyl groupoid $\mathcal{W}(\mathcal{C})$ admits a length function $\ell : \mathcal{W}(\mathcal{C}) \to \mathbb{N}_0$ such that

$$\ell(w) = \min\{k \in \mathbb{N}_0 \mid \exists i_1, \dots, i_k \in I, a \in A : w = \sigma_{i_1} \cdots \sigma_{i_k} 1^a\} \tag{2.3}$$

for all $w \in \mathcal{W}(\mathcal{C})$. If there exists a root system of type $\mathcal{C}$, then ℓ has very similar properties to the well-known length function for Weyl groups. For example,

$$\ell(w) = |w(R^a_+) \cap -R^b_+|, \tag{2.4}$$

for all $w \in \mathrm{Hom}(a, b)$ and $a, b \in A$ by [HY08, Lemma 8(iii)].

Lemma 2.7. *Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. Let $a \in A$. Then $-c^a_{ij} = \max\{m \in \mathbb{N}_0 \mid \alpha_j + m\alpha_i \in R^a_+\}$ for all $i, j \in I$ with $i \neq j$.*

Proof. By (C2) and (R3), $\sigma_i^{r_i(a)}(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \in R^a_+$. Hence $-c^a_{ij} \leq \max\{m \in \mathbb{N}_0 \mid \alpha_j + m\alpha_i \in R^a_+\}$. On the other hand, if $\alpha_j + m\alpha_i \in R^a_+$, then $\sigma_i^a(\alpha_j + m\alpha_i) = \alpha_j + (-c^a_{ij} - m)\alpha_i \in R_+^{r_i(a)}$ by (R3) and (R1), and hence $m \leq -c^a_{ij}$. This proves the lemma. □

Remark 2.8. In the context of Lemma 2.7 it is in general not true that $\alpha_j + m\alpha_i \in R^a_+$ for all $m \in \{0, 1, \dots, -c^a_{ij}\}$. Nevertheless, in the cases of our interest the sets R^a_+ will be associated to bicharacters, see Subsect. 2.2. In this case the above property holds by [Ros98, Lemma 14].

Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. We say that $\mathcal{R}$ is *finite*, if R^a is finite for all $a \in A$. The following lemmas are well-known for traditional root systems.

Lemma 2.9. [CH09, Lemma 2.11] *Let $\mathcal{C}$ be a connected Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. The following are equivalent.*

(1) *$\mathcal{R}$ is finite.*
(2) *R^a is finite for at least one $a \in A$.*
(3) *$\mathcal{W}(\mathcal{R})$ is finite.*

Lemma 2.10. [HY08, Cors. 2,5] *Let $\mathcal{C}$ be a connected Cartan scheme and $\mathcal{R}$ a finite root system of type $\mathcal{C}$. Then for all $a \in A$ there exist unique elements $b \in A$ and $w \in \mathrm{Hom}(b,a)$ such that $|R^a_+| = \ell(w) \geq \ell(w')$ for all $w' \in \mathrm{Hom}(b',a')$, $a',b' \in A$. If $w = 1^a\sigma_{i_1}\cdots\sigma_{i_n}$, where $n = \ell(w)$ and $i_1,\dots,i_n \in I$, then*

$$R^a_+ = \{1^a\sigma_{i_1}\cdots\sigma_{i_{k-1}}(\alpha_{i_k}) | k \in \{1,2,\dots,n\}\}.$$

Moreover, $\ell((w'')^{-1}w) + \ell(w'') = \ell(w)$ for all $w'' \in \mathrm{Hom}(b'',a)$, $b'' \in A$.

2.2. **The Weyl groupoid of a bicharacter.** Let I be a non-empty finite set. Recall that a bicharacter on $\mathbb{Z}^I$ with values in $\Bbbk^\times$ is a map $\chi : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times$ such that

$$\chi(a+b,c) = \chi(a,c)\chi(b,c), \qquad \chi(c,a+b) = \chi(c,a)\chi(c,b) \tag{2.5}$$

for all $a,b,c \in \mathbb{Z}^I$. Then $\chi(0,a) = \chi(a,0) = 1$ for all $a \in \mathbb{Z}^I$. Let $\mathcal{X}$ be the set of bicharacters on $\mathbb{Z}^I$. If $\chi \in \mathcal{X}$, then

$$\chi^{\mathrm{op}} : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times, \qquad \chi^{\mathrm{op}}(a,b) = \chi(b,a), \tag{2.6}$$

$$\chi^{-1} : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times, \qquad \chi^{-1}(a,b) = \chi(a,b)^{-1}, \tag{2.7}$$

and for all $w \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$ the map

$$w^*\chi : \mathbb{Z}^I \times \mathbb{Z}^I \to \Bbbk^\times, \qquad w^*\chi(a,b) = \chi(w^{-1}(a), w^{-1}(b)), \tag{2.8}$$

are bicharacters on $\mathbb{Z}^I$. The equation

$$(ww')^*\chi = w^*(w'^*\chi) \tag{2.9}$$

holds for all $w,w' \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$ and all $\chi \in \mathcal{X}$.

Definition 2.11. Let $\chi \in \mathcal{X}$, $p \in I$, and $q_{ij} = \chi(\alpha_i,\alpha_j)$ for all $i,j \in I$. We say that χ is *p-finite*, if for all $j \in I \setminus \{p\}$ there exists $m \in \mathbb{N}_0$ such that $(m+1)_{q_{pp}} = 0$ or $q_{pp}^m q_{pj} q_{jp} = 1$.

Assume that χ is p-finite. Let $c^\chi_{pp} = 2$, and for all $j \in I \setminus \{p\}$ let

$$c^\chi_{pj} = -\min\{m \in \mathbb{N}_0 \,|\, (m+1)_{q_{pp}}(q_{pp}^m q_{pj} q_{jp} - 1) = 0\}.$$

If χ is i-finite for all $i \in I$, then the matrix $C^\chi = (c^\chi_{ij})_{i,j\in I}$ is called the *Cartan matrix* associated to χ. It is a generalized Cartan matrix, see Sect. 2.1.

Remark 2.12. Let $\chi \in \mathcal{X}$, $p \in I$, and $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$. Let us discuss in more detail the meaning of Def. 2.11. If $q_{pp}^m = 1$ and $q_{pp} \neq 1$, then χ is p-finite and $-c_{pj}^\chi \le m-1$ for all $j \in I$. If $q_{pp} = 1$ then χ is p-finite if and only if the characteristic of $\Bbbk$ is positive or if $q_{pj}q_{jp} = 1$ for all $j \in I \setminus \{p\}$. Finally, if $q_{pp}^m \neq 1$ for all $m \in \mathbb{N}$ then χ is p-finite if and only if for all $j \in I \setminus \{p\}$ there exists $m \in \mathbb{N}_0$ such that $q_{pj}q_{jp} = q_{pp}^{-m}$.

For all $p \in I$ and $\chi \in \mathcal{X}$, where χ is p-finite, let $\sigma_p^\chi \in \mathrm{Aut}_\mathbb{Z}(\mathbb{Z}^I)$,

$$\sigma_p^\chi(\alpha_j) = \alpha_j - c_{pj}^\chi \alpha_p \quad \text{for all } j \in I.$$

Towards the definition of the Weyl groupoid of a bicharacter, we define bijections $r_p : \mathcal{X} \to \mathcal{X}$ for all $p \in I$. Namely, let

$$r_p : \mathcal{X} \to \mathcal{X}, \quad r_p(\chi) = \begin{cases} (\sigma_p^\chi)^*\chi & \text{if } \chi \text{ is } p\text{-finite}, \\ \chi & \text{otherwise.} \end{cases}$$

Let $p \in I$, $\chi \in \mathcal{X}$, $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$. If χ is p-finite, then

$$\begin{aligned} r_p(\chi)(\alpha_p, \alpha_p) &= q_{pp}, & r_p(\chi)(\alpha_p, \alpha_j) &= q_{pj}^{-1} q_{pp}^{c_{pj}^\chi}, \\ r_p(\chi)(\alpha_i, \alpha_p) &= q_{ip}^{-1} q_{pp}^{c_{pi}^\chi}, & r_p(\chi)(\alpha_i, \alpha_j) &= q_{ij} q_{ip}^{-c_{pj}^\chi} q_{pj}^{-c_{pi}^\chi} q_{pp}^{c_{pi}^\chi c_{pj}^\chi} \end{aligned} \tag{2.10}$$

for all $i, j \in I \setminus \{p\}$. It is a small exercise to check that then $(\sigma_p^\chi)^*\chi$ is p-finite, and

$$c_{pj}^{r_p(\chi)} = c_{pj}^\chi \quad \text{for all } j \in I, \qquad r_p^2(\chi) = \chi. \tag{2.11}$$

The involutions r_p, where $p \in I$, generate a subgroup

$$\mathcal{G} = \langle r_p \,|\, p \in I \rangle$$

of the group of bijections of the set $\mathcal{X}$. For all $\chi \in \mathcal{X}$ let $\mathcal{G}(\chi)$ denote the $\mathcal{G}$-orbit of χ under the action of $\mathcal{G}$.

Let $\chi \in \mathcal{X}$ such that χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$. By Eq. (2.11) we obtain that

$$\mathcal{C}(\chi) = \mathcal{C}(I, \mathcal{G}(\chi), (r_p)_{p \in I}, (C^{\chi'})_{\chi' \in \mathcal{G}(\chi)})$$

is a connected Cartan scheme. The Weyl groupoid of χ is then the Weyl groupoid of the Cartan scheme $\mathcal{C}(\chi)$ and is denoted by $\mathcal{W}(\chi)$. Clearly, $\mathcal{C}(\chi) = \mathcal{C}(\chi')$ and $\mathcal{W}(\chi) = \mathcal{W}(\chi')$ for all $\chi' \in \mathcal{G}(\chi)$.

Example 2.13. Let $C = (c_{ij})_{i,j \in I}$ be a generalized Cartan matrix. Let $\chi \in \mathcal{X}$, $q_{ij} = \chi(\alpha_i, \alpha_j)$ for all $i, j \in I$, and assume that $q_{ii}^{c_{ij}} = q_{ij}q_{ji}$ for all $i, j \in I$, and that $(m+1)_{q_{ii}} \neq 0$ for all $i \in I$ and $m \in \mathbb{N}_0$ with $m < \max\{-c_{ij} \,|\, j \in I \setminus \{i\}\}$. (The latter is not an essential assumption, since if it fails, then one can replace C by another generalized Cartan matrix $\tilde{C}$, such that χ has this property with respect to $\tilde{C}$.) One says that χ is of *Cartan type*. Then χ is i-finite for all $i \in I$, and $c_{ij}^\chi = c_{ij}$ for all $i, j \in I$. Eq. (2.10) gives that

$$\begin{aligned} r_p(\chi)(\alpha_i, \alpha_i) &= q_{ii} = \chi(\alpha_i, \alpha_i), \\ r_p(\chi)(\alpha_i, \alpha_j)\, r_p(\chi)(\alpha_j, \alpha_i) &= q_{ij}q_{ji} = r_p(\chi)(\alpha_i, \alpha_i)^{c_{pi}} \end{aligned}$$

for all $p, i, j \in I$. Hence $r_p(\chi)$ is again of Cartan type with the same Cartan matrix C. Thus χ' is i-finite for all $\chi' \in \mathcal{G}(\chi)$ and $i \in I$.

Let $C = (c_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix, and for all $i \in I$ let $d_i \in \mathbb{N}$ such that $d_i c_{ij} = d_j c_{ji}$ for all $i, j \in I$. Let $q \in \Bbbk^\times$ such that $(m+1)_{q^{2d_i}} \neq 0$ for all $m \in \mathbb{N}_0$ with $m < -c_{ij}$ for some $j \in I$. Define $\chi \in \mathcal{X}$ by $\chi(\alpha_i, \alpha_j) = q^{d_i c_{ij}}$. Then χ is of Cartan type, hence χ is p-finite for all $p \in I$. Eq. (2.10) implies that $r_p(\chi) = \chi$ for all $p \in I$, and hence $\mathcal{G}(\chi)$ consists of precisely one element. In this case the Weyl groupoid $\mathcal{W}(\chi)$ is a group, which is precisely the Weyl group associated to the generalized Cartan matrix C. We will study this example in Sect. 8 under the assumption that C is of finite type.

2.3. **Roots.** Let $\chi \in \mathcal{X}$. There exists a canonical root system of type $\mathcal{C}(\chi)$ which we describe in this subsection. It is based on the construction of a restricted PBW basis of Nichols algebras of diagonal type. Nichols algebras are braided Hopf algebras defined by a universal property. More details can be found in [AS02] on braided Hopf algebras and Nichols algebras, in [Kha99] on the PBW basis, and in [Hec06] on the root system.

Let $V \in {}^{\Bbbk\mathbb{Z}^I}_{\Bbbk\mathbb{Z}^I}\mathcal{YD}$ be a $|I|$-dimensional Yetter–Drinfel'd module of diagonal type. Let $\delta : V \to \Bbbk\mathbb{Z}^I \otimes V$ and $\cdot : \Bbbk\mathbb{Z}^I \otimes V \to V$ denote the left coaction and the left action of $\Bbbk\mathbb{Z}^I$ on V, respectively. Fix a basis $\{x_i \,|\, i \in I\}$ of V, elements g_i, where $i \in I$, and a matrix $(q_{ij})_{i,j\in I} \in (\Bbbk^\times)^{I\times I}$, such that

$$\delta(x_i) = g_i \otimes x_i, \quad g_i \cdot x_j = q_{ij} x_j \quad \text{for all } i, j \in I.$$

Assume that $\chi(\alpha_i, \alpha_j) = q_{ij}$ for all $i, j \in I$. For all $\alpha \in \mathbb{Z}^I$ define the "bound function"

$$b^\chi(\alpha) = \begin{cases} \min\{m \in \mathbb{N} \,|\, (m)_{\chi(\alpha,\alpha)} = 0\} & \text{if } (m)_{\chi(\alpha,\alpha)} = 0 \\ & \text{for some } m \in \mathbb{N}, \\ \infty & \text{otherwise.} \end{cases} \tag{2.12}$$

Let us see, what this means. By the definition of the number $(m)_{\chi(\alpha,\alpha)}$, if $\chi(\alpha, \alpha) = 1$ and the characteristic of $\Bbbk$ is positive then $b^\chi(\alpha)$ coincides with the characteristic of $\Bbbk$. If $\chi(\alpha, \alpha) = 1$ and the characteristic of $\Bbbk$ is zero, then $b^\chi(\alpha) = \infty$. If $\chi(\alpha, \alpha)$ is a primitive mth root of 1 for some integer $m > 1$, then $b^\chi(\alpha) = m$. Otherwise $b^\chi(\alpha) = \infty$.

If $p \in I$ such that χ is p-finite, then

$$b^{r_p(\chi)}(\sigma_p^\chi(\alpha)) = b^\chi(\alpha) \quad \text{for all } \alpha \in \mathbb{Z}^I \tag{2.13}$$

by Eq. (2.8).

The tensor algebra $T(V)$ admits a universal braided Hopf algebra quotient $\mathfrak{B}(V)$, called the *Nichols algebra of* V. As an algebra, $\mathfrak{B}(V)$ has a unique $\mathbb{Z}^I$-grading

$$\mathfrak{B}(V) = \oplus_{\alpha\in\mathbb{Z}^I} \mathfrak{B}(V)_\alpha \tag{2.14}$$

such that $\deg x_i = \alpha_i$ for all $i \in I$. This is also a coalgebra grading. There exists a totally ordered index set $(L, \leq)$ and a family $(y_l)_{l\in L}$ of $\mathbb{Z}^I$-homogeneous elements

$y_l \in \mathfrak{B}(V)$ such that the set

$$\begin{aligned}\{y_{l_1}^{m_1} y_{l_2}^{m_2} \cdots y_{l_k}^{m_k} \,|\, k \geq 0,\, l_1, \dots, l_k \in L,\, l_1 > l_2 > \cdots > l_k,\\ m_i \in \mathbb{N},\, m_i < b^\chi(\deg y_{l_i}) \quad \text{for all } i \in I\}\end{aligned} \tag{2.15}$$

forms a vector space basis of $\mathfrak{B}(V)$. Let

$$R_+^\chi = \{\deg y_l \,|\, l \in L\} \subset \mathbb{Z}^I. \tag{2.16}$$

In particular, $|R_+^\chi| \leq |L|$ and equality holds if and only if the degrees $\deg y_l$ are pairwise distinct. The set R_+^χ depends on the matrix $(q_{ij})_{i,j\in I}$, but not on the choice of the basis $\{x_i \,|\, i \in I\}$, the set L, and the elements g_i, $i \in I$, and y_l, $l \in L$. Let

$$R^\chi = R_+^\chi \cup -R_+^\chi.$$

Let $m \in \mathbb{N}$ and $i, j \in I$ with $i \neq j$. Then

$$\begin{aligned}\prod_{s=1}^{m} \big((s)_{q_{ii}}(1 - q_{ii}^{s-1} q_{ij} q_{ji})\big) \neq 0\\ \Leftrightarrow s\alpha_i + \alpha_j \in R_+^\chi \text{ for all } s \in \{0, 1, 2, \dots, m\}\end{aligned} \tag{2.17}$$

by [Ros98, Lemma 14, Cor. 18]. Further, if χ is p-finite, then

$$\sigma_p^\chi(R_+^\chi \setminus \{\alpha_p\}) = R_+^{r_p(\chi)} \setminus \{\alpha_p\} \tag{2.18}$$

by [Hec06, Prop. 1], so $\sigma_p^\chi(R^\chi) = R^{r_p(\chi)}$.

Theorem 2.14. [Hec10, Thm. 3.13] *Let $\chi \in \mathcal{X}$ such that χ' is p-finite for all $p \in I$. $\chi' \in \mathcal{G}(\chi)$. Then $\mathcal{R}(\chi) = \mathcal{R}(\mathcal{C}(\chi), (R^{\chi'})_{\chi' \in \mathcal{G}(\chi)})$ is a root system of type $\mathcal{C}(\chi)$.*

Roots with finite bounds often play a distinguished role. For all $\chi \in \mathcal{X}$ let

$$R_{+\mathrm{fin}}^\chi = \{\beta \in R_+^\chi \,|\, b^\chi(\beta) < \infty\}, \quad R_{+\infty}^\chi = R_+^\chi \setminus R_{+\mathrm{fin}}^\chi. \tag{2.19}$$

We will use several finiteness properties of bicharacters.

$$\mathcal{X}_1 = \{\chi \in \mathcal{X} \,|\, \chi \text{ is } p\text{-finite for all } p \in I\}, \tag{2.20}$$

$$\mathcal{X}_2 = \{\chi \in \mathcal{X} \,|\, \chi' \text{ is } p\text{-finite for all } \chi' \in \mathcal{G}(\chi),\, p \in I\}, \tag{2.21}$$

$$\mathcal{X}_3 = \{\chi \in \mathcal{X} \,|\, R^\chi \text{ is finite}\}, \tag{2.22}$$

$$\mathcal{X}_4 = \{\chi \in \mathcal{X} \,|\, R^\chi \text{ is finite},\, R_+^\chi = R_{+\mathrm{fin}}^\chi\}, \tag{2.23}$$

$$\mathcal{X}_5 = \{\chi \in \mathcal{X}_4 \,|\, \chi(\alpha, \alpha) \neq 1 \text{ for all } \alpha \in R_+^\chi\}. \tag{2.24}$$

Clearly, $\mathcal{X}_i \supset \mathcal{X}_j$ for $1 \leq i < j \leq 5$, where the inclusion $\mathcal{X}_2 \supset \mathcal{X}_3$ follows from (2.17) and (R3). Eq. (2.12) implies that $\chi \in \mathcal{X}_5$ if and only if R^χ is finite and $\chi(\alpha, \alpha)$ is a root of 1 different from 1 for all $\alpha \in R_+^\chi$.

Lemma 2.15. *Let $\chi, \chi' \in \mathcal{X}_2$.*

(i) If $R_+^\chi = R_+^{\chi'}$, then $C^{w^*\chi} = C^{w^*\chi'}$ for all $w \in \mathrm{Hom}(\chi, \_) \subset \mathcal{W}(\chi)$.

(ii) Assume that $\chi, \chi' \in \mathcal{X}_3$. If $C^{w^*\chi} = C^{w^*\chi'}$ for all $w \in \mathrm{Hom}(\chi, \_) \subset \mathcal{W}(\chi)$, then $R_+^\chi = R_+^{\chi'}$.

Proof. By Thm. 2.14, $\mathcal{R}(\chi)$ is a root system of type $\mathcal{C}(\chi)$.

(i) Assume that $R^\chi_+ = R^{\chi'}_+$. Then $C^\chi = C^{\chi'}$ by Lemma 2.7. Therefore $\sigma_i^\chi = \sigma_i^{\chi'}$ in $\mathrm{Aut}(\mathbb{Z}^I)$. Since $\chi, \chi' \in \mathcal{X}_2$, by induction it follows that $\sigma_{i_1} \cdots \sigma_{i_k}^\chi = \sigma_{i_1} \cdots \sigma_{i_k}^{\chi'}$ in $\mathrm{Aut}(\mathbb{Z}^I)$ and $C^{(\sigma_{i_1} \cdots \sigma_{i_k}^\chi)^* \chi} = C^{(\sigma_{i_1} \cdots \sigma_{i_k}^{\chi'})^* \chi'}$ for all $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$. Hence $C^{w^*\chi} = C^{w^*\chi'}$ for all $w \in \mathrm{Hom}(\chi, \_) \subset \mathcal{W}(\chi)$.

(ii) Since $\chi \in \mathcal{X}_3$, $R^\chi = \{w^{-1}(\alpha_i) \,|\, w \in \mathrm{Hom}(\chi, \_) \subset \mathcal{W}(\chi)\}$ by [CH09, Prop. 2.12]. By assumption on the Cartan matrices, $\sigma_{i_1} \cdots \sigma_{i_k}^\chi = \sigma_{i_1} \cdots \sigma_{i_k}^{\chi'}$ in $\mathrm{Aut}(\mathbb{Z}^I)$ for all $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$. Hence $R^\chi = R^{\chi'}$, and the lemma holds by (R1). □

For our study of Drinfel'd doubles we will use an analog of the sum of fundamental weights, commonly known as ρ. More precisely, we define a character version of the linear form $(2\rho, \cdot)$, where $(\cdot, \cdot)$ is the usual bilinear form on the weight lattice.

Let $\widehat{\mathbb{Z}^I} = \mathrm{Hom}(\mathbb{Z}^I, \Bbbk^\times)$ denote the group of characters of $\mathbb{Z}^I$ with values in $\Bbbk^\times$.

Definition 2.16. Let $\chi \in \mathcal{X}$. Let $\rho^\chi \in \widehat{\mathbb{Z}^I}$ such that

$$\rho^\chi(\alpha_i) = \chi(\alpha_i, \alpha_i) \quad \text{for all } i \in I.$$

Lemma 2.17. *Let $\chi \in \mathcal{X}$, $p \in I$, and $b = b^\chi(\alpha_p)$. Assume that $b < \infty$. Then χ is p-finite and*

$$\chi(\alpha_p, \beta)^{b-1} \chi(\beta, \alpha_p)^{b-1} = \frac{\rho^{r_p(\chi)}(\sigma_p^\chi(\beta))}{\rho^\chi(\beta)}$$

for all $\beta \in \mathbb{Z}^I$.

Proof. Define $\xi_1, \xi_2 : \mathbb{Z}^I \to \Bbbk^\times$ by

$$\xi_1(\beta) = \chi(\alpha_p, \beta)^{b-1} \chi(\beta, \alpha_p)^{b-1}, \qquad \xi_2(\beta) = \frac{\rho^{r_p(\chi)}(\sigma_p^\chi(\beta))}{\rho^\chi(\beta)}.$$

Then $\xi_1, \xi_2 \in \widehat{\mathbb{Z}^I}$. Thus it suffices to prove that $\xi_1(\alpha_j) = \xi_2(\alpha_j)$ for all $j \in I$. Let $q_{jk} = \chi(\alpha_j, \alpha_k)$ for all $j, k \in I$. Then $q_{pp}^b = 1$ since $(b)_{q_{pp}} = 0$. Moreover,

$$\begin{aligned} \rho^{r_p(\chi)}(\alpha_j) =& r_p(\chi)(\alpha_j, \alpha_j) \\ =& \chi(\alpha_j - c_{pj}^\chi \alpha_p, \alpha_j - c_{pj}^\chi \alpha_p) = q_{jj} (q_{pj} q_{jp})^{-c_{pj}^\chi} q_{pp}^{c_{pj}^\chi c_{pj}^\chi} \end{aligned}$$

for all $j \in I$.

By assumption, χ is p-finite, and hence for all $j \in I \setminus \{p\}$ we have $q_{pp}^{c_{pj}^\chi} = q_{pj} q_{jp}$ or $q_{pp}^{1 - c_{pj}^\chi} = 1$. Let $j \in I$. If $q_{pp}^{c_{pj}^\chi} = q_{pj} q_{jp}$, then

$$\rho^{r_p(\chi)}(\alpha_j) = q_{jj},$$

$$\xi_2(\alpha_j) = q_{jj}^{-1} \rho^{r_p(\chi)}(\sigma_p^\chi(\alpha_j)) = q_{jj}^{-1} \rho^{r_p(\chi)}(\alpha_j - c_{pj}^\chi \alpha_p) = q_{pp}^{-c_{pj}^\chi},$$

$$\xi_1(\alpha_j) = q_{pj}^{b-1} q_{jp}^{b-1} = q_{pp}^{(b-1) c_{pj}^\chi} = q_{pp}^{-c_{pj}^\chi},$$

and hence $\xi_1(\alpha_j)=\xi_2(\alpha_j)$. Otherwise $b=1-c^\chi_{pj}$, $q_{pp}^{c^\chi_{pj}}=q_{pp}$, and then

$$\rho^{r_p(\chi)}(\alpha_p)=q_{pp},\quad \rho^{r_p(\chi)}(\alpha_j)=q_{jj}(q_{pj}q_{jp})^{-c^\chi_{pj}}q_{pp},$$
$$\xi_2(\alpha_j)=q_{jj}^{-1}\rho^{r_p(\chi)}(\sigma^\chi_p(\alpha_j))=q_{jj}^{-1}\rho^{r_p(\chi)}(\alpha_j-c^\chi_{pj}\alpha_p)=(q_{pj}q_{jp})^{-c^\chi_{pj}},$$
$$\xi_1(\alpha_j)=q_{pj}^{b-1}q_{jp}^{b-1}=(q_{pj}q_{jp})^{-c^\chi_{pj}}.$$

Hence $\xi_1(\alpha_j)=\xi_2(\alpha_j)$ also in this case. This proves the lemma. □

3. Multiparameter Drinfel'd doubles

In this paper we study Verma modules for a class of Hopf algebras introduced in [Hec10]. This class contains multiparameter quantizations of semisimple Lie algebras and basic classical Lie superalgebras. The precise definition is given in Eq. (3.14). It uses the Drinfel'd double construction and the theory of Nichols algebras.

The Drinfel'd double [Jos95, Sect. 3.2] can be defined via a skew-Hopf pairing of two Hopf algebras or as the quotient of a free associative algebra by a certain ideal, see also Rem. 3.3. The first approach is more technical, but also more powerful. We present here the second definition. For proofs see [Hec10].

Let I be a non-empty finite set, χ a bicharacter on $\mathbb{Z}^I$ with values in $\Bbbk^\times$, and $q_{ij}=\chi(\alpha_i,\alpha_j)$ for all $i,j\in I$. Let $\mathcal{U}(\chi)$ be the unital associative $\Bbbk$-algebra with generators K_i, K_i^{-1}, L_i, L_i^{-1}, E_i, and F_i, where $i\in I$, and defining relations

$$XY=YX \quad \text{for all } X,Y\in\{K_i,K_i^{-1},L_i,L_i^{-1}\,|\,i\in I\}, \tag{3.1}$$
$$K_iK_i^{-1}=1, \qquad L_iL_i^{-1}=1, \tag{3.2}$$
$$K_iE_jK_i^{-1}=q_{ij}E_j, \qquad L_iE_jL_i^{-1}=q_{ji}^{-1}E_j, \tag{3.3}$$
$$K_iF_jK_i^{-1}=q_{ij}^{-1}F_j, \qquad L_iF_jL_i^{-1}=q_{ji}F_j, \tag{3.4}$$
$$E_iF_j-F_jE_i=\delta_{i,j}(K_i-L_i), \tag{3.5}$$

where $i,j\in I$, and $\delta_{i,j}$ denotes Kronecker's δ. The algebra $\mathcal{U}(\chi)$ can be given a Hopf algebra structure in many different ways. We will use the unique Hopf algebra structure determined by

$$\begin{cases} \varepsilon(K_i)=1,\quad \varepsilon(E_i)=0, & \varepsilon(L_i)=1,\quad \varepsilon(F_i)=0,\\ \Delta(K_i)=K_i\otimes K_i, & \Delta(L_i)=L_i\otimes L_i,\\ \Delta(K_i^{-1})=K_i^{-1}\otimes K_i^{-1}, & \Delta(L_i^{-1})=L_i^{-1}\otimes L_i^{-1},\\ \Delta(E_i)=E_i\otimes 1+K_i\otimes E_i, & \Delta(F_i)=1\otimes F_i+F_i\otimes L_i \end{cases} \tag{3.6}$$

for all $i\in I$.

Let $\mathcal{U}^{+0},\mathcal{U}^{-0}$, and $\mathcal{U}^0$ denote the commutative cocommutative Hopf subalgebras of $\mathcal{U}(\chi)$ generated by $\{K_i,K_i^{-1}\,|\,i\in I\}$, $\{L_i,L_i^{-1}\,|\,i\in I\}$, and $\{K_i,K_i^{-1},L_i,L_i^{-1}\,|\,i\in I\}$, respectively. For any $\alpha=\sum_{i\in I}m_i\alpha_i\in\mathbb{Z}^I$ let $K_\alpha=\prod_{i\in I}K_i^{m_i}$ and $L_\alpha=\prod_{i\in I}L_i^{m_i}$. Then the set

$$\{K_\alpha L_\beta|\alpha,\beta\in\mathbb{Z}^I\}$$

is a $\Bbbk$-basis of $\mathcal{U}^0$.

Let $\mathcal{U}^+(\chi)$, $\mathcal{V}^+(\chi)$, $\mathcal{U}^-(\chi)$, and $\mathcal{V}^-(\chi)$ denote the subalgebras of $\mathcal{U}(\chi)$ generated by $\{E_i \,|\, i \in I\}$, $\{E_i, K_i, K_i^{-1} \,|\, i \in I\}$, $\{F_i \,|\, i \in I\}$, and $\{F_i, L_i, L_i^{-1} \,|\, i \in I\}$, respectively. Then $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ are Hopf subalgebras of $\mathcal{U}(\chi)$.

The algebra $\mathcal{U}(\chi)$ admits a unique $\mathbb{Z}^I$-grading

$$\begin{gathered}\mathcal{U}(\chi) = \oplus_{\beta\in\mathbb{Z}^I}\mathcal{U}(\chi)_\beta,\\ 1 \in \mathcal{U}(\chi)_0, \quad \mathcal{U}(\chi)_\beta\mathcal{U}(\chi)_\gamma \subset \mathcal{U}(\chi)_{\beta+\gamma} \quad \text{for all } \beta,\gamma \in \mathbb{Z}^I,\end{gathered} \tag{3.7}$$

such that $K_i, K_i^{-1}, L_i, L_i^{-1} \in \mathcal{U}(\chi)_0$, $E_i \in \mathcal{U}(\chi)_{\alpha_i}$, and $F_i \in \mathcal{U}(\chi)_{-\alpha_i}$ for all $i \in I$. Let

$$\mathbb{N}_0^I = \Big\{ \sum_{i\in I} a_i\alpha_i \,|\, a_i \in \mathbb{N}_0 \Big\} \subset \mathbb{Z}^I,$$

and for any subspace $\mathcal{U}' \subset \mathcal{U}(\chi)$ and any $\beta \in \mathbb{Z}^I$ let $\mathcal{U}'_\beta = \mathcal{U}' \cap \mathcal{U}(\chi)_\beta$. Then

$$\mathcal{U}^+(\chi) = \oplus_{\beta\in\mathbb{N}_0^I}\, \mathcal{U}^+(\chi)_\beta, \qquad \mathcal{U}^-(\chi) = \oplus_{\beta\in\mathbb{N}_0^I}\, \mathcal{U}^-(\chi)_{-\beta}.$$

For all $\beta \in \mathbb{Z}^I$ let

$$|\beta| = \sum_{i\in I} a_i \in \mathbb{Z}, \qquad \text{where } \beta = \sum_{i\in I} a_i\alpha_i. \tag{3.8}$$

The decomposition

$$\mathcal{U}(\chi) = \oplus_{m\in\mathbb{Z}}\mathcal{U}(\chi)_m, \quad \text{where} \quad \mathcal{U}(\chi)_m = \oplus_{\beta:|\beta|=m}\mathcal{U}(\chi)_\beta, \tag{3.9}$$

gives a $\mathbb{Z}$-grading of $\mathcal{U}(\chi)$ called the *standard grading*.

Proposition 3.1. *Let $\chi \in \mathcal{X}$.*

(1) Let $\underline{a} = (a_i \,|\, i \in I) \in (\Bbbk^\times)^I$. Then there exists a unique algebra automorphism $\varphi_{\underline{a}}$ of $\mathcal{U}(\chi)$ such that

$$\varphi_{\underline{a}}(K_i) = K_i,\ \varphi_{\underline{a}}(L_i) = L_i,\ \varphi_{\underline{a}}(E_i) = a_iE_i,\ \varphi_{\underline{a}}(F_i) = a_i^{-1}F_i. \tag{3.10}$$

(2) There is a unique algebra antiautomorphism Ω of $\mathcal{U}(\chi)$ such that

$$\Omega(K_i) = K_i, \qquad \Omega(L_i) = L_i, \qquad \Omega(E_i) = F_i, \qquad \Omega(F_i) = E_i. \tag{3.11}$$

It satisfies the relation $\Omega^2 = \mathrm{id}$.

Lemma 3.2. *For all $i \in I$ there exist unique linear maps $\partial_i^K, \partial_i^L \in \mathrm{End}_\Bbbk(\mathcal{U}^+(\chi))$ such that*

$$[E, F_i] = \partial_i^K(E)K_i - L_i\partial_i^L(E) \quad \textit{for all } E \in \mathcal{U}^+(\chi).$$

The maps $\partial_i^K, \partial_i^L \in \mathrm{End}_\Bbbk(\mathcal{U}^+(\chi))$ are skew-derivations. More precisely,

$$\partial_i^K(1) = \partial_i^L(1) = 0, \quad \partial_i^K(E_j) = \partial_i^L(E_j) = \delta_{i,j}, \tag{3.12}$$

$$\begin{aligned}\partial_i^K(EE') &= \partial_i^K(E)(K_i.E') + E\partial_i^K(E'),\\ \partial_i^L(EE') &= \partial_i^L(E)E' + (L_i^{-1}.E)\partial_i^L(E')\end{aligned} \tag{3.13}$$

for all $i,j \in I$ and $E, E' \in \mathcal{U}^+(\chi)$.

Let $\mathcal{I}^+(\chi)$ be the unique maximal ideal of $\mathcal{U}^+(\chi)$ such that $\mathcal{I}^+(\chi) \subset \ker \varepsilon$ and $\partial_i^K(\mathcal{I}^+(\chi)) \subset \mathcal{I}^+(\chi)$ for all $i \in I$. Equivalently, $\mathcal{I}^+(\chi)$ is the unique maximal ideal of $\mathcal{U}^+(\chi)$ such that $\mathcal{I}^+(\chi) \subset \ker \varepsilon$ and $\partial_i^L(\mathcal{I}^+(\chi)) \subset \mathcal{I}^+(\chi)$ for all $i \in I$, see [Hec10, Prop. 5.4]. Let $\mathcal{I}^-(\chi) = \Omega(\mathcal{I}^+(\chi))$. Let

$$U^+(\chi) = \mathcal{U}^+(\chi)/\mathcal{I}^+(\chi), \qquad U^-(\chi) = \mathcal{U}^-(\chi)/\mathcal{I}^-(\chi),$$
$$V^+(\chi) = \mathcal{V}^+(\chi)/\mathcal{I}^+(\chi)\mathcal{U}^{+0}, \qquad V^-(\chi) = \mathcal{V}^-(\chi)/\mathcal{I}^-(\chi)\mathcal{U}^{-0},$$

and

$$U(\chi) = \mathcal{U}(\chi)/(\mathcal{I}^+(\chi), \mathcal{I}^-(\chi)). \tag{3.14}$$

The canonical inclusions $\mathcal{U}^\pm(\chi) \subset \mathcal{U}(\chi)$, $\mathcal{U}^0 \subset \mathcal{U}(\chi)$ induce maps

$$\iota_+ : U^+(\chi) \to U(\chi), \quad \iota_0 : \mathcal{U}^0 \to U(\chi), \quad \iota_- : U^-(\chi) \to U(\chi).$$

Remark 3.3. (i) The vector space $V = \oplus_{i \in I} \Bbbk E_i$ is a Yetter–Drinfel'd module over the group algebra $\Bbbk\mathbb{Z}^I \simeq \Bbbk[K_i, K_i^{-1} \,|\, i \in I] \subset \mathcal{U}^0$, where the left action $\cdot : \Bbbk\mathbb{Z}^I \otimes V \to V$ and the left coaction $\delta : V \to \Bbbk\mathbb{Z}^I \otimes V$ are defined by

$$K_i \cdot E_j = q_{ij} E_j, \qquad \delta(E_i) = K_i \otimes E_i$$

for all $i, j \in I$. The algebra $U^+(\chi)$ is commonly known as the *Nichols algebra* of the Yetter–Drinfel'd module V.

(ii) There are various descriptions of the ideal $\mathcal{I}^+(\chi)$, see *e. g.* [AS02]. In case of quantized enveloping algebras, see Sect. 8, Serre relations generate the ideal $\mathcal{I}^+(\chi)$. A more general case is studied by Angiono [Ang09]. For quantized Lie superalgebras the defining relations are determined in [Yam99, Yam01]. It is in general an open problem to give a nice set of generators of $\mathcal{I}^+(\chi)$, see [And02, Question 5.9].

Proposition 3.4. *(Triangular decomposition) The map*

$$\mathrm{m}(\iota_- \otimes \iota_0 \otimes \iota_+) : U^-(\chi) \otimes \mathcal{U}^0 \otimes U^+(\chi) \to U(\chi)$$

is an isomorphism of $\mathbb{Z}^I$-graded vector spaces, where m *denotes the multiplication map. In particular,*

$$(\mathcal{I}^+(\chi), \mathcal{I}^-(\chi)) = \mathcal{I}^-(\chi)\mathcal{U}^0\mathcal{U}^+(\chi) + \mathcal{U}^-(\chi)\mathcal{U}^0\mathcal{I}^+(\chi).$$

Proof. The claim follows from the characterization of triangular decomposability in [Hec10, Prop. 4.17, Rem. 4.19]. The equivalent conditions for $\mathcal{I}^+(\chi)$ and $\mathcal{I}^-(\chi)$ are proven in [Hec10, Sect. 5], where $\mathcal{I}^+(\chi)$ is denoted by $\mathcal{S}^+(\chi)$. □

Following the convention in [Jos95, Sect. 3.2.1], a skew-Hopf pairing $\eta : A \times B \to \Bbbk$, $(x, y) \mapsto \eta(x, y)$, of two Hopf algebras A, B is a bilinear map satisfying the equations

$$\eta(1, y) = \varepsilon(y), \qquad \eta(x, 1) = \varepsilon(x), \tag{3.15}$$
$$\eta(xx', y) = \eta(x', y_{(1)})\eta(x, y_{(2)}), \qquad \eta(x, yy') = \eta(x_{(1)}, y)\eta(x_{(2)}, y'), \tag{3.16}$$
$$\eta(S(x), y) = \eta(x, S^{-1}(y)) \tag{3.17}$$

for all $x, x' \in A$ and $y, y' \in B$.

Proposition 3.5. *(i) There exists a unique skew-Hopf pairing η of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ such that for all $i,j\in I$ one has*

$$\eta(E_i,F_j)=-\delta_{i,j},\quad \eta(E_i,L_j)=0,\quad \eta(K_i,F_j)=0,\quad \eta(K_i,L_j)=q_{ij}.$$

(ii) The skew-Hopf pairing η satisfies the equations

$$\eta(EK,FL)=\eta(E,F)\eta(K,L)$$

for all $E\in\mathcal{U}^+(\chi)$, $F\in\mathcal{U}^-(\chi)$, $K\in\mathcal{U}^{+0}$, and $L\in\mathcal{U}^{-0}$.

(iii) If $\beta,\gamma\in\mathbb{N}_0^I$ with $\beta\neq\gamma$, $E\in\mathcal{U}^+(\chi)_\beta$, $F\in\mathcal{U}^-(\chi)_{-\gamma}$, then $\eta(E,F)=0$.

(iv) The restriction of η to $\mathcal{U}^+(\chi)\times\mathcal{U}^-(\chi)$ induces a non-degenerate pairing $\eta: U^+(\chi)\times U^-(\chi)\to\Bbbk$.

Proof. (i) and (ii) are [Hec10, Prop. 4.3]. (iii) follows from the definition of η and since Δ is a $\mathbb{Z}^I$-homogeneous map. (iv) was proven in [Hec10, Thm. 5.8]. □

By the general theory, see [Jos95, 3.2.2], the pairing η in Prop. 3.5 can be used to describe commutation rules in $\mathcal{U}(\chi)$ and $U(\chi)$. Namely,

$$yx=\eta(x_{(1)},S(y_{(1)}))x_{(2)}y_{(2)}\eta(x_{(3)},y_{(3)}),\tag{3.18}$$
$$xy=\eta(x_{(1)},y_{(1)})y_{(2)}x_{(2)}\eta(x_{(3)},S(y_{(3)}))\tag{3.19}$$

for all $x\in\mathcal{V}^+(\chi)$ and $y\in\mathcal{V}^-(\chi)$. Note that the second formula follows from the first one and Eqs. (3.15)–(3.16).

Later we will also need some other general facts about $U(\chi)$. Some of them are collected here. Let

$$U^+_{i,K}(\chi)=\ker\partial_i^K\subset U^+(\chi),\qquad U^+_{i,L}(\chi)=\ker\partial_i^L\subset U^+(\chi),\tag{3.20}$$
$$U^-_{i,K}(\chi)=\Omega(U^+_{i,K}),\qquad U^-_{i,L}(\chi)=\Omega(U^+_{i,L}).\tag{3.21}$$

Recall the definition of b^χ in Eq. (2.12).

Lemma 3.6. *Let $i\in I$.*

(i) Let $m\in\mathbb{N}$. The following are equivalent.

- *$E_i^m=0$ in $U(\chi)$,*
- *$F_i^m=0$ in $U(\chi)$,*
- *$m\geq b^\chi(\alpha_i)$.*

(ii) Let $\Bbbk[E_i]$ and $\Bbbk[F_i]$ be the subalgebras of $U(\chi)$ generated by E_i and F_i, respectively. The multiplication maps

$$U^+_{i,K}(\chi)\otimes\Bbbk[E_i]\to U^+(\chi),\qquad \Bbbk[E_i]\otimes U^+_{i,K}(\chi)\to U^+(\chi),$$
$$U^+_{i,L}(\chi)\otimes\Bbbk[E_i]\to U^+(\chi),\qquad \Bbbk[E_i]\otimes U^+_{i,L}(\chi)\to U^+(\chi),$$
$$U^-_{i,K}(\chi)\otimes\Bbbk[F_i]\to U^-(\chi),\qquad \Bbbk[F_i]\otimes U^-_{i,K}(\chi)\to U^-(\chi),$$
$$U^-_{i,L}(\chi)\otimes\Bbbk[F_i]\to U^-(\chi),\qquad \Bbbk[F_i]\otimes U^-_{i,L}(\chi)\to U^-(\chi),$$

are isomorphisms of $\mathbb{Z}^I$-graded algebras.

Proof. (i) is standard in the theory of Nichols algebras. It follows from Eqs. (3.12), (3.13) and the definitions of $\mathcal{I}^+(\chi)$ and $\mathcal{I}^-(\chi)$. The proof of (ii) for $U^+(\chi)$ can be performed as in [Hec06]. The formulas with $U^-(\chi)$ follow from those with $U^+(\chi)$ and Eqs. (3.20), (3.21). □

Lemma 3.7. *Let $m, n \in \mathbb{N}_0$ and $p \in I$. Then*

$$E_p^m F_p^n = \sum_{i=0}^{\min\{m,n\}} \frac{(m)^!_{q_{pp}}(n)^!_{q_{pp}}}{(i)^!_{q_{pp}}(m-i)^!_{q_{pp}}(n-i)^!_{q_{pp}}} F_p^{n-i} \prod_{j=1}^{i} (q_{pp}^{i+j-m-n} K_p - L_p) E_p^{m-i}.$$

Proof. For $n = 0$ the claim is trivial. By [Hec10, Cor. 5.4],

$$E_p^m F_p - F_p E_p^m = (m)_{q_{pp}} (q_{pp}^{1-m} K_p - L_p) E_p^{m-1}.$$

Hence the lemma holds for $n = 1$. It suffices to check the claim for $m \geq n$, since then it also holds for $m < n$ using the algebra antiisomorphism Ω. The proof of the lemma for $m \geq n$ is a standard calculation by induction on n. □

4. An analogue of Lusztig's PBW basis

Let $\chi \in \mathcal{X}$ and $p \in I$. Assume that χ is p-finite. Let $q_{ij} = \chi(\alpha_i, \alpha_j)$ and $c_{pi} = c^\chi_{pi}$ for all $i, j \in I$.

For all $m \in \mathbb{N}_0$ and $i \in I \setminus \{p\}$ define recursively $E^\pm_{i,m} \in U^+_{\alpha_i + m\alpha_p}$, $F^\pm_{i,m} \in U^-_{\alpha_i + m\alpha_p}$ by

$$\begin{aligned}
E^+_{i,0} &= E_i, & E^+_{i,m+1} &= E_p E^+_{i,m} - (K_p . E^+_{i,m}) E_p,\\
E^-_{i,0} &= E_i, & E^-_{i,m+1} &= E_p E^-_{i,m} - (L_p . E^-_{i,m}) E_p,\\
F^+_{i,0} &= F_i, & F^+_{i,m+1} &= F_p F^+_{i,m} - (L_p . F^+_{i,m}) F_p,\\
F^-_{i,0} &= F_i, & F^-_{i,m+1} &= F_p F^-_{i,m} - (K_p . F^-_{i,m}) F_p.
\end{aligned}$$

We also define $E^+_{i,-1} = E^-_{i,-1} = F^+_{i,-1} = F^-_{i,-1} = 0$. The above definitions depend essentially on p. If we want to emphasize this, we will write $E^\pm_{i,m(p)}$ and $F^\pm_{i,m(p)}$ instead of $E^\pm_{i,m}$ and $F^\pm_{i,m}$, respectively.

For all $i \in I \setminus \{p\}$ define

$$\lambda_i^\chi = (-c_{pi})^!_{q_{pp}} \prod_{j=0}^{-c_{pi}-1} (q_{pp}^j q_{pi} q_{ip} - 1).$$

Then $\lambda_i^\chi \neq 0$ by definition of $c_{pi} = c^\chi_{pi}$.

The next theorem was proven in [Hec10, Thm. 6.11].

Theorem 4.1. *Let $\chi \in \mathcal{X}$ and $p \in I$. Assume that χ is p-finite. Let $c_{pi} = c^\chi_{pi}$ for all $i \in I$.*

(i) There exist unique algebra isomorphisms $T_p, T_p^- : U(\chi) \to U(r_p(\chi))$ such that

$$\begin{aligned} T_p(K_p) &= T_p^-(K_p) = K_p^{-1}, & T_p(K_i) &= T_p^-(K_i) = K_i K_p^{-c_{pi}}, \\ T_p(L_p) &= T_p^-(L_p) = L_p^{-1}, & T_p(L_i) &= T_p^-(L_i) = L_i L_p^{-c_{pi}}, \\ T_p(E_p) &= F_p L_p^{-1}, & T_p(E_i) &= E^+_{i,-c_{pi}}, \\ T_p(F_p) &= K_p^{-1} E_p, & T_p(F_i) &= \lambda_i(r_p(\chi))^{-1} F^+_{i,-c_{pi}}, \\ T_p^-(E_p) &= K_p^{-1} F_p, & T_p^-(E_i) &= \lambda_i(r_p(\chi^{-1}))^{-1} E^-_{i,-c_{pi}}, \\ T_p^-(F_p) &= E_p L_p^{-1}, & T_p^-(F_i) &= (-1)^{c_{pi}} F^-_{i,-c_{pi}}. \end{aligned}$$

(ii) The maps T_p, T_p^- satisfy $T_p T_p^- = T_p^- T_p = \mathrm{id}_{U(\chi)}$.
(iii) There exists a unique $\underline{a} \in (\Bbbk^\times)^I$ such that $T_p \Omega = \Omega T_p^- \varphi_{\underline{a}}$ in $\mathrm{Hom}(U(\chi), U(r_p(\chi)))$.

Note that $T_p T_p^-$ is an automorphism of $U(\chi)$ if one regards T_p^- as a map from $U(\chi)$ to $U(r_p(\chi))$ and T_p as a map from $U(r_p(\chi))$ to $U(r_p r_p(\chi)) = U(\chi)$.

Proposition 4.2. *Let $\chi \in \mathcal{X}$ and $p \in I$. Assume that χ is p-finite. Then*

$$T_p(U(\chi)_\alpha) = U(r_p(\chi))_{\sigma_p^\chi(\alpha)} \quad \textit{for all } \alpha \in \mathbb{Z}^I.$$

Proof. The maps $T_p : U(\chi) \to U(r_p(\chi))$ and $T_p^- : U(r_p(\chi)) \to U(\chi)$ are mutually inverse algebra isomorphisms, and send generators of degree α into the homogeneous component of degree $\sigma_p(\alpha)$. □

Lemma 4.3. *Let $\chi \in \mathcal{X}$ and $p \in I$. Assume that χ is p-finite. Then*

$$\begin{aligned} T_p(U^+_{p,L}(\chi)) &= U^+_{p,K}(r_p(\chi)), & T_p(U^-_{p,K}(\chi)) &= U^-_{p,L}(r_p(\chi)), \\ T_p^-(U^+_{p,K}(\chi)) &= U^+_{p,L}(r_p(\chi)), & T_p^-(U^-_{p,L}(\chi)) &= U^-_{p,K}(r_p(\chi)). \end{aligned}$$

Proof. Since χ and $r_p(\chi)$ are p-finite, [Hec10, Prop. 5.10] and [Hec10, Prop. 6.7(d)] give that

$$T_p(U^+_{p,L}(\chi)) \subset U^+_{p,K}(r_p(\chi)), \qquad T_p^-(U^+_{p,K}(r_p(\chi))) \subset U^+_{p,L}(\chi).$$

Thus $T_p(U^+_{p,L}(\chi)) = U^+_{p,K}(r_p(\chi))$ by Thm. 4.1(ii). Similar arguments yield that $T_p^-(U^+_{p,K}(\chi)) = U^+_{p,L}(r_p(\chi))$. The remaining two equations can be obtained from these and Thm. 4.1(iii). □

In the rest of the section assume that $\chi \in \mathcal{X}_3$. Let $n = |R^\chi_+| \in \mathbb{N}$. The following construction generalizes the Poincaré-Birkhoff-Witt basis of quantized enveloping algebras given by Lusztig.

Let $i_1, i_2, \dots, i_n \in I$ such that $\ell(1^\chi \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}) = n$. For all $\nu \in \{1, 2, \dots, n\}$ let

$$\beta^\chi_\nu = 1^\chi \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_{\mu-1}}(\alpha_{i_\nu}). \tag{4.1}$$

Then the elements β^χ_ν, $1 \le \nu \le n$, are pairwise different and

$$R^\chi_+ = \{\beta^\chi_\nu \,|\, 1 \le \nu \le n\} \tag{4.2}$$

by [CH09, Prop. 2.12]. For all $\nu \in \{1,2,\dots,n\}$ let

$$E_{\beta_\nu} = E^\chi_{\beta_\nu} = T_{i_1} \dots T_{i_{\nu-1}}(E_{i_\nu}), \quad F_{\beta_\nu} = F^\chi_{\beta_\nu} = T_{i_1} \dots T_{i_{\nu-1}}(F_{i_\nu}), \tag{4.3}$$

$$\bar{E}_{\beta_\nu} = \bar{E}^\chi_{\beta_\nu} = T^-_{i_1} \dots T^-_{i_{\nu-1}}(E_{i_\nu}), \quad \bar{F}_{\beta_\nu} = \bar{F}^\chi_{\beta_\nu} = T^-_{i_1} \dots T^-_{i_{\nu-1}}(F_{i_\nu}), \tag{4.4}$$

where $E_{i_\nu}, F_{i_\nu} \in U(r_{i_{\nu-1}} \dots r_{i_2} r_{i_1}(\chi))$. Then

$$E_{\beta_\nu}, \bar{E}_{\beta_\nu} \in U^+(\chi)_{\beta_\nu}, \qquad F_{\beta_\nu}, \bar{F}_{\beta_\nu} \in U^-(\chi)_{-\beta_\nu} \tag{4.5}$$

for all $\nu \in \{1,\dots,n\}$ by [Hec10, Thm. 6.20], Thm. 4.1(iii) and Prop. 4.2.

Lemma 4.4. *Assume that $\chi \in \mathcal{X}_3$. Let $\nu \in \{1,2,\dots,n\}$. If $b^\chi(\beta_\nu) < \infty$, then $E_{\beta_\nu}^{b^\chi(\beta_\nu)} = F_{\beta_\nu}^{b^\chi(\beta_\nu)} = 0$ and $E^m_{\beta_\nu} \neq 0$, $F^m_{\beta_\nu} \neq 0$ in $U(\chi)$ for $1 \le m \le b^\chi(\beta_\nu) - 1$. If $b^\chi(\beta_\nu) = \infty$, then $E^m_{\beta_\nu} \neq 0$, $F^m_{\beta_\nu} \neq 0$ in $U(\chi)$ for all $m \in \mathbb{N}$.*

Proof. By Eq. (2.13), the assumption $\chi \in \mathcal{X}_3$, and since T_i is an isomorphism for each $i \in I$, it suffices to prove for all $m \in \mathbb{N}$ and all $\chi' \in \mathcal{X}$ that $E^m_i = 0$ in $U(\chi')$ if and only if $F^m_i = 0$ in $U(\chi')$ if and only if $m \ge b^{\chi'}(\alpha_i)$ for all $\chi' \in \mathcal{X}$ and $i \in I$ with $b^{\chi'}(\alpha_i) < \infty$. This follows from Lemma 3.6(i). □

Theorem 4.5. *Assume that $\chi \in \mathcal{X}_3$. Let $n = |R^\chi_+| \in \mathbb{N}$. Both sets*

$$\{E^{m_1}_{\beta_1} E^{m_2}_{\beta_2} \cdots E^{m_n}_{\beta_n} \mid 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{1,2,\dots,n\}\}, \tag{4.6}$$

$$\{\bar{E}^{m_1}_{\beta_1} \bar{E}^{m_2}_{\beta_2} \cdots \bar{E}^{m_n}_{\beta_n} \mid 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{1,2,\dots,n\}\} \tag{4.7}$$

form vector space bases of $U^+(\chi)$.

Proof. We prove the claim for the basis in Eq. (4.6). For the other set the proof is analogous. By Eqs. (2.16) and (4.2),

$$\dim U^+(\chi)_\alpha = \Big|\{(m_1,m_2,\dots,m_n) \in \mathbb{N}_0^n \mid \sum_{\nu=1}^n m_\nu \beta_\nu = \alpha,$$
$$m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{1,\dots,n\}\}\Big|$$

for all $\alpha \in \mathbb{N}_0^I$. Since $\deg E_{\beta_\nu} = \beta_\nu$ for all $\nu \in \{1,2,\dots,n\}$, it suffices to show that for all $\mu \in \{1,2,\dots,n+1\}$ the elements of the set

$$\{E^{m_\mu}_{\beta_\mu} E^{m_{\mu+1}}_{\beta_{\mu+1}} \cdots E^{m_n}_{\beta_n} \mid 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{\mu,\mu+1,\dots,n\}\}$$

are linearly independent. We proceed by induction on $n+1-\mu$. If $\mu = n+1$, then the above set is empty, and hence its elements are linearly independent.

Let now $\mu \in \{1,2,\dots,n\}$. For all $m_\mu,\dots,m_n \in \mathbb{N}_0$ with $m_\nu < b^\chi(\beta_\nu)$ for all $\nu \in \{\mu,\mu+1,\dots,n\}$ let $a_{m_\mu,\dots,m_n} \in \Bbbk$. Assume that

$$\sum_{m_\mu,\dots,m_n} a_{m_\mu,\dots,m_n} E^{m_\mu}_{\beta_\mu} E^{m_{\mu+1}}_{\beta_{\mu+1}} \cdots E^{m_n}_{\beta_n} = 0 \tag{4.8}$$

in $U^+(\chi)$. Let $T^- = T^-_{i_\mu} \cdots T^-_{i_2} T^-_{i_1}$. Since $T^-(E_{\beta_\mu}) = T^-_{i_\mu}(E_{i_\mu}) = K^{-1}_{i_\mu} F_{i_\mu}$, we obtain that

$$\sum_{m_\mu,\dots,m_n} a_{m_\mu,\dots,m_n} (K^{-1}_{i_\mu} F_{i_\mu})^{m_\mu} T^-(E_{\beta_{\mu+1}})^{m_{\mu+1}} \cdots T^-(E_{\beta_n})^{m_n} = 0.$$

Since $T^-(E_{\beta_\nu}) \in U^+(r_{i_\mu}\cdots r_{i_2}r_{i_1}(\chi))$ for all $\nu \in \{\mu+1,\mu+2,\dots,n\}$, Prop. 3.4 implies that

$$\sum_{m_{\mu+1},\dots,m_n} a_{m_\mu,m_{\mu+1},\dots,m_n} T^-(E_{\beta_{\mu+1}})^{m_{\mu+1}}\cdots T^-(E_{\beta_n})^{m_n} = 0$$

for all $m_\mu \in \mathbb{N}_0$, $m_\mu < b^\chi(\beta_\mu)$. Therefore

$$\sum_{m_{\mu+1},\dots,m_n} a_{m_\mu,m_{\mu+1},\dots,m_n} E_{\beta_{\mu+1}}^{m_{\mu+1}}\cdots E_{\beta_n}^{m_n} = 0$$

for all $m_\mu \in \mathbb{N}_0$, $m_\mu < b^\chi(\beta_\mu)$. Then $a_{m_\mu,m_{\mu+1},\dots,m_n} = 0$ for all $(m_\mu,\dots,m_n)$ by induction hypothesis, which proves the induction step. Thus the theorem holds. □

Lemma 4.6. *Assume that $\chi \in \mathcal{X}_3$. Then $\ker(\partial^K_{i_1} : U^+(\chi) \to U^+(\chi))$ coincides with the subalgebra of $U^+(\chi)$ generated by the elements E_{β_ν}, $\nu \in \{2,3,\dots,n\}$. The set*

$$\{E_{\beta_2}^{m_2}E_{\beta_3}^{m_3}\cdots E_{\beta_n}^{m_n} \,|\, 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{2,3,\dots,n\}\} \tag{4.9}$$

forms a vector space basis of $\ker \partial^K_{i_1}$.

Proof. Let $\nu \in \{2,3,\dots,n\}$. By [Hec10, Lemma 4.31, Prop. 5.10] and Lemma 4.4 there exist $m \in \mathbb{N}_0$ and $x_0,x_1,\dots,x_m \in \ker \partial^K_{i_1}$ such that $m < b^\chi(\alpha_{i_1})$ and $E_{\beta_\nu} = \sum_{\mu=0}^m x_\mu E_{i_1}^\mu$. Then

$$T^-_{i_1}(E_{\beta_\nu}) = \sum_{\mu=0}^m T^-_{i_1}(x_\mu)(K_{i_1}^{-1}F_{i_1})^\mu.$$

Moreover, $T^-_{i_1}(x_\mu) \in U^+(r_{i_1}(\chi))$ for all $\mu \in \{0,1,\dots,m\}$ by [Hec10, Prop. 5.10, Lemma 6.7(d)]. Since $T^-_{i_1}(E_{\beta_\nu}) \in U^+(r_{i_1}(\chi))$, triangular decomposition of $U(r_{i_1}(\chi))$ implies that $x_\mu = 0$ for all $\mu > 0$. Hence $E_{\beta_\nu} = x_0 \in \ker \partial^K_{i_1}$. Then the claim of the lemma follows from the inclusions

$$\langle E_{\beta_\kappa} \,|\, \kappa \in \{2,3,\dots,n\}\rangle \subset \ker \partial^K_{i_1} \subset \bigoplus_{\substack{(m_2,m_3,\dots,m_n)\\ 0\le m_\kappa<b^\chi(\beta_\kappa) \text{ for all } \kappa}} \Bbbk E_{\beta_2}^{m_2}\cdots E_{\beta_n}^{m_n},$$

where the second inclusion is obtained from Thm. 4.5 and the formula

$$\partial^K_{i_1}(E_{\beta_1}^{m_1}E_{\beta_2}^{m_2}\cdots E_{\beta_n}^{m_n}) = (m_1)_{q_{i_1i_1}} E_{\beta_1}^{m_1-1}K_{i_1}\cdot(E_{\beta_2}^{m_2}\cdots E_{\beta_n}^{m_n}).$$

□

Remark 4.7. The analogous version of Lemma 4.6 for $\ker \partial^L_{i_1}$ is obtained by replacing E_{β_ν} by $\bar{E}_{\beta_\nu}$ for all $\nu \in \{2,3,\dots,n\}$.

Theorem 4.8. *Assume that $\chi \in \mathcal{X}_3$. Let $n = |R^\chi_+| \in \mathbb{N}$. Then*

$$\begin{aligned}
E_{\beta_\mu}E_{\beta_\nu} - \chi(\beta_\mu,\beta_\nu)E_{\beta_\nu}E_{\beta_\mu} &\in \langle E_{\beta_\kappa} \,|\, \mu<\kappa<\nu\rangle \subset U^+(\chi),\\
\bar{E}_{\beta_\mu}\bar{E}_{\beta_\nu} - \chi^{-1}(\beta_\nu,\beta_\mu)\bar{E}_{\beta_\nu}\bar{E}_{\beta_\mu} &\in \langle \bar{E}_{\beta_\kappa} \,|\, \mu<\kappa<\nu\rangle \subset U^+(\chi),\\
F_{\beta_\mu}F_{\beta_\nu} - \chi(\beta_\nu,\beta_\mu)F_{\beta_\nu}F_{\beta_\mu} &\in \langle F_{\beta_\kappa} \,|\, \mu<\kappa<\nu\rangle \subset U^-(\chi),\\
\bar{F}_{\beta_\mu}\bar{F}_{\beta_\nu} - \chi^{-1}(\beta_\mu,\beta_\nu)\bar{F}_{\beta_\nu}\bar{F}_{\beta_\mu} &\in \langle \bar{F}_{\beta_\kappa} \,|\, \mu<\kappa<\nu\rangle \subset U^-(\chi)
\end{aligned}$$

for all $\mu,\nu \in \{1,2,\dots,n\}$ with $\mu<\nu$.

Proof. The proof will be achieved in a standard way, see e. g. [Bec94, Prop. 7].

We prove the first relation for $\mu = 1$ and all $\nu \in \{2, 3, \dots, n\}$. Then the first relation for $\mu > 1$ follows from

$$E_{\beta_\mu} E_{\beta_\nu} - \chi(\beta_\mu, \beta_\nu) E_{\beta_\nu} E_{\beta_\mu} = T_{i_1} \cdots T_{i_{\mu-1}}(E_{i_\mu} E'_\nu - \chi(\beta_\mu, \beta_\nu) E'_\nu E_{i_\mu}),$$

where $E'_\nu = T_{i_\mu} \cdots T_{i_{\nu-1}}(E_{i_\nu})$, by using the case $\mu = 1$, Eq. (2.8) and the first relation in (4.5). The proof of the second relation of the theorem is similar. The third and fourth relations can be obtained from the first two by applying Ω and using the formulas

$$\Omega(E_{\beta_\kappa}) \in \Bbbk^\times \bar{F}_{\beta_\kappa}, \quad \Omega(\bar{E}_{\beta_\kappa}) \in \Bbbk^\times F_{\beta_\kappa}, \qquad \kappa \in \{1, 2, \dots, n\}, \tag{4.10}$$

which follow from Thm. 4.1(iii).

Let $\nu \in \{2, 3, \dots, n\}$. For all $(m_1, m_2, \dots, m_n) \in \mathbb{N}_0^I$ with $m_\kappa < b^\chi(\beta_\kappa)$ for all $\kappa \in \{1, 2, \dots, n\}$ let $a_{m_1,\dots,m_n} \in \Bbbk$ be such that

$$E_{i_1} E_{\beta_\nu} - \chi(\alpha_{i_1}, \beta_\nu) E_{\beta_\nu} E_{i_1} = \sum_{m_1,\dots,m_n} a_{m_1,\dots,m_n} E_{\beta_1}^{m_1} \cdots E_{\beta_n}^{m_n}. \tag{4.11}$$

The numbers $a_{m_1,\dots,m_n} \in \Bbbk$ exist and are unique by Thm. 4.5. Let $\chi_\nu = r_{i_\nu} \cdots r_{i_2} r_{i_1}(\chi)$. Apply to Eq. (4.11) the isomorphism $T^- = T^-_{i_\nu} \cdots T^-_{i_2} T^-_{i_1} \in \mathrm{Hom}(U(\chi), U(\chi_\nu))$. For all $\kappa \in \{1, 2, \dots, \nu\}$,

$$\begin{aligned} T^-_{i_\nu} \cdots T^-_{i_2} T^-_{i_1}(E_{\beta_\kappa}) =& T^-_{i_\nu} \cdots T^-_{i_{\kappa+1}} T^-_{i_\kappa}(E_{i_\kappa}) \\ =& T^-_{i_\nu} \cdots T^-_{i_{\kappa+1}}(K^{-1}_{i_\kappa} F_{i_\kappa}) \in U^-(\chi_\nu)\mathcal{U}^0 \end{aligned}$$

by Eq. (4.5). Hence

$$\begin{aligned} \sum_{m_1,\dots,m_n} & a_{m_1,\dots,m_n} T^-(E_{\beta_1}^{m_1} \cdots E_{\beta_\nu}^{m_\nu}) T^-(E_{\beta_{\nu+1}}^{m_{\nu+1}} \cdots E_{\beta_n}^{m_n}) \\ &= T^-(E_{i_1} E_{\beta_\nu} - \chi(\alpha_{i_1}, \beta_\nu) E_{\beta_\nu} E_{i_1}) \in U^-(\chi_\nu)\mathcal{U}^0. \end{aligned}$$

By triangular decomposition of $U(\chi)$ it follows that $a_{m_1,\dots,m_n} = 0$ for all $(m_1, \dots, m_n)$ with $m_\kappa > 0$ for some $\kappa \in \{\nu+1, \nu+2, \dots, n\}$.

By Lemma 4.6, $E_{\beta_\nu} \in \ker \partial^K_{i_1}$. Hence $E_{i_1} E_{\beta_\nu} - \chi(\alpha_{i_1}, \beta_\nu) E_{\beta_\nu} E_{i_1} \in \ker \partial^K_{i_1}$ by Lemma 3.2. Thus Lemma 4.6 implies that $a_{m_1,\dots,m_n} = 0$ whenever $m_1 > 0$.

Suppose that there exists $(m_1, \dots, m_n)$ with $a_{m_1,\dots,m_n} \neq 0$ and $m_\kappa = 0$ for all $\kappa \in \{1\} \cup \{\nu+1, \nu+2, \dots, n\}$. Since $E_{i_1} E_{\beta_\nu} - \chi(\alpha_{i_1}, \beta_\nu) E_{\beta_\nu} E_{i_1}$ is $\mathbb{Z}^I$-homogeneous of degree $\alpha_{i_1} + \beta_\nu$, the equality $\alpha_{i_1} + \beta_\nu = \sum_{\kappa'=2}^{\nu} m_{\kappa'} \beta_{\kappa'}$ holds, which implies $m_\nu = 0$. Thus the theorem is proven. □

Next we prove a generalization of Thm. 4.5.

Theorem 4.9. *Assume that $\chi \in \mathcal{X}_3$. Let $n = |R^\chi_+|$ and let τ be a permutation of the set $\{1, 2, \dots, n\}$. Then the sets*

$$\{E_{\beta_{\tau(1)}}^{m_{\tau(1)}} E_{\beta_{\tau(2)}}^{m_{\tau(2)}} \cdots E_{\beta_{\tau(n)}}^{m_{\tau(n)}} \mid 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{1, 2, \dots, n\}\},$$
$$\{\bar{E}_{\beta_{\tau(1)}}^{m_{\tau(1)}} \bar{E}_{\beta_{\tau(2)}}^{m_{\tau(2)}} \cdots \bar{E}_{\beta_{\tau(n)}}^{m_{\tau(n)}} \mid 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{1, 2, \dots, n\}\}$$

form vector space bases of $U^+(\chi)$*, and the sets*

$$\{F^{m_{\tau(1)}}_{\beta_{\tau(1)}} F^{m_{\tau(2)}}_{\beta_{\tau(2)}} \cdots F^{m_{\tau(n)}}_{\beta_{\tau(n)}} \,|\, 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{1,2,\dots,n\}\},$$
$$\{\bar F^{m_{\tau(1)}}_{\beta_{\tau(1)}} \bar F^{m_{\tau(2)}}_{\beta_{\tau(2)}} \cdots \bar F^{m_{\tau(n)}}_{\beta_{\tau(n)}} \,|\, 0 \le m_\nu < b^\chi(\beta_\nu) \text{ for all } \nu \in \{1,2,\dots,n\}\}$$

form vector space bases of $U^-(\chi)$.

Proof. It suffices to prove that the first set is a basis of $U^+(\chi)$. Indeed, the proof for the second set can be obtained by using the maps T_i^-, where $i \in I$, instead of T_i. The second part of the claim follows from the first part by applying the algebra antiautomorphism Ω and using Eq. (4.10).

For any $\underline{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ let $|\underline{m}| = \sum_{\mu=1}^n m_\mu |\beta_\mu|$, where $|\alpha| = \sum_{j\in I} a_j$ for all $\alpha = \sum_{j\in I} a_j\alpha_j \in \mathbb{N}_0^I$. Let N be the (additive) monoid $\mathbb{N}_0^n$ equipped with the following ordering:

$$\underline{m}' \le \underline{m} \quad\Leftrightarrow\quad |\underline{m}'| < |\underline{m}| \quad\text{or}\quad |\underline{m}'| = |\underline{m}|,\ \underline{m}' \le_{\mathrm{lex}} \underline{m},$$

where $\le_{\mathrm{lex}}$ means lexicographical ordering. We use the convention $\underline{m} \le_{\mathrm{lex}} \underline{m}$. The ordering $\le$ is a total ordering. For all $\underline{m} \in \mathbb{N}_0^n$ define

$$\mathcal{F}^{\underline{m}} U^+(\chi) = \bigoplus_{\underline{m}'\in N, \underline{m}' \le \underline{m}} \Bbbk E^{m'_1}_{\beta_1} E^{m'_2}_{\beta_2} \cdots E^{m'_n}_{\beta_n} \subset U^+(\chi).$$

The vector spaces $\mathcal{F}^{\underline{m}} U^+(\chi)$, where $\underline{m} \in N$, are finite-dimensional, since the degrees of their elements are bounded. Moreover,

$$\mathcal{F}^0 U^+(\chi) = \Bbbk 1, \qquad \mathcal{F}^{\underline{m}} U^+(\chi) \mathcal{F}^{\underline{m}'} U^+(\chi) \subset \mathcal{F}^{\underline{m}+\underline{m}'} U^+(\chi)$$

for all $\underline{m}, \underline{m}' \in N$ by Thm. 4.8 and since $U^+(\chi)$ is $\mathbb{Z}^I$-graded. Thus $\mathcal{F}$ defines a filtration of $U^+(\chi)$ by the monoid N, and the corresponding graded algebra

$$\bigoplus_{\underline{m}\in N} \Big(\mathcal{F}^{\underline{m}} U^+(\chi) / \sum_{\underline{m}' \le \underline{m}, \underline{m}' \ne \underline{m}} \mathcal{F}^{\underline{m}'} U^+(\chi) \Big)$$

is a skew-polynomial ring in n variables by Thm. 4.8. By a standard conclusion we obtain that the first set in the claim of the theorem is indeed a basis of $U^+(\chi)$. □

5. Verma modules and morphisms

We consider Verma modules for the algebras $U(\chi)$, $\chi \in \mathcal{X}$. We observe that the fundaments of the theory of Verma modules for quantized enveloping algebras can be carried over to a great extent to $U(\chi)$. New phenomena appear if some generators of $U(\chi)$ are nilpotent.

Let $\mathbb{K}$ be a field extension of $\Bbbk$. Although the $\mathbb{K}$-Hopf algebra $U(\chi) \otimes_\Bbbk \mathbb{K}$ can be identified with the one defined over $\mathbb{K}$, we use it deliberately. Let $\mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$ denote the set of $\mathbb{K}^\times$-valued characters (algebra maps from $\mathcal{U}^0$ to $\mathbb{K}^\times$) of the group algebra $\mathcal{U}^0$. For all $\chi \in \mathcal{X}$ there is a natural group homomorphism

$$\zeta^\chi : \mathbb{Z}^I \to \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times), \quad \zeta^\chi(\alpha)(K_\beta L_{\beta'}) = \chi(\beta,\alpha)\chi(\alpha,\beta')^{-1} \tag{5.1}$$

for all $\alpha, \beta, \beta' \in \mathbb{Z}^I$. If $\chi \in \mathcal{X}$ and $p \in I$ such that χ is p-finite, then

$$\zeta^{r_p(\chi)}(\sigma_p^\chi(\alpha))(K_{\sigma_p^\chi(\beta)} L_{\sigma_p^\chi(\beta')}) = \zeta^\chi(\alpha)(K_\beta L_{\beta'}) \tag{5.2}$$

by Eq. (2.8).

Let $\chi \in \mathcal{X}$. Given a $\mathbb{K}^\times$-valued character $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$, one can regard $\mathbb{K}$ as a $U^+(\chi)\mathcal{U}^0$-module with generator $1_\Lambda = 1$ via

$$uE1_\Lambda = \varepsilon(E)\Lambda(u)1_\Lambda \qquad \text{for all } E \in U^+(\chi),\, u \in \mathcal{U}^0. \tag{5.3}$$

We write $\mathbb{K}_\Lambda$ for this module.

Definition 5.1. A *Verma module* of $U(\chi)$ is a $U(\chi)$-module of the form

$$M^\chi(\Lambda) = U(\chi) \otimes_{U^+(\chi)\mathcal{U}^0} \mathbb{K}_\Lambda, \quad \text{where } \Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times).$$

We write v_Λ^χ or just v_Λ for $1 \otimes 1_\Lambda \in M^\chi(\Lambda)$.

Any Verma module is also a $\mathbb{K}$-module via the $\mathbb{K}$-module structure of the second tensor factor.

Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. Triangular decomposition of $U(\chi)$ gives the following standard fact.

Lemma 5.2. *The map $U^-(\chi) \otimes_\Bbbk \mathbb{K} \to M^\chi(\Lambda)$, $u \otimes x \mapsto uxv_\Lambda = u \otimes x1_\Lambda$, is an isomorphism of vector spaces over $\mathbb{K}$.*

The isomorphism in Lemma 5.2 and the $\mathbb{Z}^I$-grading of $U^-(\chi)$ induce a unique $\mathbb{Z}^I$-grading on $M^\chi(\Lambda)$ such that

$$M^\chi(\Lambda)_\alpha = U^-(\chi)_\alpha \otimes_\Bbbk \mathbb{K}_\Lambda \quad \text{for all } \alpha \in \mathbb{Z}^I. \tag{5.4}$$

Then

$$M^\chi(\Lambda)_0 = \mathbb{K}v_\Lambda,\ U(\chi)_\alpha M^\chi(\Lambda)_\beta \subset M^\chi(\Lambda)_{\alpha+\beta} \text{ for all } \alpha, \beta \in \mathbb{Z}^I. \tag{5.5}$$

Moreover, $M^\chi(\Lambda)_\alpha \neq 0$ implies that $-\alpha \in \mathbb{N}_0^I$.

The group algebra $\mathcal{U}^0$ acts on $M^\chi(\Lambda)$ via left multiplication. This action is given by characters:

$$uv = (\Lambda + \zeta^\chi(\alpha))(u)v \quad \text{for all } u \in \mathcal{U}^0,\ \alpha \in \mathbb{Z}^I,\ v \in M^\chi(\Lambda)_\alpha, \tag{5.6}$$

see Eqs. (5.1), (5.3), (3.3), and (3.4).

Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. The family of those $U(\chi) \otimes_\Bbbk \mathbb{K}$-submodules of $M^\chi(\Lambda)$, which are contained in $\oplus_{\alpha\neq 0} M^\chi(\Lambda)_\alpha$, have a unique maximal element $I^\chi(\Lambda)$. Let

$$L^\chi(\Lambda) = M^\chi(\Lambda)/I^\chi(\Lambda) \tag{5.7}$$

be the quotient $U(\chi)$-module. For all $\alpha \in \mathbb{Z}^I$ let

$$I^\chi(\Lambda)_\alpha = M^\chi(\Lambda)_\alpha \cap I^\chi(\Lambda). \tag{5.8}$$

The maximality of $I^\chi(\Lambda)$ implies that the inclusions $I^\chi(\Lambda)_\alpha \to I^\chi(\Lambda)$ induce isomorphisms

$$I^\chi(\Lambda) \simeq \oplus_{\alpha\in\mathbb{Z}^I} I^\chi(\Lambda)_\alpha, \quad L^\chi(\Lambda) \simeq \oplus_{\alpha\in\mathbb{Z}^I} M^\chi(\Lambda)_\alpha / I^\chi(\Lambda)_\alpha,$$

and hence $I^\chi(\Lambda)$ and $L^\chi(\Lambda)$ are $\mathbb{Z}^I$-graded. Since $M^\chi(\Lambda)_0 = \mathbb{K}v_\Lambda$, and any $\mathbb{Z}^I$-graded quotient of $M^\chi(\Lambda)$ by a $U(\chi) \otimes_\Bbbk \mathbb{K}$-submodule containing v_Λ is zero, $L^\chi(\Lambda)$ is the unique simple $\mathbb{Z}^I$-graded $U(\chi) \otimes_\Bbbk \mathbb{K}$-module quotient of $M^\chi(\Lambda)$.

Definition 5.3. Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$ and V a $\mathbb{Z}^I$-graded subquotient of $M^\chi(\Lambda)$. The *(formal) character of* V is the sum

$$\mathrm{ch}\, V = \sum_{\alpha \in \mathbb{N}_0^I} (\dim V_{-\alpha}) e^{-\alpha},$$

where e is a formal variable.

Eq. (5.4) implies that

$$\mathrm{ch}\, M^\chi(\Lambda) = \sum_{\alpha \in \mathbb{N}_0^I} \dim U^-(\chi)_{-\alpha} e^{-\alpha} \tag{5.9}$$

for all $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$.

Remark 5.4. For all $\alpha, \beta \in \mathbb{Z}^I$ we let $e^\alpha e^\beta = e^{\alpha+\beta}$. Thus we can consider formal characters as elements of the ring $\cup_{\alpha \in \mathbb{N}_0^I} e^\alpha \mathbb{Z}[[e^{-\alpha_i} \,|\, i \in I]]$, where $e^\alpha \mathbb{Z}[[e^{-\alpha_i} \,|\, i \in I]] \subset e^{\alpha+\beta}\mathbb{Z}[[e^{-\alpha_i} \,|\, i \in I]]$ for all $\alpha, \beta \in \mathbb{N}_0^I$ in the natural way.

From now on until Lemma 5.15 let $\chi \in \mathcal{X}$, $p \in I$, *and* $b = b^\chi(\alpha_p) = b^{r_p(\chi)}(\alpha_p)$, *and assume that* $b < \infty$. Then χ and $r_p(\chi)$ are p-finite. We deduce some phenomena which arise from the finiteness assumption on b.

For all $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$ define $t_p^\chi(\Lambda) \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$ by

$$t_p^\chi(\Lambda)(K_\alpha L_\beta) = \Lambda(K_{\sigma_p^{r_p(\chi)}(\alpha)} L_{\sigma_p^{r_p(\chi)}(\beta)}) \frac{r_p(\chi)(\alpha, \alpha_p)^{b-1}}{r_p(\chi)(\alpha_p, \beta)^{b-1}} \tag{5.10}$$

for all $\alpha, \beta \in \mathbb{Z}^I$. By Eq. (2.8) this is equivalent to

$$t_p^\chi(\Lambda)(K_{\sigma_p^\chi(\alpha)} L_{\sigma_p^\chi(\beta)}) = \Lambda(K_\alpha L_\beta) \frac{\chi(\alpha_p, \beta)^{b-1}}{\chi(\alpha, \alpha_p)^{b-1}} \tag{5.11}$$

for all $\alpha, \beta \in \mathbb{Z}^I$.

Recall the definition of ρ^χ from Def. 2.16.

Lemma 5.5. *Let* $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. *Then*

$$\rho^{r_p(\chi)}(\sigma_p^\chi(\alpha))\, t_p^\chi(\Lambda)(K_{\sigma_p^\chi(\alpha)} L^{-1}_{\sigma_p^\chi(\alpha)}) = \rho^\chi(\alpha)\, \Lambda(K_\alpha L_\alpha^{-1})$$

for all $\alpha \in \mathbb{Z}^I$.

Proof. Insert Eq. (5.11) and use Lemma 2.17. □

Example 5.6. Let C be a symmetrizable Cartan matrix and $q \in \Bbbk^\times$, $\chi \in \mathcal{X}$ as in the second part of Ex. 2.13. In particular, $\chi(\alpha_i, \alpha_j) = q^{d_i c_{ij}}$. Let $p \in I$. Assume that $b = b^\chi(\alpha_p) < \infty$. Then $q^{2d_p b} = 1$, and hence $q^{2b(\alpha, \alpha_p)} = 1$ for all $\alpha \in \mathbb{Z}^I$. Further, $r_p(\chi) = \chi$.

Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. Assume that $\Lambda(K_\alpha L_\alpha^{-1}) = q^{2(\alpha,\lambda)}$ for some λ in the weight lattice. Then

$$\begin{aligned} t_p^\chi(\Lambda)(K_\alpha L_\alpha^{-1}) =& \Lambda(K_{\sigma_p^\chi(\alpha)} L^{-1}_{\sigma_p^\chi(\alpha)}) q^{2(b-1)(\alpha,\alpha_p)} \\ =& q^{2(\sigma_p^\chi(\alpha),\lambda)} q^{-2(\alpha,\alpha_p)} = q^{2(\alpha, \sigma_p^\chi(\lambda) - \alpha_p)}, \end{aligned}$$

which recovers the dot action of the Weyl group on the weight lattice.

If we consider a composition $t_i^{\chi'} t_j^{\chi''}$, where $i, j \in I$, $\chi', \chi'' \in \mathcal{X}$, then we will always assume that

$$\chi' = r_j(\chi'').$$

For simplicity, we will omit the upper index χ' if it is uniquely determined by another bicharacter in the expression.

Lemma 5.7. *Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. Then $t_p t_p^\chi(\Lambda) = \Lambda$.*

Proof. By Eqs. (5.10) and (5.11), and since $r_p^2(\chi) = \chi$,

$$t_p^{r_p(\chi)} t_p^\chi(\Lambda)(K_\alpha L_\beta) = t_p^\chi(\Lambda)(K_{\sigma_p^\chi(\alpha)} L_{\sigma_p^\chi(\beta)}) \frac{\chi(\alpha, \alpha_p)^{b-1}}{\chi(\alpha_p, \beta)^{b-1}} = \Lambda(K_\alpha L_\beta)$$

for all $\alpha, \beta \in \mathbb{Z}^I$. This proves the lemma. □

Lemma 5.8. *Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. There exist unique $\mathbb{K}$-linear maps*

$$\hat{T}_p = \hat{T}_{p,\Lambda}^\chi,\ \hat{T}_p^- = \hat{T}_{p,\Lambda}^{\chi,-} :\ M^{r_p(\chi)}(t_p^\chi(\Lambda)) \to M^\chi(\Lambda),$$

such that for all $u \in U(r_p(\chi))$ we have

$$\hat{T}_p(u v_{t_p^\chi(\Lambda)}) = T_p(u) F_p^{b-1} v_\Lambda, \quad \hat{T}_p^-(u v_{t_p^\chi(\Lambda)}) = T_p^-(u) F_p^{b-1} v_\Lambda.$$

If $V \subset M^{r_p(\chi)}(t_p^\chi(\Lambda))$ is a $U(r_p(\chi))$-submodule, then $\hat{T}_p(V), \hat{T}_p^-(V)$ are $U(\chi)$-submodules of $M^\chi(\Lambda)$.

Proof. The uniqueness of the maps $\hat{T}_p$, $\hat{T}_p^-$ is clear. We prove that $\hat{T}_p$ is well-defined. The proof for $\hat{T}_p^-$ is analogous.

Let $\chi' = r_p(\chi)$ and $\Lambda' = t_p^\chi(\Lambda)$. By Lemma 5.2 and Thm. 4.9,

$$M^\chi(\Lambda)_{\alpha_j + a\alpha_p} = 0 = M^\chi(\Lambda)_{-b\alpha_p} \quad \text{for all } a \in \mathbb{Z},\ j \in I \setminus \{p\}.$$

Thus, since $\deg T_p(E_j) = \sigma_p^{\chi'}(\alpha_j)$ for $j \in I$, we conclude that

$$T_p(E_j) F_p^{b-1} v_\Lambda \in M^\chi(\Lambda)_{\alpha_j + (1-b-c_{pj}^{\chi'})\alpha_p} = 0 \quad \text{for all } j \in I.$$

Moreover, Eqs. (3.4), (5.11) give that

$$\begin{aligned} K_\alpha L_\beta F_p^{b-1} v_\Lambda &= \chi(\alpha, \alpha_p)^{1-b} \chi(\alpha_p, \beta)^{b-1} \Lambda(K_\alpha L_\beta) F_p^{b-1} v_\Lambda \\ &= \Lambda'(K_{\sigma_p^\chi(\alpha)} L_{\sigma_p^\chi(\beta)}) F_p^{b-1} v_\Lambda \end{aligned}$$

for all $\alpha, \beta \in \mathbb{Z}^I$. Hence $T_p(u) F_p^{b-1} v_\Lambda = \hat{T}_p(\Lambda'(u) v_{\Lambda'})$ for all $u \in \mathcal{U}^0$. Therefore $\hat{T}_p$ is well-defined.

The last claim of the lemma follows from the equations

$$\hat{T}_p(uv) = T_p(u)\hat{T}_p(v), \quad \hat{T}_p^-(uv) = T_p^-(u)\hat{T}_p^-(v), \tag{5.12}$$

where $u \in U(r_p(\chi))$ and $v \in M^{r_p(\chi)}(t_p^\chi(\Lambda))$, and from the fact that $T_p(U(r_p(\chi))) = T_p^-(U(r_p(\chi))) = U(\chi)$. □

Lemma 5.9. *Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$ and $k \in I\setminus\{p\}$. Assume that $m^\chi_{p,k} = |(\mathbb{N}_0\alpha_p + \mathbb{N}_0\alpha_k) \cap R^\chi_+| < \infty$ and that $b^\chi(\alpha) < \infty$ for all $\alpha \in (\mathbb{N}_0\alpha_p + \mathbb{N}_0\alpha_k) \cap R^\chi_+$. Then*

$$(t_p t_k)^{m^\chi_{p,k}-1} t_p t^\chi_k(\Lambda) = \Lambda. \tag{5.13}$$

Proof. Let $m = m^\chi_{p,k}$, and let $i_0, i_1, \dots, i_m \in \{p, k\}$ such that $i_\nu = p$ if ν is even and $i_\nu = k$ if ν is odd. Let $\chi' = r_{i_{m-1}} \cdots r_{i_1} r_{i_0}(\chi) = r_{i_m} \cdots r_{i_2} r_{i_1}(\chi)$ (by (R4)), and

$$\Lambda' = t_{i_{m-1}} \cdots t_{i_1} t^\chi_{i_0}(\Lambda), \quad \Lambda'' = t_{i_m} \cdots t_{i_2} t^\chi_{i_1}(\Lambda).$$

These definitions make sense, since $b^\chi(\alpha) < \infty$ for all $\alpha \in (\mathbb{N}_0\alpha_p + \mathbb{N}_0\alpha_k) \cap R^\chi_+$. Lemma 5.7 implies that the claim of the lemma is equivalent to $\Lambda' = \Lambda''$.

Let $\hat{T}' = \hat{T}_{i_0}\hat{T}_{i_1} \cdots \hat{T}_{i_{m-1}} : M^{\chi'}(\Lambda') \to M^\chi(\Lambda)$ and $\hat{T}'' = \hat{T}_{i_1}\hat{T}_{i_2} \cdots \hat{T}_{i_m} : M^{\chi'}(\Lambda'') \to M^\chi(\Lambda)$. For all $\nu \in \{1, 2, \dots, m\}$ let

$$\beta'_\nu = 1^\chi \sigma_{i_0}\sigma_{i_1} \cdots \sigma_{i_{\nu-2}}(\alpha_{i_{\nu-1}}), \quad \beta''_\nu = 1^\chi \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_{\nu-1}}(\alpha_{i_\nu}).$$

By definition of $\hat{T}_p$ and $\hat{T}_k$,

$$\begin{aligned}\hat{T}''(v_{\Lambda''}) =& \hat{T}_{i_1} \cdots \hat{T}_{i_{m-1}}(F_{i_m}^{b^{\chi'}(\alpha_{i_m})-1} v_{t^{\chi'}_{i_m}(\Lambda'')}) = \cdots \\ =& F_{\beta''_m}^{b^\chi(\beta''_m)-1} \cdots F_{\beta''_2}^{b^\chi(\beta''_2)-1} F_{\beta''_1}^{b^\chi(\beta''_1)-1} v_\Lambda, \\ \hat{T}'(v_{\Lambda'}) =& F_{\beta'_m}^{b^\chi(\beta'_m)-1} \cdots F_{\beta'_2}^{b^\chi(\beta'_2)-1} F_{\beta'_1}^{b^\chi(\beta'_1)-1} v_\Lambda.\end{aligned}$$

Both expressions are nonzero by Thm. 4.9. Since

$$\{\beta'_\nu \,|\, 1 \le \nu \le m\} = \{\beta''_\nu \,|\, 1 \le \nu \le m\} = R^\chi_+ \cap (\mathbb{N}_0\alpha_p + \mathbb{N}_0\alpha_k),$$

we obtain that

($*$) $\hat{T}'(\mathbb{K}v_{\Lambda'})$ and $\hat{T}''(\mathbb{K}v_{\Lambda''})$ are isomorphic $\mathcal{U}^0$-modules.

By Thm. 2.6,

$$T_{i_0}T_{i_1} \cdots T_{i_{m-1}}(u_0) = T_{i_1}T_{i_2} \cdots T_{i_m}(u_0) \quad \text{for all } u_0 \in \mathcal{U}^0.$$

Hence Lemma 5.8 yields that

$$\begin{aligned}\Lambda'(u_0)\hat{T}'(v_{\Lambda'}) = \hat{T}'(u_0 v_{\Lambda'}) =& T_{i_0}T_{i_1} \cdots T_{i_{m-1}}(u_0)\hat{T}'(v_{\Lambda'}), \\ \Lambda''(u_0)\hat{T}''(v_{\Lambda''}) = \hat{T}''(u_0 v_{\Lambda''}) =& T_{i_0}T_{i_1} \cdots T_{i_{m-1}}(u_0)\hat{T}''(v_{\Lambda''}).\end{aligned}$$

Thus $\Lambda' = \Lambda''$ by ($*$). This proves the lemma. □

Remark 5.10. In view of Thm. 2.6 and Lemmas 5.7, 5.9 we can say that Eq. (5.10) defines an action of the groupoid $\mathcal{W}(\chi)$ on $\mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. Then Lemma 5.5 says that the numbers $\rho^\chi(\alpha)\Lambda(K_\alpha L_\alpha^{-1})$, where $\alpha \in \mathbb{Z}^I$, are invariants of this action.

In general, the maps $\hat{T}_p$ and $\hat{T}^-_p$ are not isomorphisms.

Proposition 5.11. *Assume that $\Lambda(K_p L_p^{-1}) \neq \chi(\alpha_p, \alpha_p)^{t-1}$ for all $t \in \{1, 2, \dots, b-1\}$. Then $\hat{T}_p, \hat{T}^-_p : M^{r_p(\chi)}(t^\chi_p(\Lambda)) \to M^\chi(\Lambda)$ are isomorphisms of vector spaces over $\mathbb{K}$.*

Proof. Let $q = \chi(\alpha_p, \alpha_p)$, $\chi' = r_p(\chi)$, $\Lambda' = t_p^\chi(\Lambda)$, and $\hat{T}' = \hat{T}^\chi_{p,\Lambda}\hat{T}^{\chi',-}_{p,\Lambda'}$. By Lemma 5.7 and since $r_p^2(\chi) = \chi$, $\hat{T}'$ is a $U(\chi)$-module endomorphism of $M^\chi(\Lambda)$. We calculate $\hat{T}'(v_\Lambda)$.

$$\begin{aligned}\hat{T}'(v_\Lambda) =&\hat{T}_p(F_p^{b-1}v_{\Lambda'}) = T_p(F_p^{b-1})F_p^{b-1}v_\Lambda\\ =&(K_p^{-1}E_p)^{b-1}F_p^{b-1}v_\Lambda = q^{(b-2)(b-1)/2}\Lambda(K_p^{1-b})E_p^{b-1}F_p^{b-1}v_\Lambda\\ =&q^{(b-2)(b-1)/2}\Lambda(K_p^{1-b})(b-1)_q^!\prod_{t=1}^{b-1}(q^{t+1-b}\Lambda(K_p)-\Lambda(L_p))v_\Lambda\end{aligned}$$

by Lemma 3.7. By assumption, $\hat{T}'(v_\Lambda) \neq 0$, and hence $\hat{T}'$ is a nonzero multiple of $\mathrm{id}_{M^\chi(\Lambda)}$. Therefore $\hat{T}_p$ is an isomorphism. The proof for $\hat{T}_p^-$ is analogous. □

Lemma 5.12. *Let $t \in \{1, 2, \dots, b-1\}$. Let $q = \chi(\alpha_p, \alpha_p)$. Assume that $\Lambda(K_pL_p^{-1}) = q^{t-1}$. Then in $M^\chi(\Lambda)$*

$$E_pF_p^m v_\Lambda = (m)_q\Lambda(L_p)(q^{t-m}-1)F_p^{m-1}v_\Lambda \quad \textit{for all } m \in \mathbb{N}_0. \tag{5.14}$$

In particular, if $q \neq 1$, then $EF_p^m v_\Lambda = \varepsilon(E)F_p^m v_\Lambda$ for all $E \in U^+(\chi)$ if and only if $m = 0$, $m = t$ or $m \geq b$. If $q = 1$, then $EF_p^m v_\Lambda = \varepsilon(E)F_p^m v_\Lambda$ for all $E \in U^+(\chi)$, $m \in \mathbb{N}_0$.

Proof. Eq. (5.14) follows from Lemma 3.7. By definition of $b = b^\chi(\alpha_p)$, either $q \neq 1$ and q is a primitive b-th root of 1, or $q = 1$ and $b = \mathrm{char}\,\Bbbk$. Therefore, if $q \neq 1$ and $m \in \{0, 1, \dots, b-1\}$, then $q^{t-m} = 1$ if and only if $t = m$. If $E = E_i$ with $i \neq p$, then $EF_p^m v_\Lambda = 0$ by Eqs. (5.4), (5.5). The rest is a consequence of Lemma 3.6(i). □

Proposition 5.13. *Assume that $\Lambda(K_pL_p^{-1}) = \chi(\alpha_p, \alpha_p)^{t-1}$ for some $t \in \{1, 2, \dots, b-1\}$.*

(i) If $\chi(\alpha_p, \alpha_p) \neq 1$, then t is unique, and

$$\begin{aligned}\ker \hat{T}^\chi_{p,\Lambda} = \ker \hat{T}^{\chi,-}_{p,\Lambda} &= U^-(r_p(\chi))F_p^{b-t} \otimes \Bbbk_{t_p^\chi(\Lambda)},\\ \mathrm{Im}\, \hat{T}^\chi_{p,\Lambda} = \mathrm{Im}\, \hat{T}^{\chi,-}_{p,\Lambda} &= U^-(\chi)F_p^t \otimes \Bbbk_\Lambda.\end{aligned}$$

(ii) If $\chi(\alpha_p, \alpha_p) = 1$, then $\mathrm{char}\,\Bbbk = b > 0$ and

$$\begin{aligned}\ker \hat{T}^\chi_{p,\Lambda} = \ker \hat{T}^{\chi,-}_{p,\Lambda} &= U^-(r_p(\chi))F_p \otimes \Bbbk_{t_p^\chi(\Lambda)},\\ \mathrm{Im}\, \hat{T}^\chi_{p,\Lambda} = \mathrm{Im}\, \hat{T}^{\chi,-}_{p,\Lambda} &= U^-(\chi)F_p^{b-1} \otimes \Bbbk_\Lambda.\end{aligned}$$

Proof. Let $q = \chi(\alpha_p, \alpha_p)$, $\chi' = r_p(\chi)$, and $\Lambda' = t_p^\chi(\Lambda)$. We prove the claims about $\hat{T}^\chi_{p,\Lambda}$ in (i). The rest is analogous.

Let $m \in \{0,1,\dots,b-1\}$. By Lemma 3.7 we obtain that

$$\begin{aligned}\hat{T}^{\chi}_{p,\Lambda}(F_p^m v_{\Lambda'}) =&T_p(F_p^m)F_p^{b-1}v_\Lambda = (K_p^{-1}E_p)^m F_p^{b-1}v_\Lambda\\ =&aE_p^m F_p^{b-1}v_\Lambda\\ =&a'F_p^{b-1-m}\prod_{j=1}^{m}(q^{j+1-b}\Lambda(K_pL_p^{-1})-1)v_\Lambda\\ =&a'F_p^{b-1-m}\prod_{j=1}^{m}(q^{j+t-b}-1)v_\Lambda\end{aligned} \tag{5.15}$$

for some $a,a' \in \Bbbk^\times$, where the last equation follows from the assumption on Λ. Thus, $\hat{T}^{\chi}_{p,\Lambda}(F_p^m v_{\Lambda'})=0$ if and only if $j=b-t$ for some $j\in\{1,2,\dots,m\}$. The latter is equivalent to $m\ge b-t$.

Let $F\in U^-(\chi')$. Lemma 3.6(ii) implies that for all integers $m\in\{0,1,\dots,b-1\}$ there exist unique elements $F'_m\in U^-_{p,K}(\chi')$ such that $F=\sum_{m=0}^{b-1}F'_mF_p^m$. Recall that $q\in\Bbbk$. Hence by Lemma 4.3 and Eq. (5.15) there exist unique $a_m\in\Bbbk^\times$ with $m\in\{0,1,\dots,b-1-t\}$ such that

$$\hat{T}^{\chi}_{p,\Lambda}(Fv_{\Lambda'})=\sum_{m=0}^{b-1-t}a_mT_p(F'_m)F_p^{b-1-m}v_\Lambda\in U^-(\chi)\otimes 1_\Lambda.$$

By Lemmas 3.6(ii) and 4.3, the latter expression is zero if and only if $T_p(F'_m)=0$ for all $m\in\{0,1,\dots,b-1-t\}$. Therefore $\hat{T}^{\chi}_{p,\Lambda}(Fv_{\Lambda'})=0$ if and only if $F'_m=0$ for all $m\in\{0,1,\dots,b-1-t\}$. This and Eq. (5.15) imply that $\ker\hat{T}^{\chi}_{p,\Lambda}=U^-(\chi')F_p^{b-t}\otimes\mathbb{K}_{\Lambda'}$ and $\operatorname{Im}\hat{T}^{\chi}_{p,\Lambda}=U^-(\chi)F_p^t\otimes\mathbb{K}_\Lambda$. □

For all $w\in\operatorname{Aut}(\mathbb{Z}^I)$ and $\alpha\in\mathbb{Z}^I$ let $w(e^\alpha)=e^{w(\alpha)}$, and extend this definition linearly on formal characters. We investigate the effect of the maps $\hat{T}_p,\hat{T}_p^-$ on formal characters. For all $\chi'\in\mathcal{G}(\chi)$ and $i\in I$ with $b^{\chi'}(\alpha_i)<\infty$ let $\dot{\sigma}_i^{\chi'}:\mathbb{Z}^I\to\mathbb{Z}^I$ be the affine transformation

$$\dot{\sigma}_i^{\chi'}(\alpha)=\sigma_i^{\chi'}(\alpha)+(1-b^{\chi'}(\alpha_i))\alpha_i. \tag{5.16}$$

Note that then $\dot{\sigma}_i^{r_i(\chi')}\dot{\sigma}_i^{\chi'}(\alpha)=\alpha$ for all $\alpha\in\mathbb{Z}^I$.

Lemma 5.14. *Let $\Lambda\in\operatorname{Hom}(\mathcal{U}^0,\mathbb{K}^\times)$ and $\alpha\in\mathbb{Z}^I$. Then*

$$\hat{T}_p(M^{r_p(\chi)}(t_p^\chi(\Lambda))_\alpha)\subset M^\chi(\Lambda)_{\dot{\sigma}_p^{r_p(\chi)}(\alpha)}, \tag{5.17}$$

$$\hat{T}_p^-(M^{r_p(\chi)}(t_p^\chi(\Lambda))_\alpha)\subset M^\chi(\Lambda)_{\dot{\sigma}_p^{r_p(\chi)}(\alpha)}. \tag{5.18}$$

In particular,

$$\operatorname{ch}M^\chi(\Lambda)=\dot{\sigma}_p^{r_p(\chi)}(\operatorname{ch}M^{r_p(\chi)}(t_p^\chi(\Lambda))). \tag{5.19}$$

Proof. Let $\Lambda'=t_p^\chi(\Lambda)$ and $u\in U(r_p(\chi))_\alpha$. Then

$$\hat{T}_p(uv_{\Lambda'})=T_p(u)F_p^{b-1}v_\Lambda\in U(\chi)_{\sigma_p^{r_p(\chi)}(\alpha)}U(\chi)_{(1-b)\alpha_p}v_\Lambda$$

by Prop. 4.2. This proves Eq. (5.17), since $v_\Lambda \in M^\chi(\Lambda)_0$, The proof of Eq. (5.18) is similar. By Eq. (5.9), $\mathrm{ch}\, M^\chi(\Lambda)$ does not depend on Λ. Hence by Prop. 5.11 we may assume that $\hat{T}_p$ is an isomorphism. Then Eq. (5.19) follows from Eq. (5.17). □

Lemma 5.15. *Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$ and $t \in \{1, 2, \dots, b-1\}$. Assume that $\Lambda(K_pL_p^{-1}) = \chi(\alpha_p, \alpha_p)^{t-1}$. Then $V = U^-(\chi)F_p^t \otimes \mathbb{K}_\Lambda$ is a $U(\chi) \otimes \mathbb{K}$-submodule of $M^\chi(\Lambda)$ with*

$$\mathrm{ch}\, V = \mathrm{ch}\, M^\chi(\Lambda)\frac{e^{-t\alpha_p} - e^{-b\alpha_p}}{1 - e^{-b\alpha_p}}.$$

Proof. The formal character of the subspace $\oplus_{m=t}^{b-1} F_p^m$ of $M^\chi(\Lambda)$ is

$$e^{-t\alpha_p} + e^{-(t+1)\alpha_p} + \cdots + e^{-(b-1)\alpha_p} = \frac{e^{-t\alpha_p} - e^{-b\alpha_p}}{1 - e^{-\alpha_p}}.$$

Thus the lemma is a consequence of Lemmas 5.12, 5.2 and Thm. 4.9. □

In the rest of this section assume that $\chi \in \mathcal{X}_4$. Let $n = |R_+^\chi|$ and $i_1, \dots, i_n \in I$ with $\ell(1^\chi \sigma_{i_1} \cdots \sigma_{i_n}) = n$. Recall the definitions of β_ν and F_{β_ν}, where $1 \le \nu \le n$, from Eq. (4.3). Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. We characterize irreducible Verma modules (see also Lemma 6.7).

Proposition 5.16. *Assume that*

$$\prod_{\nu=1}^{n} \prod_{t=1}^{b^\chi(\beta_\nu)-1} \left(\rho^\chi(\beta_\nu)\Lambda(K_{\beta_\nu}L_{\beta_\nu}^{-1}) - \chi(\beta_\nu, \beta_\nu)^t\right) \neq 0. \tag{5.20}$$

Then $I^\chi(\Lambda) = 0$.

Proof. For all $\nu \in \{1, 2, \dots, n\}$ let $\chi_\nu = r_{i_{\nu-1}} \cdots r_{i_2} r_{i_1}(\chi)$ and $\Lambda_\nu = t_{i_{\nu-1}} \cdots t_{i_2} t_{i_1}^\chi(\Lambda)$. By Lemma 5.5 and Eq. (2.13), Eq. (5.20) is equivalent to

$$\rho^{\chi_\nu}(\alpha_{i_\nu})\Lambda_\nu(K_{i_\nu}L_{i_\nu}^{-1}) \neq \chi_\nu(\alpha_{i_\nu}, \alpha_{i_\nu})^t$$

for all $\nu \in \{1, 2, \dots, n\}$, $t \in \{1, 2, \dots, b^{\chi_\nu}(\alpha_{i_\nu}) - 1\}$. Hence, by Prop. 5.11, the map

$$\hat{T}_{i_1}\hat{T}_{i_2} \cdots \hat{T}_{i_n} : M^{r_{i_n}(\chi_n)}(t_{i_n}(\Lambda_n)) \to M^\chi(\Lambda)$$

is an isomorphism. Thus $v = F_{\beta_n}^{b^\chi(\beta_n)-1} \cdots F_{\beta_2}^{b^\chi(\beta_2)-1} F_{\beta_1}^{b^\chi(\beta_1)-1} v_\Lambda \neq 0$ and $(U^+(\chi) \otimes_\Bbbk \mathbb{K})v = M^\chi(\Lambda)$. Since v is contained in any nonzero $U(\chi) \otimes_\Bbbk \mathbb{K}$-submodule of $M^\chi(\Lambda)$ by Thms. 4.8, 4.9, it follows that $I^\chi(\Lambda) = 0$. □

6. The Shapovalov form

We discuss the analog of the Shapovalov form for the algebras $U(\chi)$ following the construction in [Jos95, 3.4.10].

Let $\chi \in \mathcal{X}$. By Prop. 3.4, there exists a decomposition

$$\mathcal{U}(\chi) = \Big(\sum_{i \in I} F_i\mathcal{U}(\chi) + \sum_{i \in I} \mathcal{U}(\chi)E_i\Big) \oplus \mathcal{U}^0$$

and hence a unique projection

$$\theta^\chi : \mathcal{U}(\chi) \to \mathcal{U}^0$$

with kernel $\sum_{i\in I} F_i\mathcal{U}(\chi) + \sum_{i\in I}\mathcal{U}(\chi)E_i$. This map is commonly known as the *Harish-Chandra map*. By definition, θ^χ satisfies the property

$$\theta^\chi(u_-uu_+) = \varepsilon(u_-)\theta^\chi(u)\varepsilon(u_+) \tag{6.1}$$

for all $u_-\in U^-(\chi)$, $u\in U(\chi)$, $u_+\in U^+(\chi)$. Since $\Omega(u)=u$ for all $u\in\mathcal{U}^0$,

$$\theta^\chi(\Omega(u)) = \theta^\chi(u) \qquad \text{for all } u\in\mathcal{U}(\chi). \tag{6.2}$$

The bilinear map

$$\mathrm{Sh}:\mathcal{U}(\chi)\times\mathcal{U}(\chi)\to\mathcal{U}^0, \qquad \mathrm{Sh}(u,v)=\theta^\chi(\Omega(u)v), \tag{6.3}$$

is called the *Shapovalov form*. By Eq. (6.2) and since $\Omega^2=\mathrm{id}$,

$$\mathrm{Sh}(u,v)=\mathrm{Sh}(v,u) \quad \text{for all } u,v\in\mathcal{U}(\chi). \tag{6.4}$$

Moreover, by definition of Sh and θ^χ,

$$\mathrm{Sh}(u,v)=0 \quad \text{if } u\in\sum_{i\in I}\mathcal{U}(\chi)E_i \text{ or } v\in\sum_{i\in I}\mathcal{U}(\chi)E_i. \tag{6.5}$$

Recall the definitions of $U(\chi)$, $\mathcal{I}^+(\chi)$ and $\mathcal{I}^-(\chi)$ from Sect. 3. Since $\mathcal{U}(\chi)\mathcal{I}^+(\chi)\mathcal{U}(\chi)+\mathcal{U}(\chi)\mathcal{I}^-(\chi)\mathcal{U}(\chi)\subset\ker\theta^\chi$, θ^χ and Sh induce maps

$$\theta^\chi: U(\chi)\to\mathcal{U}^0, \qquad \mathrm{Sh}: U(\chi)\times U(\chi)\to\mathcal{U}^0.$$

The map θ^χ is $\mathbb{Z}^I$-homogeneous, that is, $\theta^\chi(u)=0$ for all $u\in U(\chi)_\alpha$, where $\alpha\in\mathbb{Z}^I\setminus\{0\}$. The map Ω reverses degrees, that is, $\Omega(u)\in U(\chi)_{-\alpha}$ for all $u\in U(\chi)_\alpha$, where $\alpha\in\mathbb{Z}^I$. Therefore, for all $\alpha,\beta\in\mathbb{Z}^I$, where $\alpha\neq\beta$, we get

$$\mathrm{Sh}(u,v)=0 \quad \text{for all } u\in U(\chi)_\alpha,\ v\in U(\chi)_\beta \quad (\alpha\neq\beta). \tag{6.6}$$

Definition 6.1. The family of determinants

$$\det{}^\chi_\alpha = \det \mathrm{Sh}(F'_i,F'_j)_{i,j\in\{1,\dots,k\}}\in\mathcal{U}^0/\Bbbk^\times,$$

where $\alpha\in\mathbb{N}_0^I$, $k=\dim U^-(\chi)_{-\alpha}$, and $\{F'_1,F'_2,\dots,F'_k\}$ is a basis of $U^-(\chi)_{-\alpha}$, is called the *Shapovalov determinant* of $U(\chi)$.

Remark 6.2. Let $\alpha\in\mathbb{N}_0^I$ and $k=\dim U^-(\chi)_{-\alpha}$. By the above considerations, $\mathrm{Sh}: U^-(\chi)_{-\alpha}\times U^-(\chi)_{-\alpha}\to\mathcal{U}^0$ is a symmetric bilinear form for all $\alpha\in\mathbb{N}_0^I$. Let $F'=\{F'_1,F'_2,\dots,F'_k\}$ be a basis of $U^-(\chi)_{-\alpha}$, and let $d(F')=\det\mathrm{Sh}(F'_i,F'_j)_{i,j\in\{1,\dots,k\}}\in\mathcal{U}^0$. Then $d(A'F')=(\det A')^2d(F')$ for all $A'\in\mathrm{GL}(k,\Bbbk)$, and hence $\det^\chi_\alpha = d(F')/\Bbbk^\times$ does not depend on the choice of the basis F' of $U^-(\chi)_{-\alpha}$.

Lemma 6.3. *Let $\chi\in\mathcal{X}$. Let J be an ideal of $\mathcal{U}^0$. Assume that J is contained in the center of $U(\chi)$. Let J_U be the ideal of $U(\chi)$ generated by J. Then* $\mathrm{Sh}: U(\chi)\times U(\chi)\to\mathcal{U}^0$ *induces a map* $\mathrm{Sh}: U(\chi)/J_U\times U(\chi)/J_U\to\mathcal{U}^0/J$.

Proof. Since J is contained in the center of $U(\chi)$, triangular decomposition of $U(\chi)$ yields that $J_U=U^-(\chi)JU^+(\chi)$. This and $\Omega(J)=J$ imply the claim of the lemma. □

Lemma 6.4. *Let $\chi \in \mathcal{X}$, $\alpha \in \mathbb{Z}^I$, $E \in U^+(\chi)_\alpha$, and $F \in U^-(\chi)_{-\alpha}$. Then*

$$\mathrm{Sh}(\Omega(E), F) \in \sum_{\beta,\gamma\in\mathbb{N}_0^I,\, \beta+\gamma=\alpha} \Bbbk K_\beta L_\gamma$$

Moreover, the coefficients of K_α and L_α of $\mathrm{Sh}(\Omega(E), F)$ are $\eta(E, S(F))$ and $\eta(E, F)$, respectively.

Proof. If $\alpha \notin \mathbb{N}_0^I$ or $\alpha = 0$, then $U^+(\chi)_\alpha = 0$ or $U^+(\chi)_\alpha = \Bbbk$. In this case the claim of the lemma holds by definition of Sh. Assume now that $\alpha \in \mathbb{N}_0^I \setminus \{0\}$. Using Eqs. (3.1)–(3.5), by induction on β and β' one can show that

$$E'F' \in \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4\in\mathbb{N}_0^I,\gamma_4-\gamma_1=\beta-\beta',\gamma_2+\gamma_3+\gamma_4=\beta} U^-(\chi)_{-\gamma_1} K_{\gamma_2} L_{\gamma_3} U^+(\chi)_{\gamma_4}$$

for all $\beta, \beta' \in \mathbb{N}_0^I$ and $E' \in U^+(\chi)_\beta$, $F' \in U^-(\chi)_{-\beta'}$. This implies the first claim of the lemma by letting $\beta = \beta' = \alpha$ and $E' = E$, $F' = F$. The second claim follows from

$$\Delta(E) - K_\alpha \otimes E - E \otimes 1 \in \bigoplus_{\beta,\gamma\in\mathbb{N}_0^I,\beta+\gamma=\alpha,\, \beta,\gamma\neq 0} U^+(\chi)_\beta K_\gamma \otimes U^+(\chi)_\gamma,$$

$$\Delta(F) - 1 \otimes F - F \otimes L_\alpha \in \bigoplus_{\beta,\gamma\in\mathbb{N}_0^I,\beta+\gamma=\alpha,\beta,\gamma\neq 0} U^-(\chi)_{-\beta} \otimes U^-(\chi)_{-\gamma} L_\beta,$$

and from Eq. (3.19) (with $x = E$, $y = F$) and Prop. 3.5(iii). □

Let $\mathbb{K}$ be a field extension of $\Bbbk$. The importance of the Shapovalov form arises from the fact that it induces a form on the Verma modules $M^\chi(\Lambda)$ and on their simple quotients $L^\chi(\Lambda)$, where $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$.

Let $\Lambda \in \mathrm{Hom}(\mathcal{U}^0, \mathbb{K}^\times)$. Define

$$\Lambda\mathrm{Sh} : U(\chi) \times U(\chi) \to \mathbb{K}, \quad (u, v) \mapsto \Lambda(\mathrm{Sh}(u, v)). \tag{6.7}$$

By Eq. (6.1),

$$\begin{aligned}\Lambda\mathrm{Sh}(u_-u_0u_+, v_-v_0v_+) =&\varepsilon(u_+)\varepsilon(v_+)\Lambda(u_0\mathrm{Sh}(u_-, v_-)v_0)\\ =&\Lambda(u_0)\Lambda(v_0)\varepsilon(u_+)\varepsilon(v_+)\Lambda\mathrm{Sh}(u_-, v_-).\end{aligned}$$

Thus, by Eq. (5.3), $\Lambda\mathrm{Sh}$ induces a $\mathbb{K}$-bilinear form on $M^\chi(\Lambda)$ by letting

$$\Lambda\mathrm{Sh} : M^\chi(\Lambda) \times M^\chi(\Lambda) \to \mathbb{K}, \quad (u \otimes 1_\Lambda, v \otimes 1_\Lambda) \mapsto \Lambda\mathrm{Sh}(u, v)$$

for all $u, v \in U(\chi)$. Moreover, Eq. (6.6) gives that

$$\Lambda\mathrm{Sh}(u \otimes 1_\Lambda, v) = \Lambda\mathrm{Sh}(1 \otimes 1_\Lambda, \Omega(u)v) = 0 \tag{6.8}$$

for all $u \in U(\chi)$ and $v \in I^\chi(\Lambda)$, since $\Omega(u)v \in I^\chi(\Lambda) \subset \oplus_{\alpha\neq 0} M^\chi(\Lambda)_\alpha$. Thus by Eq. (6.4), $\Lambda\mathrm{Sh}$ induces a symmetric bilinear form on $L^\chi(\Lambda)$, also denoted by $\Lambda\mathrm{Sh}$. The radical of this form is a $\mathbb{Z}^I$-graded $U(\chi) \otimes_\Bbbk \mathbb{K}$-submodule of $L^\chi(\Lambda)$, but does not contain $1 \otimes 1_\Lambda$, and hence it is zero. Thus $\Lambda\mathrm{Sh}$ is a nondegenerate symmetric bilinear form on $L^\chi(\Lambda)$.

For the rest of this section let $\chi \in \mathcal{X}_4$, $n = |R_+^\chi|$, and $i_1, \dots, i_n \in I$ with $\ell(1^\chi\sigma_{i_1} \cdots \sigma_{i_n}) = n$. For all $\nu \in \{1, 2, \dots, n\}$ let

$$\beta_\nu = 1^\chi\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_{\nu-1}}(\alpha_{i_\nu}), \qquad \chi_\nu = r_{i_{\nu-1}} \cdots r_{i_2}r_{i_1}(\chi).$$

For all $\nu \in \{1,2,\dots,n\}$, $\alpha \in \mathbb{N}_0^I$, and $t \in \{1,2,\dots,b^\chi(\beta_\nu)-1\}$ let

$$P^\chi(\alpha,\beta_\nu;t) = \Big|\Big\{(m_1,\dots,m_n) \in \mathbb{N}_0^n \mid \sum_{\mu=1}^n m_\mu\beta_\mu = \alpha,\ m_\nu \geq t,\\ m_\mu < b^\chi(\beta_\mu) \quad \text{for all } \mu \in \{1,2,\dots,n\}\Big\}\Big|. \tag{6.9}$$

We will use two important facts on the function P^χ.

Lemma 6.5. *For all $\nu \in \{1,2,\dots,n\}$ and $t \in \{1,2,\dots,b^\chi(\beta_\nu)-1\}$,*

$$\sum_{\alpha\in\mathbb{N}_0^I} P^\chi(\alpha,\beta_\nu;t)e^{-\alpha} = \frac{e^{-t\beta_\nu} - e^{-b^\chi(\beta_\nu)\beta_\nu}}{1-e^{-\beta_\nu}} \prod_{\mu\in\{1,\dots,n\},\,\mu\neq\nu} \frac{1-e^{-b^\chi(\beta_\mu)\beta_\mu}}{1-e^{-\beta_\mu}}.$$

Proof. By Thm. 4.9, the two sides of the equation are two different expressions for the formal character of the subspace of $U^-(\chi)\otimes\mathbb{K}$ spanned by the elements

$$\prod_{\substack{m_1,\dots,m_n\\ m_\nu\geq t,\,0\leq m_\mu<b^\chi(\beta_\mu)\text{ for all }\mu}} F_{\beta_1}^{m_1}F_{\beta_2}^{m_2}\cdots F_{\beta_n}^{m_n}\otimes 1.$$

□

Lemma 6.6. *For all $\alpha \in \mathbb{N}_0^I$,*

$$\alpha \dim U^-(\chi)_{-\alpha} = \sum_{\nu=1}^n \sum_{t=1}^{b^\chi(\beta_\nu)-1} P^\chi(\alpha,\beta_\nu;t)\beta_\nu.$$

Proof. By Thm. 4.9, for each $\alpha \in \mathbb{N}_0^I$ there is a basis of $U^-(\chi)_{-\alpha}$ parametrized by the set

$$\Big\{(m_1,\dots,m_n)\in\mathbb{N}_0^n \mid \sum_{\mu=1}^n m_\mu\beta_\mu = \alpha,\ m_\mu < b^\chi(\beta_\mu) \text{ for all } \mu\Big\}. \tag{6.10}$$

Each $(m_1,\dots,m_n)$ in this set contributes to $P^\chi(\alpha,\beta_\nu;t)$ with a summand 1, for all $\nu \in \{1,2,\dots,n\}$ and $t \in \{1,2,\dots,m_\nu\}$. Thus the claim of the lemma follows from the decomposition of the $P^\chi(\alpha,\beta_\nu;t)$ into $1+1+\cdots+1$ by reordering the summands. □

Lemma 6.7. *Let $\nu \in \{1,2,\dots,n\}$, $t \in \{1,2,\dots,b^\chi(\beta_\nu)-1\}$, and $\Lambda \in \mathrm{Hom}(\mathcal{U}^0,\mathbb{K}^\times)$. Assume that $\rho^\chi(\beta_\nu)\Lambda(K_{\beta_\nu}L_{\beta_\nu}^{-1}) = \chi(\beta_\nu,\beta_\nu)^t$ and*

$$\prod_{\mu=1}^{\nu-1}\prod_{m=1}^{b^\chi(\beta_\mu)-1}(\rho^\chi(\beta_\mu)\Lambda(K_{\beta_\mu}L_{\beta_\mu}^{-1}) - \chi(\beta_\mu,\beta_\mu)^m) \neq 0.$$

Then $M^\chi(\Lambda)$ contains a $U(\chi)\otimes\mathbb{K}$-submodule V with

$$\mathrm{ch}\,V = \sum_{\alpha\in\mathbb{N}_0^I} P^\chi(\alpha,\beta_\nu;t)e^{-\alpha}. \tag{6.11}$$

In particular, $0 \neq V \subset I^\chi(\Lambda)$.

Proof. We proceed by induction on ν. Let first $\nu = 1$. By Lemma 5.15, $V = U^-(\chi)F^t_{i_1} \otimes \mathbb{K}_\Lambda$ is a $U(\chi)\otimes\mathbb{K}$-submodule of $M^\chi(\Lambda)$. Then Eq. (6.11) follows from Thm. 4.9.

Assume now that $\nu \in \{2, 3, \dots, n\}$ and that the lemma holds for $\nu - 1$. Let $\chi_\mu = r_{i_{\mu-1}} \cdots r_{i_2} r_{i_1}(\chi)$ and $\Lambda_\mu = t_{i_{\mu-1}} \cdots t_{i_2} t^\chi_{i_1}(\Lambda)$ for all $\mu \in \{1, 2, \dots, \nu\}$. By Lemma 5.5, the assumptions on Λ are equivalent to the relations

$$\prod_{\mu=1}^{\nu-1} \prod_{m=1}^{b^{\chi_\mu}(\alpha_{i_\mu})-1} (\Lambda_\mu(K_{i_\mu}L^{-1}_{i_\mu}) - \rho^{\chi_\mu}(\alpha_{i_\mu})^{m-1}) \neq 0$$

and $\Lambda_\nu(K_{i_\nu}L^{-1}_{i_\nu}) = \rho^{\chi_\nu}(\alpha_{i_\nu})^{t-1}$. Let

$$\beta'_\nu = \sigma^\chi_{i_1}(\beta_\nu) = 1^{\chi_2}\sigma_{i_2}\sigma_{i_3}\cdots\sigma_{i_{\nu-1}}(\alpha_{i_\nu}).$$

By induction hypothesis there exists a $U(\chi_2)$-submodule V' of $M^{\chi_2}(\Lambda_2)$ with

$$\operatorname{ch} V' = \sum_{\alpha\in\mathbb{N}_0^I} P^{\chi_2}(\alpha, \beta'_\nu; t)e^{-\alpha}. \tag{6.12}$$

Moreover, $\Lambda(K_{i_1}L^{-1}_{i_1}) \neq \rho^\chi(\alpha_{i_1})^{m-1}$ for all $m \in \{1, 2, \dots, b^\chi(\alpha_{i_1}) - 1\}$, and hence $\hat{T}_{i_1} : M^{\chi_2}(\Lambda_2) \to M^\chi(\Lambda)$ is an isomorphism. Let $V = \hat{T}_{i_1}(V')$. By Lemmas 5.8 and 5.14, V is a $U(\chi)$-submodule of $M^\chi(\Lambda)$ and

$$\operatorname{ch} V = \dot\sigma^{\chi_2}_{i_1}(\operatorname{ch} V') = \sum_{\alpha\in\mathbb{N}_0^I} \dot\sigma^{\chi_2}_{i_1}(P^{\chi_2}(\alpha, \beta'_\nu; t)e^{-\alpha}). \tag{6.13}$$

Thus, by Lemma 6.5,

$$\operatorname{ch} V = e^{(1-b^\chi(\alpha_{i_1}))\alpha_{i_1}}\sigma^{\chi_2}_{i_1}\Big(\frac{e^{-t\beta'_\nu} - e^{-b^{\chi_2}(\beta'_\nu)\beta'_\nu}}{1 - e^{-\beta'_\nu}} \prod_{\beta\in R^{\chi_2}_+\setminus\{\beta'_\nu\}} \frac{1 - e^{-b^{\chi_2}(\beta)\beta}}{1 - e^{-\beta}}\Big).$$

Recall that $\beta'_\nu \neq \alpha_{i_1}$, since $\nu > 1$. Moreover,

$$\begin{aligned} &e^{(1-b^\chi(\alpha_{i_1}))\alpha_{i_1}}\sigma^{\chi_2}_{i_1}\Big(\frac{1 - e^{-b^{\chi_2}(\alpha_{i_1})\alpha_{i_1}}}{1 - e^{-\alpha_{i_1}}}\Big) \\ &\quad = e^{(1-b^\chi(\alpha_{i_1}))\alpha_{i_1}}\frac{1 - e^{b^\chi(\alpha_{i_1})\alpha_{i_1}}}{1 - e^{\alpha_{i_1}}} = \frac{1 - e^{-b^\chi(\alpha_{i_1})\alpha_{i_1}}}{1 - e^{-\alpha_{i_1}}}. \end{aligned}$$

Therefore

$$\operatorname{ch} V = \frac{e^{-t\beta_\nu} - e^{-b^\chi(\beta_\nu)\beta_\nu}}{1 - e^{-\beta_\nu}} \prod_{\beta\in R^\chi_+\setminus\{\beta_\nu\}} \frac{1 - e^{-b^\chi(\beta)\beta}}{1 - e^{-\beta}} = \sum_{\alpha\in\mathbb{N}_0^I} P^\chi(\alpha, \beta_\nu; t)e^{-\alpha}$$

by Lemma 6.5. This proves Eq. (6.11).

Since $t > 0$, $\operatorname{ch} V \neq \operatorname{ch} M^\chi(\Lambda)$. By assumption on t, $V_{t\beta_\nu} \neq 0$, and hence $V \neq 0$. Since V is a $\mathbb{Z}^I$-graded $U(\chi)$-submodule of $M^\chi(\Lambda)$, the lemma is proven. □

Theorem 6.8. *Let $\chi \in \mathcal{X}_5$. For all $\alpha \in \mathbb{N}_0^I$, the Shapovalov determinant of $U(\chi)$ is the family $(\det_\alpha^\chi)_{\alpha\in\mathbb{N}_0^I}$, where*

$$\det_\alpha^\chi = \prod_{\beta\in R_+^\chi} \prod_{t=1}^{b^\chi(\beta_\nu)-1} (\rho^\chi(\beta)K_\beta - \chi(\beta,\beta)^t L_\beta)^{P^\chi(\alpha,\beta;t)}. \tag{6.14}$$

Proof. Let $\alpha \in \mathbb{N}_0^I$, $k = \dim U^-(\chi)_{-\alpha}$, and let $\{F_1', F_2', \dots, F_k'\}$ be a basis of $U^-(\chi)_{-\alpha}$. Then $\mathrm{Sh}(F_i', F_j') \in \sum_{\beta,\gamma\in\mathbb{N}_0^I,\beta+\gamma=\alpha} \Bbbk K_\beta L_\gamma$ by Lemma 6.4, and hence

$$\det_\alpha^\chi \in \sum_{\beta,\gamma\in\mathbb{N}_0^I,\beta+\gamma=k\alpha} \Bbbk K_\beta L_\gamma. \tag{6.15}$$

The polynomials

$$\rho^\chi(\beta_\nu)K_{\beta_\nu} - \chi(\beta_\nu,\beta_\nu)^t L_{\beta_\nu} = T_{i_1}\cdots T_{i_{\nu-1}}(\rho^\chi(\beta_\nu)K_{i_\nu} - \chi(\beta_\nu,\beta_\nu)^t L_{i_\nu})$$

are irreducible and pairwise distinct for all $\nu \in \{1,2,\dots,n\}$ and $t \in \{1,2,\dots,b^\chi(\beta_\nu)-1\}$. Thus by Lemma 6.6 it suffices to prove that $\det_\alpha^\chi \neq 0$ and that the polynomials $(\rho^\chi(\beta_\nu)K_{\beta_\nu} - \chi(\beta_\nu,\beta_\nu)^t L_{\beta_\nu})^{P^\chi(\alpha,\beta_\nu;t)}$ are factors of $\det_\alpha^\chi$.

Let $\bar{\Bbbk}$ be the algebraic closure of $\Bbbk$ and $\mathbb{T} = \mathrm{maxspec}\,\mathcal{U}^0 \otimes_\Bbbk \bar{\Bbbk}$ the algebraic torus. The points of $\mathbb{T}$ are just the $\bar{\Bbbk}^\times$-valued characters of $\mathcal{U}^0$. The equation $\det_\alpha^\chi = 0$ defines a closed affine subvariety $\mathbb{T}_\alpha' \subset \mathbb{T}$. Let $\Lambda \in \mathbb{T}$. By definition, $\Lambda \in \mathbb{T}_\alpha'$ if and only if $\Lambda\mathrm{Sh} : U^-(\chi)_{-\alpha} \times U^-(\chi)_{-\alpha} \to \bar{\Bbbk}$ is a degenerate symmetric bilinear form, that is, if $I^\chi(\Lambda)_{-\alpha} \neq 0$. Thus, by Prop. 5.16, $\mathbb{T}_\alpha'$ is a subset of the finite union of irreducible varieties

$$\mathbb{T}_{\alpha,\nu,t} = \mathrm{maxspec}\,(\mathcal{U}^0 \otimes_\Bbbk \bar{\Bbbk})/(\rho^\chi(\beta_\nu)K_{\beta_\nu} - \chi(\beta_\nu,\beta_\nu)^t L_{\beta_\nu}),$$

where $\nu \in \{1,2,\dots,n\}$ and $t \in \{1,2,\dots,b^\chi(\beta_\nu)-1\}$. By Eq. (6.15)

$$\det_\alpha^\chi = f \prod_{\nu=1}^{n} \prod_{t=1}^{b^\chi(\beta_\nu)-1} (\rho^\chi(\beta_\nu)K_{\beta_\nu} - \chi(\beta_\nu,\beta_\nu)^t L_{\beta_\nu})^{N_{\nu,t}}$$

for some $N_{\nu,t} \in \mathbb{N}_0$ and an element $f \in \Bbbk[K_i, L_i \,|\, i \in I]$ which is invertible on $\mathbb{T}$. In particular, $\det_\alpha^\chi \neq 0$. We finish the proof of the theorem by showing that $N_{\nu,t} \geq P^\chi(\alpha,\beta_\nu;t)$ for all $\nu \in \{1,2,\dots,n\}$ and $t \in \{1,2,\dots,b^\chi(\beta_\nu)-1\}$. The essential ingredients will be Lemmas 6.7 and 9.2.

Let $\nu \in \{1,2,\dots,n\}$ and $t \in \{1,2,\dots,b^\chi(\beta_\nu)-1\}$. Let $w = 1^\chi\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{\nu-1}}$. Then

$$\mathcal{U}^0 = \Bbbk[K_{w(\alpha_j)}, K_{w(\alpha_j)}^{-1}, L_{w(\alpha_j)}, L_{w(\alpha_j)}^{-1} \,|\, j \in I]$$

and $w(\alpha_{i_\nu}) = \beta_\nu$. Let

$$B = \Bbbk[L_{\beta_\nu}, L_{\beta_\nu}^{-1}, K_{w(\alpha_j)}, K_{w(\alpha_j)}^{-1}, L_{w(\alpha_j)}, L_{w(\alpha_j)}^{-1} \,|\, j \in I \setminus \{i_\nu\}]$$

and $x = \rho^\chi(\beta_\nu)K_{\beta_\nu} - \chi(\beta_\nu,\beta_\nu)^t L_{\beta_\nu}$. Then

$$\mathcal{U}^0 \simeq B[x, (x + \chi(\beta_\nu,\beta_\nu)^t L_{\beta_\nu})^{-1}].$$

Let $X' = (x_{ij}')_{i,j\in\{1,2,\dots,k\}} \in (\mathcal{U}^0)^{k\times k}$ with

$$x_{ij}' = \mathrm{Sh}(F_i', F_j') \quad \text{for all } i,j \in \{1,2,\dots,k\}.$$

Let $l \in \mathbb{Z}$ such that $K_{\beta_\nu}^l X' \in B[x]^{k\times k}$, and let $X = K_{\beta_\nu}^l X'$. By Lemma 6.7 and Eq. (6.8) there is a non-empty open subset of the variety of $B \simeq \mathcal{U}^0/(x)$ such that $\operatorname{rk} X(0)_p \le k - P^\chi(\alpha,\beta_\nu;t)$ for all p in this set. By Lemma 9.2, $\det X = x^{P^\chi(\alpha,\beta_\nu;t)} b'$ for some $b' \in B[x]$. In particular, $x^{P^\chi(\alpha,\beta_\nu;t)}$ is a factor of $\det^\chi_\alpha$, and the proof of the theorem is complete. □

7. Shapovalov determinants for bicharacters with finite root systems

In Sect. 6 we mainly considered bicharacters $\chi \in \mathcal{X}_5$. Here we extend our results to all $\chi \in \mathcal{X}_3$ with $\chi(\beta,\beta) \ne 1$ for all $\beta \in R^\chi_+$.

In what follows let $\overline{\mathcal{X}}$ denote the set of $\bar{\Bbbk}^\times$-valued bicharacters on $\mathbb{Z}^I$. Identify $\overline{\mathcal{X}}$ with $(\bar{\Bbbk}^\times)^{I\times I}$ via $\chi \mapsto (\chi(\alpha_i,\alpha_j))_{i,j\in I}$ for all $\chi \in \overline{\mathcal{X}}$. For all $i \in \{1,2,3,4,5\}$ define $\overline{\mathcal{X}}_i \subset \overline{\mathcal{X}}$ in analogy to Eqs. (2.20)–(2.24). Note that $\mathcal{X}_i = \mathcal{X} \cap \overline{\mathcal{X}}_i$ for all $i \in \{1,2,3,4,5\}$.

For all $\beta,\beta' \in \mathbb{Z}^I$ let $f_{\beta,\beta'}$ be the rational function on the affine variety $\overline{\mathcal{X}} = (\bar{\Bbbk}^\times)^{I\times I}$ such that

$$f_{\beta,\beta'}(\chi) = \chi(\beta,\beta') \quad \text{for all } \chi \in \overline{\mathcal{X}}.$$

Clearly, the functions $f_{\beta,\beta'}$ with $\beta,\beta' \in \{\alpha_i, -\alpha_i \,|\, i \in I\}$ generate the algebra $\bar{\Bbbk}[\overline{\mathcal{X}}]$. Recall that a subset of $\overline{\mathcal{X}}$ is locally closed, if it is the intersection of an open and a closed subset of $\overline{\mathcal{X}}$.

Proposition 7.1. *Let $\chi \in \overline{\mathcal{X}}_3$. Assume that $\chi(\beta,\beta) \ne 1$ for all $\beta \in R^\chi_+$. Let $\underline{n} = (n_\beta)_{\beta\in R^\chi_{+\infty}}$ with $n_\beta \in \mathbb{N}$ for all $\beta \in R^\chi_{+\infty}$. Then there exists an ideal $J \subsetneq \bar{\Bbbk}[\overline{\mathcal{X}}]$ generated by products of polynomials of the form*

$$q - \prod_{i,j\in I} f^{m_{ij}}_{\alpha_i,\alpha_j}, \quad q \text{ is a root of } 1,\ m_{ij} \in \mathbb{Z} \text{ for all } i,j \in I,$$

such that the set

$$V^\chi_{\underline{n}} = \{\chi' \in \overline{\mathcal{X}} \,|\, R^{\chi'}_+ = R^\chi_+,\ b^{\chi'}(\beta) = b^\chi(\beta) \text{ for all } \beta \in R^\chi_{+\mathrm{fin}},\\ \chi'(\beta,\beta)^n \ne 1 \text{ for all } \beta \in R^\chi_{+\infty},\ 1 \le n \le n_\beta\} \tag{7.1}$$

is an open subset of $\operatorname{maxspec} \bar{\Bbbk}[\overline{\mathcal{X}}]/J$.

Proof. We use Lemma 2.15 and Def. 2.11 to reformulate the equation $R^\chi_+ = R^{\chi'}_+$.

Let $\chi' \in \mathcal{G}(\chi)$. Since $\chi \in \overline{\mathcal{X}}_3$, χ' is p-finite for all $p \in I$. Further, $\chi'(\alpha_p,\alpha_p) \ne 1$ since $\chi(\beta,\beta) \ne 1$ for all $\beta \in R^\chi_+$, see Eq. (2.8). Thus

$$(\chi'(\alpha_p,\alpha_p)^{-c^{\chi'}_{pj}}\chi'(\alpha_p,\alpha_j)\chi'(\alpha_j,\alpha_p) - 1)(\chi'(\alpha_p,\alpha_p)^{1-c^{\chi'}_{pj}} - 1) = 0$$

for all $p,j \in I$ with $p \ne j$. Let $w \in \operatorname{Hom}(\chi,\chi') \subset \mathcal{W}(\chi)$. Identify w with the corresponding element in $\operatorname{Aut}(\mathbb{Z}^I)$ in the usual way. Then $\chi' = w^*\chi$ and hence

$$(\chi(\gamma_p,\gamma_p)^{-c^{\chi'}_{pj}}\chi(\gamma_p,\gamma_j)\chi(\gamma_j,\gamma_p) - 1)(\chi(\gamma_p,\gamma_p)^{1-c^{\chi'}_{pj}} - 1) = 0 \tag{7.2}$$

for all $p,j\in I$ with $p\neq j$, where $\gamma_p=w^{-1}(\alpha_p)$ and $\gamma_j=w^{-1}(\alpha_j)$. Let

$$\begin{aligned} J'=\big((f_{\gamma_p,\gamma_p}^{-c_{pj}^{w^*\chi}}f_{\gamma_p,\gamma_j}f_{\gamma_j,\gamma_p}-1)(f_{\gamma_p,\gamma_p}^{1-c_{pj}^{w^*\chi}}-1)\,|\\ j,p\in I, j\neq p,\, w\in \mathrm{Hom}(\chi,\_),\, \gamma_p=w^{-1}(\alpha_p),\, \gamma_j=w^{-1}(\alpha_j)\big)\end{aligned} \tag{7.3}$$

and

$$J=J'+\big(f_{\beta,\beta}^{b^\chi(\beta)}-1\,|\,\beta\in R^\chi_{+\mathrm{fin}}\big). \tag{7.4}$$

Then, by Lemma 2.15, Def. 2.11, and Eq. (2.12), $V^\chi_{\underline{n}}$ is the set of points $\chi''\in \mathrm{maxspec}\,\bar{\Bbbk}[\overline{\mathcal{X}}]/J$ such that

- $f^n_{\beta,\beta}(\chi'')\neq 1$ for all $\beta\in R^\chi_{+\infty}$, $1\le n\le n_\beta$ and
- $(f^m_{\gamma_p,\gamma_p}f_{\gamma_p,\gamma_j}f_{\gamma_j,\gamma_p}-1)(\chi'')\,(f^{m+1}_{\gamma_p,\gamma_p}-1)(\chi'')\neq 0$ for all $j,p\in I$, $w\in \mathrm{Hom}(\chi,\_)$, and $m\in\{0,1,\dots,-c_{pj}^{w^*\chi}-1\}$, where $j\neq p$ and $\gamma_p=w^{-1}(\alpha_p)$, $\gamma_j=w^{-1}(\alpha_j)$.

This is clearly an open subset, which proves the proposition. □

Proposition 7.2. *Let $\chi\in\overline{\mathcal{X}}_3$. Assume that $\chi(\beta,\beta)\neq 1$ for all $\beta\in R^\chi_+$. Let $\underline{n}=(n_\beta)_{\beta\in R^\chi_{+\infty}}$, where $n_\beta\in\mathbb{N}$ for all $\beta\in R^\chi_{+\infty}$. Let $V^\chi_{\underline{n}}$ be as in Prop. 7.1. Then $\overline{\mathcal{X}}_5\cap V^\chi_{\underline{n}}$ is Zariski dense in $V^\chi_{\underline{n}}$.*

Proof. Prop. 7.1 gives that $V^\chi_{\underline{n}}\subset\overline{\mathcal{X}}_3$ satisfies the conditions on V in Lemma 9.4, where $k=|I|^2$ and $\{x_i\,|\,i=1,2,\dots,k\}=\{f_{\alpha_i,\alpha_j}\,|\,i,j\in I\}$. Since $\overline{\mathcal{X}}_4$ contains all finite sets $V_{n_1,\dots,n_k}$ in Lemma 9.4, and $\overline{\mathcal{X}}_5\cap V^\chi_{\underline{n}}=\overline{\mathcal{X}}_4\cap V^\chi_{\underline{n}}$ by definition of $V^\chi_{\underline{n}}$, the proof is completed. □

Similarly to Eq. (6.9) define $P^\chi(\alpha,\beta_\nu;t)$ for all $\chi\in\overline{\mathcal{X}}_3$, $\alpha\in\mathbb{N}_0^I$, $\beta_\nu\in R^\chi_+$, and $t\in\mathbb{N}$ with $t<b^\chi(\beta_\nu)$ by

$$\begin{aligned} P^\chi(\alpha,\beta_\nu;t)=\Big|\Big\{(m_1,\dots,m_n)\in\mathbb{N}_0^n\,\Big|\,\sum_{\mu=1}^n m_\mu\beta_\mu=\alpha,\, m_\nu\ge t,\\ m_\mu<b^\chi(\beta_\mu)\quad\text{for all }\mu\in\{1,2,\dots,n\}\Big\}\Big|.\end{aligned} \tag{7.5}$$

Theorem 7.3. *Let $\chi\in\mathcal{X}_3$. Assume that $\chi(\beta,\beta)\neq 1$ for all $\beta\in R^\chi_+$. The Shapovalov determinant of $U(\chi)$ is the family $(\det^\chi_\alpha)_{\alpha\in\mathbb{N}_0^I}$, where*

$$\det{}^\chi_\alpha=\prod_{\beta\in R^\chi_+}\prod_{t=1}^{b^\chi(\beta)-1}(\rho^\chi(\beta)K_\beta-\chi(\beta,\beta)^tL_\beta)^{P^\chi(\alpha,\beta;t)}. \tag{7.6}$$

Proof. Let $\alpha\in\mathbb{N}_0^I$. Choose a basis $\{F'_1,\dots,F'_k\}$ of $U^-(\chi)_{-\alpha}$ consisting of monomials $F_{i_1}F_{i_2}\cdots F_{i_l}$, where $k,l\in\mathbb{N}_0$ and $i_1,\dots,i_l\in I$. Identify $\oplus_{\beta,\gamma\in\mathbb{N}_0^I,\,\beta+\gamma=\alpha}\bar{\Bbbk}K_\beta L_\gamma$ with $\bar{\Bbbk}^N$ for an appropriate $N\in\mathbb{N}$. By the commutation relations (3.1)–(3.5) and the definition of Sh, the map

$$d:\overline{\mathcal{X}}\to\bar{\Bbbk}^N,\quad \chi'\mapsto\det(\mathrm{Sh}(F'_i,F'_j))_{i,j\in\{1,2,\dots,k\}}$$

is a morphism of affine varieties. Further, $d(\chi) \neq 0$ by Lemma 6.4, the choice of $\{F'_1, \dots, F'_k\}$, and the nondegeneracy of the pairing η, see Prop. 3.5(iv). Recall the definition of $|\beta|$, $\beta \in \mathbb{Z}^I$, from Eq. (3.8). Restrict d to the set $V^\chi_{\underline{n}}$ defined in Prop. 7.1, with $n_\beta = |\alpha|/|\beta|$ for all $\beta \in R^\chi_{+\infty}$. The set

$$V' = \{\chi' \in V^\chi_{\underline{n}} \,|\, d(\chi') \neq 0\}$$

is open in $V^\chi_{\underline{n}}$ and contains χ. Thus by Prop. 7.2 the set

$$V'' = \{\chi' \in \overline{\mathcal{X}}_5 \cap V^\chi_{\underline{n}} \,|\, d(\chi') \neq 0\}$$

is Zariski dense in all irreducible components of $V^\chi_{\underline{n}}$ containing χ. The definition of $V^\chi_{\underline{n}}$ and the choice of $\underline{n}$ yield that $R^{\chi'}_+ = R^\chi_+$ and $b^{\chi'}(\beta) \leq b^\chi(\beta)$ for all $\chi' \in V^\chi_{\underline{n}}$ and $\beta \in R^\chi_+$. Thus $\dim U(\chi')_{-\alpha} \leq \dim U(\chi)_{-\alpha}$ for all $\chi' \in V^\chi_{\underline{n}}$ by Eqs. (2.15), (2.16). Hence $d(\chi')$ is a multiple of $\det^{\chi'}_\alpha$ for all $\chi' \in V^\chi_{\underline{n}}$. By Thm. 6.8,

$$d(\chi') = a(\chi') \prod_{\beta \in R^\chi_+} \prod_{t=1}^{b^\chi(\beta_\nu)-1} (\rho^\chi(\beta)K_\beta - \chi(\beta,\beta)^t L_\beta)^{P^\chi(\alpha,\beta;t)} \tag{7.7}$$

for all $\chi' \in V''$, where $a(\cdot)$ is some regular function on $\overline{\mathcal{X}}$ which does not vanish on V''. By the density of V'', Eq. (7.7) holds for all $\chi' \in V'$ in the irreducible components of $V^\chi_{\underline{n}}$ containing χ, and $a(\chi') \neq 0$ for all $\chi' \in V'$ by definition of V'. In particular, Eq. (7.7) holds for $\chi' = \chi$. Thus the theorem is proven. □

8. Quantized enveloping algebras

We adapt our main result to quantized enveloping algebras.

Let I be a finite set and let $C = (c_{ij})_{i,j \in I}$ be a symmetrizable Cartan matrix of finite type. Let $\mathfrak{g}$ be the associated semisimple Lie algebra and R_+ the set of positive roots. For all $i \in I$ let $d_i \in \mathbb{N}$ such that $d_i c_{ij} = d_j c_{ji}$ for all $i, j \in I$. Assume that the numbers d_i, where $i \in I$, are relatively prime. Identify $\mathbb{Z}^I$ with the root lattice by considering $\{\alpha_i \,|\, i \in I\}$ as the set of simple roots. Let $(\cdot,\cdot) : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ be the (positive definite) symmetric bilinear form defined by $(\alpha_i, \alpha_j) = d_i c_{ij}$. Let $\rho : \mathbb{Z}^I \to \mathbb{Z}$ be the linear form defined by $\rho(\alpha_i) = d_i$ for all $i \in I$.

Let $\Bbbk$ be a field, and let $q \in \Bbbk^\times$. Assume that $q^{2m} \neq 1$ for all $m \in \mathbb{N}$ with $m \leq \max\{d_i \,|\, i \in I\}$. The quantized enveloping algebra of $\mathfrak{g}$ is the associative algebra $U_q(\mathfrak{g})$ generated by the elements E_i, F_i, K_i, and K_i^{-1}, where $i \in I$, and defined by the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$
$$K_i E_j K_i^{-1} = q^{d_i c_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-d_i c_{ij}} F_j,$$
$$E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1}),$$
$$(\mathrm{ad} E_i)^{1-c_{ij}}(E_j) = 0, \quad (\mathrm{ad} F_i)^{1-c_{ij}}(F_j) = 0 \quad (i \neq j)$$

for all $i, j \in I$. Here ad denotes adjoint action:

$$(\mathrm{ad} E_i)(x) = E_i x - K_i x K_i^{-1} E_i, \quad (\mathrm{ad} F_i)(x) = x F_i - F_i K_i^{-1} x K_i$$

for all $i \in I$ and $x \in \langle E_j, F_j, K_j, K_j^{-1} \,|\, j \in I\rangle$. Traditionally, in the third line of the defining relations of $U_q(\mathfrak{g})$ one inserts a denominator $q^{d_i} - q^{-d_i}$ on the right hand side, but this denominator can be eliminated by rescaling *e. g.* the variables E_i, $i \in I$.

Assume first that q is not a root of 1. Then, by [Lus93, Ch. 1] and [AS02, Prop. 2.10], $U_q(\mathfrak{g}) \simeq U(\chi)/(K_iL_i - 1 \,|\, i \in I)$, where $\chi \in \mathcal{X}$ with $\chi(\alpha_i, \alpha_j) = q^{d_ic_{ij}}$ for all $i, j \in I$.

Assume now that q is a root of 1. Let again $\chi \in \mathcal{X}$ with $\chi(\alpha_i, \alpha_j) = q^{d_ic_{ij}}$ for all $i, j \in I$. Then

$$U(\chi)/(K_iL_i - 1, K_\beta^{b(\beta)} - 1 \,|\, i \in I, \beta \in R_+^\chi),$$

where $b(\beta)$ is the order of $q^{(\beta,\beta)}$ for all $\beta \in R_+$, is isomorphic to Lusztig's small quantum group $u_q(\mathfrak{g})$. This was observed *e. g.* in [AS02, Thm. 4.3] by referring to results of Lusztig, de Concini, Procesi, Rosso, and Müller.

Similarly to Eq. (6.3) and Def. 6.1 one defines the Shapovalov form and the Shapovalov determinant $(\det_\alpha)_{\alpha \in \mathbb{N}_0^I}$ of $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$, respectively. Alternatively, since K_iL_i for $i \in I$ and $K_\beta^{b(\beta)}$ for $\beta \in R_+^\chi$ (the latter only if q is a root of 1) are central elements in $U(\chi)$ for all $i \in I$, the Shapovalov form can also be obtained from the definition in Sect. 6 via Lemma 6.3.

Theorem 8.1. *Let I, C, $(d_i)_{i \in I}$, and $\mathfrak{g}$ as above. Let $q \in \Bbbk^\times$. Assume that $q^{2m} \neq 1$ for all $m \in \mathbb{N}$ with $m \leq \max\{d_i \,|\, i \in I\}$.*

(i) [CK90] If q is not a root of 1, then the Shapovalov determinant of $U_q(\mathfrak{g})$ is the family $(\det_\alpha)_{\alpha \in \mathbb{N}_0^I}$, where

$$\det_\alpha = \prod_{\beta \in R_+} \prod_{t=1}^{\infty} (q^{2\rho(\beta)} K_\beta - q^{t(\beta,\beta)} K_\beta^{-1})^{P(\alpha,\beta;t)}. \tag{8.1}$$

(ii) Assume that q is a root of 1. Then the Shapovalov determinant of $u_q(\mathfrak{g})$ is the family $(\det_\alpha)_{\alpha \in \mathbb{N}_0^I}$, where

$$\det_\alpha = \prod_{\beta \in R_+} \prod_{t=1}^{b(\beta)-1} (q^{2\rho(\beta)} K_\beta - q^{t(\beta,\beta)} K_\beta^{-1})^{P(\alpha,\beta;t)}. \tag{8.2}$$

Proof. Let $\chi \in \mathcal{X}$ with $\chi(\alpha_i, \alpha_j) = q^{d_ic_{ij}}$ for all $i, j \in I$. Choose the ideal J in Lemma 6.3 as explained above. Then one gets the Shapovalov determinants of $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$ from the one of $U(\chi)$ in Thm. 7.3. □

The second part of Thm. 8.1 was proved in [KL97] in the case when the order of q is prime and $\Bbbk$ is the cyclotomic field $\mathbb{Q}[q]$.

9. Appendix

For the proofs of Thms. 6.8 and 7.3 we need some commutative algebra which is considered here. Let $\bar{\Bbbk}$ be an algebraically closed field.

Lemma 9.1. *Let B be an integral domain, x an indeterminate, $k \in \mathbb{N}$, and $X \in B[x]^{k\times k}$. Then there exist $s \in \{0,1,\dots,k\}$, $D_1, D_2 \in B^{k\times k}$, $D_0 \in B[x]^{k\times k}$ and $b \in B \setminus \{0\}$ such that* $\det D_1, \det D_2 \neq 0$,

$$D_1 X D_2 = x D_0 + b \operatorname{diag}(\underbrace{1,\dots,1}_{s},0,\dots,0). \tag{9.1}$$

Proof. Let $\operatorname{Frac}(B)$ be the field of fractions of B. Then there exist $s \in \{0,1,\dots,k\}$ and $D_1', D_2' \in \operatorname{Frac}(B)^{k\times k}$ such that $\det D_1', \det D_2' \neq 0$ and

$$D_1' X(0) D_2' = \operatorname{diag}(\underbrace{1,\dots,1}_{s},0,\dots,0).$$

Let $b_1, b_2 \in B \setminus \{0\}$ such that $b_1 D_1', b_2 D_2' \in B[x]^{k\times k}$. Let $b = b_1 b_2$, $D_1 = b_1 D_1'$, and $D_2 = b_2 D_2'$. Then

$$D_1 X(0) D_2 = b \operatorname{diag}(\underbrace{1,\dots,1}_{s},0,\dots,0),$$

and hence the lemma holds for $D_0 = D_1 X' D_2$, where $X' \in B[x]^{k\times k}$ such that $X = X(0) + xX'$. □

Lemma 9.2. *Let B be a finitely generated integral domain over $\bar{\Bbbk}$, x an indeterminate, $k \in \mathbb{N}$, $r \in \{0,1,\dots,k\}$, and $X \in B[x]^{k\times k}$. Assume that* $\operatorname{rk} X(0)_p \leq r$ *for all points p in a non-empty Zariski open subset of the affine variety of B. Then* $\det X = x^{k-r} b$ *for some $b \in B[x]$.*

Proof. By Lemma 9.1 there exist $s \in \{0,1,\dots,k\}$, $b \in B \setminus \{0\}$, $D_1, D_2 \in B^{k\times k}$, and $D_0 \in B[x]^{k\times k}$ such that $\det D_1, \det D_2 \neq 0$ and Eq. (9.1) holds. Let V be a non-empty Zariski open subset of the affine variety of $B \simeq B[x]/(x)$ such that $(\det D_1)_p, (\det D_2)_p, b_p \neq 0$ and $\operatorname{rk} X(0)_p \leq r$ for all $p \in V$. This exists by the assumption on r and since the variety of B is irreducible. Then $s \leq r$ by Eq. (9.1) in the points $(p,0)$ of the variety of $B[x]$, where $p \in V$. Therefore

$$\det D_1 \, \det X \det D_2 = x^{k-r} b'$$

for some $b' \in B[x]$. Since $\det D_1, \det D_2 \in B$ and B is an integral domain, we conclude that $\det X \in x^{k-r} B[x]$. □

For all $k \in \mathbb{N}$ let $\bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \leq i \leq k]$ denote the ring of Laurent polynomials in k variables. For all $M = (m_{ij})_{i,j\in\{1,2,\dots,k\}} \in \mathrm{GL}(k,\mathbb{Z})$ let

$$X_i^{(M)} = \prod_{j=1}^{k} x_j^{m_{ij}}, \quad 1 \leq i \leq k.$$

Then the ring endomorphism of $\bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \leq i \leq k]$ given by $x_i \mapsto X_i^{(M)}$ for all $i \in \{1,2,\dots,k\}$ is an isomorphism with inverse map given by $x_i \mapsto X_i^{(M^{-1})}$ for all $i \in \{1,2,\dots,k\}$.

Lemma 9.3. *Let $k \in \mathbb{N}$. Let $J \subsetneq \bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \le i \le k]$ be an ideal generated by elements of the form $q - \prod_{i=1}^k x_i^{m_i}$, where $m_1, \dots, m_k \in \mathbb{Z}$ and $q \in \bar{\Bbbk}^\times$ is a root of 1. Then J is a finite intersection of ideals of the form*

$$(X_1^{(M)} - q_1, X_2^{(M)} - q_2, \dots, X_l^{(M)} - q_l), \tag{9.2}$$

where $l \in \{0, 1, \dots, k\}$, $q_1, \dots, q_l \in \bar{\Bbbk}^\times$ are roots of 1, and $M \in \mathrm{GL}(k, \mathbb{Z})$.

Proof. Proceed by induction on k. If J is empty, then the claim is true. Assume now that $q - \prod_{i=1}^k x_i^{m_i}$ is one of the generators of J, where q is a root of 1 and $(m_1, \dots, m_k) \in \mathbb{Z}^k \setminus \{0\}$. Let $m_0 = \gcd(m_1, \dots, m_k)$. Let $M' \in \mathrm{GL}(k, \mathbb{Z})$ such that $m'_{1i} = m_i/m_0$ for all $i \in \{1, 2, \dots, k\}$. Then $(X_1^{(M')})^{m_0} - q \in J$, and hence J is the intersection of the (finite number of) ideals $J + (X_1^{(M')} - q')$, where $q' \in \bar{\Bbbk}$, $q'^{m_0} = q$. By assumption, $J + (X_1^{(M')} - q')$ is generated by $X_1^{(M')} - q'$ and by elements of the form $q'' - \prod_{i=2}^k (X_i^{(M')})^{m'_i}$, where $m'_2, \dots, m'_k \in \mathbb{Z}$ and $q'' \in \bar{\Bbbk}^\times$ is a root of 1. Then the claim follows by the induction hypothesis. □

The ideals in Eq. (9.2) are prime ideals of $\bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \le i \le k]$, since the quotient ring $\bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \le i \le k]/J \simeq \bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \le i \le k - l]$ is an integral domain.

Lemma 9.4. *Let $k \in \mathbb{N}$. Let $J \subsetneq \bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \le i \le k]$ be an ideal generated by polynomials of the form*

$$\big(q_1 - \prod_{i=1}^k x_i^{m_{1i}}\big)\big(q_2 - \prod_{i=1}^k x_i^{m_{2i}}\big) \cdots \big(q_l - \prod_{i=1}^k x_i^{m_{li}}\big),$$

where $l \in \mathbb{N}$, $m_{j1}, \dots, m_{jk} \in \mathbb{Z}$ and $q_j \in \bar{\Bbbk}^\times$ is a root of 1 for all $j \in \{1, 2, \dots, l\}$. Let $V \subset \operatorname{maxspec} \bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \le i \le k]/J$ be an open subset. Then the union of the subsets

$$V_{n_1,\dots,n_k} = \{p \in V \,|\, p_1^{n_1} = 1, \dots, p_k^{n_k} = 1\}, \quad n_1, \dots, n_k \in \mathbb{N},$$

is dense in V with respect to the Zariski topology.

Proof. We can assume that $V = \operatorname{maxspec} \bar{\Bbbk}[x_i, x_i^{-1} \,|\, 1 \le i \le k]/J$. Moreover, it suffices to prove the lemma for the irreducible components of V. Thus, as a first reduction, J can be assumed to be as in the assumptions of Lemma 9.3. Then by Lemma 9.3 we may assume that J is an ideal as in Eq. (9.2), where $M \in \mathrm{GL}(k, \mathbb{Z})$, $l \in \{0, 1, \dots, k\}$, and $q_1, \dots, q_l \in \bar{\Bbbk}^\times$ are roots of 1. Then $V \simeq (\bar{\Bbbk}^\times)^{k-l}$ for some $l \in \{0, 1, \dots, k\}$, and the lemma follows from the fact that infinite subsets are dense in $\bar{\Bbbk}^\times$. □

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