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PROPERTIES OF THE BIVARIATE CONFLUENT HYPERGEOMETRIC FUNCTION KIND 1 DISTRIBUTION

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Abstract. The bivariate confluent hypergeometric function kind 1 distribution is defined by the probability density function proportional to $x_1^{\nu_1-1}x_2^{\nu_2-1}{}_1F_1(\alpha;\beta;-x_1-x_2)$. In this article, we study several properties of this distribution and derive density functions of X_1/X_2, $X_1/(X_1+X_2)$, X_1+X_2 and $2\sqrt{X_1X_2}$. The density function of $2\sqrt{X_1X_2}$ is represented in terms of modified Bessel function of the second kind. We also show that for $\nu_1-\nu_2=1/2$, $2\sqrt{X_1X_2}$ follows a confluent hypergeometric function kind 1 distribution.

1. Introduction

The random variable X is said to have a confluent hypergeometric function kind 1 distribution, denoted by $X\sim \mathrm{CH}(\nu,\alpha,\beta,\text{kind } 1)$, if its probability density function (p.d.f.) is given by (Gupta and Nagar [4]),

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu)}{\Gamma(\nu)\Gamma(\beta)\Gamma(\alpha-\nu)}x^{\nu-1}{}_1F_1(\alpha;\beta;-x),\quad x>0, \tag{1.1}$$

where $\beta>\nu>0$, $\alpha>\nu>0$, and ${}_1F_1$ is the confluent hypergeometric function (Luke [8]) defined by

$${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)\,\mathrm{d}t, \quad \mathrm{Re}(c)>\mathrm{Re}(a)>0. \tag{1.2}$$

By expanding $\exp(zt)$ in (1.2) and integrating t, the series expansion for ${}_1F_1$ is obtained as

$${}_1F_1(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k!}. \tag{1.3}$$

The confluent hypergeometric function ${}_1F_1(a;c;z)$ satisfy the Kummer's relation

$${}_1F_1(a;c;-z)=\exp(-z){}_1F_1(c-a;c;z). \tag{1.4}$$

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The confluent hypergeometric function kind 1 distribution occurs as the distribution of the ratio of independent gamma and beta variables (Gupta and Nagar [4], Nadarajah and Kotz [14]). For $\alpha = \beta$, the density (1.1) reduces to a gamma density given by

$$\{\Gamma(\nu)\}^{-1} x^{\nu-1} \exp(-x), \quad x > 0.$$

The above distribution is designated by $X \sim \mathrm{Ga}(\nu)$. The bivariate generalization of the confluent hypergeometric function kind 1 distribution, denoted by $(X_1, X_2) \sim \mathrm{CH}(\nu_1, \nu_2, \alpha, \beta, \mathrm{kind}\ 1)$, is defined by the density

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} \times x_1^{\nu_1-1} x_2^{\nu_2-1} {}_1F_1(\alpha;\beta;-x_1-x_2), \quad x_1 > 0, \quad x_2 > 0, \tag{1.5}$$

where $\nu_1 > 0$, $\nu_2 > 0$, $\beta > \nu_1 + \nu_2$ and $\alpha > \nu_1 + \nu_2$. For $\alpha = \beta$, the random variables X_1 and X_2 are independent, $X_1 \sim \mathrm{Ga}(\nu_1)$ and $X_2 \sim \mathrm{Ga}(\nu_2)$.

Since, the bivariate distribution defined by the density (1.5) is a generalization of the bivariate gamma distribution, it can serve an alternative to bivariate gamma distribution and can be applied in several areas; for example, in the modeling of rainfall at two nearby rain gauges, data obtained from rainmaking experiments, the dependence between annual stream flow and areal precipitation, wind gust data and the dependence between rainfall and runoff (Nadarajah [12], Nadarajah and Gupta [13]). The bivariate generalization of the confluent hypergeometric function kind 1 distribution can also be used in reliability theory, renewal processes and stochastic routing problems.

It can easily be observed that the bivariate confluent hypergeometric function kind 1 distribution belongs to the Liouville family of distributions proposed by Marshall and Olkin [10]. Sivazlian [19] introduced Liouville distributions as generalizations of gamma and Dirichlet distributions. The Dirichlet and Liouville distributions arise in a variety of context including Bayesian analysis, modeling of multivariate data, order statistics, limit laws, multivariate analysis, reliability theory and stochastic processes. These distributions have been widely used in geology, biology, chemistry, forensic science, and statistical genetics. A comprehensive account of some applications and other aspects of these distributions can be found in Gupta and Song [5], Gupta and Richards [6], Marshall and Olkin [10], Sivazlian [19], and Song and Gupta [20]. Because of mathematical tractability of the confluent hypergeometric function and its several special cases, the bivariate confluent hypergeometric function kind 1 distribution enriches the class of Liouville distributions and may serve as an alternative to many existing distributions belonging to this class.

In this article, we study several properties of the bivariate distribution defined by (1.5).

In Section 2, we show that if $(X_1, X_2) \sim \mathrm{CH}(\nu_1, \nu_2, \alpha, \beta, \mathrm{kind}\ 1)$, then $X_1 \sim \mathrm{CH}(\nu_1, \alpha - \nu_2, \beta - \nu_2, \mathrm{kind}\ 1)$, $X_2 \sim \mathrm{CH}(\nu_2, \alpha - \nu_1, \beta - \nu_1, \mathrm{kind}\ 1)$ and compute correlation coefficient between X_1 and X_2. We also derive bivariate confluent hypergeometric function kind 1 distribution using independent gamma and beta variables. In Section 3, we derive distributions of (i) X_1/X_2, (ii)$X_1/(X_1+X_2)$, (iii) X_1+X_2 and (iv) $2\sqrt{X_1X_2}$ when $(X_1,X_2)\sim \text{CH}(\nu_1,\nu_2,\alpha,\beta,\text{kind }1)$. The density function of $2\sqrt{X_1X_2}$ is represented in terms of modified Besel function of the second kind. We also show that $2\sqrt{X_1X_2}$ for $\nu_2=\nu_1+1/2$ follows a confluent hypergeometric function kind 1 distribution.

2. Properties

In this section we study several properties of the bivariate confluent hypergeometric function kind 1 distribution defined in Section 1. We first derive marginal and conditional distributions.

Theorem 2.1. *Let* $(X_1,X_2)\sim \text{CH}(\nu_1,\nu_2,\alpha,\beta,\text{kind }1)$*. Then,* $X_1\sim \text{CH}(\nu_1,\alpha-\nu_2,\beta-\nu_2,\text{kind }1)$ *and* $X_2\sim \text{CH}(\nu_2,\alpha-\nu_1,\beta-\nu_1,\text{kind }1)$.

Proof. To find the marginal p.d.f. of X_1, we integrate (1.5) with respect to x_2 to get

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)}x_1^{\nu_1-1}\int_0^\infty x_2^{\nu_2-1}{}_1F_1(\alpha;\beta;-x_1-x_2)\,\mathrm{d}x_2.$$

Replacing ${}_1F_1\,(\alpha;\beta;-x_1-x_2)$ by its equivalent integral represenation, namely,

$$\begin{aligned}{}_1F_1\,(\alpha;\beta;-x_1-x_2)&\\ =\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}&\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left[-(x_1+x_2)t\right]\,\mathrm{d}t,\\ &\text{Re}(\beta)>\text{Re}(\alpha)>0,\end{aligned}\tag{2.1}$$

and integrating x_2, the density of X_1 is derived as

$$\begin{aligned}&\frac{\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\beta-\alpha)\Gamma(\alpha-\nu_1-\nu_2)}\\ &\qquad\times x_1^{\nu_1-1}\int_0^1 t^{\alpha-\nu_2-1}(1-t)^{\beta-\alpha-1}\exp(-x_2t)\,\mathrm{d}t.\end{aligned}$$

Now, the desired result is obtained by using (1.2). □

Using the above theorem, the conditional density function of X_1 given $X_2=x_2>0$ is obtained as

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu_1)}{\Gamma(\nu_1)\Gamma(\beta)\Gamma(\alpha-\nu_1)}\frac{x_1^{\nu_1-1}{}_1F_1(\alpha;\beta;-x_1-x_2)}{{}_1F_1(\alpha-\nu_1;\beta-\nu_1;-x_2)},\quad x_1>0.$$

The cumulative distribution function (c.d.f.) of (X_1, X_2) is derived as

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) &= \frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} \\ &\quad\times \int_0^{x_1}\int_0^{x_2} u_1^{\nu_1-1}u_2^{\nu_2-1}\,{}_1F_1(\alpha;\beta;-u_1-u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2. \\ &= \frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} x_1^{\nu_1}x_2^{\nu_2} \\ &\quad\times \int_0^{1}\int_0^{1} z_1^{\nu_1-1}z_2^{\nu_2-1}\,{}_1F_1(\alpha;\beta;-z_1x_1-z_2x_2)\,\mathrm{d}z_1\,\mathrm{d}z_2, \end{aligned} \tag{2.2}$$

where the last line has been obtained by substituting $z_i = u_i/x_i$ with $\mathrm{d}u_i = x_i\,\mathrm{d}z_i$, $i = 1, 2$. Replacing ${}_1F_1(\alpha;\beta;-z_1x_1-z_2x_2)$ by its integral representation, namely

$$\begin{aligned} &{}_1F_1(\alpha;\beta;-z_1x_1-z_2x_2) \\ &\quad= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp[-(x_1z_1+x_2z_2)t]\,\mathrm{d}t, \\ &\qquad\qquad \mathrm{Re}(\beta) > \mathrm{Re}(\alpha) > 0, \end{aligned}$$

and integrating out z_1 and z_2, the c.d.f. in (2.2) is rewritten as

$$\begin{aligned} &F_{X_1,X_2}(x_1,x_2) \\ &= \frac{\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)\Gamma(\beta-\alpha)\Gamma(\alpha-\nu_1-\nu_2)} x_1^{\nu_1}x_2^{\nu_2} \\ &\quad\times \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\,{}_1F_1(\nu_1;\nu_1+1;-x_1t)\,{}_1F_1(\nu_2;\nu_2+1;-x_2t)\,\mathrm{d}t \\ &= \frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} x_1^{\nu_1}x_2^{\nu_2} \\ &\quad\times F_{1:1;1}^{1:1;1}\left[\begin{array}{ll} \alpha:\nu_1;\nu_2; & \\ \beta:\nu_1+1;\nu_2+1; & -x_1,-x_2 \end{array}\right], \end{aligned}$$

where $F_{1:1;1}^{1:1;1}$ is the Kampé de Fériet function defined by (Sánchez, Nagar and Gupta [18], Srivastava and Karlsson [21]),

$$\begin{aligned} &F_{1:1;1}^{1:1;1}\left[\begin{array}{ll} a:b_1;b_2; & \\ c:d_1;d_2; & z_1,z_2 \end{array}\right] \\ &\quad= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\,{}_1F_1(b_1;d_1;tz_1)\,{}_1F_1(b_2;d_2;tz_2)\,\mathrm{d}t \\ &\quad= \sum_{j_1,j_2=0}^{\infty} \frac{(a)_{j_1+j_2}(b_1)_{j_1}(b_2)_{j_2}z_1^{j_1}z_2^{j_2}}{(c)_{j_1+j_2}(d_1)_{j_1}(d_2)_{j_2}\,j_1!j_2!}. \end{aligned}$$

Further, using (1.5), the joint (r,s)-th moment is obtained as

$$\begin{aligned} \mathrm{E}(X_1^r X_2^s) &= \frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} \\ &\quad \times \int_0^\infty \int_0^\infty x_1^{\nu_1+r-1} x_2^{\nu_2+s-1} {}_1F_1(\alpha;\beta;-x_1-x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \frac{\Gamma(\nu_1+r)\Gamma(\nu_2+s)\Gamma(\beta-\nu_1-\nu_2)\Gamma(\alpha-\nu_1-\nu_2-r-s)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta-\nu_1-\nu_2-r-s)\Gamma(\alpha-\nu_1-\nu_2)}, \end{aligned}$$

where $\nu_1+r>0$, $\nu_2+s>0$, $\beta>\nu_1+\nu_2+r+s>0$, and $\alpha>\nu_1+\nu_2+r+s>0$. Now, substituting appropriately, we obtain

$$\mathrm{E}(X_i) = \frac{\nu_i(\beta-\nu_1-\nu_2-1)}{\alpha-\nu_1-\nu_2-1},$$

$$\mathrm{E}(X_i^2) = \frac{\nu_i(\nu_i+1)(\beta-\nu_1-\nu_2-1)(\beta-\nu_1-\nu_2-2)}{(\alpha-\nu_1-\nu_2-1)(\alpha-\nu_1-\nu_2-2)},$$

$$\mathrm{E}(X_1X_2) = \frac{\nu_1\nu_2(\beta-\nu_1-\nu_2-1)(\beta-\nu_1-\nu_2-2)}{(\alpha-\nu_1-\nu_2-1)(\alpha-\nu_1-\nu_2-2)},$$

$$\begin{aligned} \mathrm{Var}(X_i) &= \frac{\nu_i(\beta-\nu_1-\nu_2-1)}{(\alpha-\nu_1-\nu_2-1)^2(\alpha-\nu_1-\nu_2-2)} \\ &\quad \times[\nu_i(\beta-\alpha)+(\beta-\nu_1-\nu_2-2)(\alpha-\nu_1-\nu_2-1)], \end{aligned}$$

$$\mathrm{Cov}(X_1,X_2) = \frac{\nu_1\nu_2(\beta-\nu_1-\nu_2-1)(\beta-\alpha)}{(\alpha-\nu_1-\nu_2-1)^2(\alpha-\nu_1-\nu_2-2)},$$

$$\begin{aligned} \mathrm{Corr}(X_1,X_2) &= \left[\left(1+\frac{(\beta-\nu_1-\nu_2-2)(\alpha-\nu_1-\nu_2-1)}{\nu_1(\beta-\alpha)}\right)\right. \\ &\quad \left.\times\left(1+\frac{(\beta-\nu_1-\nu_2-2)(\alpha-\nu_1-\nu_2-1)}{\nu_2(\beta-\alpha)}\right)\right]^{-1/2}. \end{aligned}$$

Bivariate gamma distributions arise as tractable "lifetime" models in many areas, including life testing and telecommunications. In the context of reliability, the stress-strength model describes the life of a component which has a random strength X_2 and is subjected to a random stress X_1. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X_2 > X_1$. Thus, $R = Pr(X_1 < X_2)$ is a measure of the component reliability. In a recent paper, Nadrajah [12] has given an extensive survey on applications and computation of R when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions. In the same paper he has computed R when X_1 and X_2 are dependent random variables from six flexible families of bivariate gamma distributions. If (X_1, X_2) has a bivariate confluent hypergeometric function kind 1 distribution, then

$$\begin{aligned} R &= \frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} \\ &\qquad \times \int_0^\infty x_1^{\nu_1-1} \int_{x_1}^\infty x_2^{\nu_2-1} {}_1F_1(\alpha;\beta;-x_1-x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1. \end{aligned}$$

Replacing ${}_1F_1\,(\alpha;\beta;-x_1-x_2)$ by its integral representation given in (2.1), integrating x_2 and x_1 using

$$\Gamma(a,x) = \int_x^\infty \exp(-t)t^{a-1}\,\mathrm{d}t$$

and (Gradshteyn and Ryzhik [3, Eq. 6.455.1])

$$\begin{aligned} \int_0^\infty x^{\mu-1}\exp(-\beta x)\Gamma(\nu,\alpha x)\,\mathrm{d}x &= \frac{\alpha^\nu\Gamma(\mu+\nu)}{\mu(\alpha+\beta)^{\mu+\nu}} {}_2F_1\left(1,\mu+\nu;\mu+1;\frac{\beta}{\alpha+\beta}\right), \\ &\qquad \mathrm{Re}(\alpha+\beta)>0,\ \mathrm{Re}(\mu)>0,\ \mathrm{Re}(\mu+\nu)>0, \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function, we obtain

$$\begin{aligned} R &= \frac{\Gamma(\beta-\nu_1-\nu_2)\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1+1)\Gamma(\nu_2)\Gamma(\alpha-\nu_1-\nu_2)\Gamma(\beta-\alpha)} \\ &\qquad \times \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-\alpha-1}}{(1+t)^{\nu_1+\nu_2}} {}_2F_1\left(1,\nu_1+\nu_2;\nu_1+1;\frac{t}{1+t}\right)\mathrm{d}t. \end{aligned}$$

Finally, expanding ${}_2F_1$ in series form and integrating t, we get

$$\begin{aligned} R &= \frac{\Gamma(\beta-\nu_1-\nu_2)}{2^{\nu_1+\nu_2}\Gamma(\nu_2)\Gamma(\alpha-\nu_1-\nu_2)} \\ &\qquad \times \sum_{i=0}^\infty \frac{\Gamma(\nu_1+\nu_2+i)\Gamma(\alpha+i)}{2^i\Gamma(\nu_1+1+i)\Gamma(\beta+i)} {}_2F_1\left(\beta-\alpha,\nu_1+\nu_2+i;\beta+i;\frac{1}{2}\right). \end{aligned}$$

In the next theorem we derive the bivariate confluent hypergeometric function kind 1 distribution using independent beta and gamma variables. First, we define beta type 1 and beta type 2 distributions. These definitions can be found in Johnson, Kotz and Balakrishnan [7].

Definition 2.2. The random variable X is said to have a beta type 1 distribution with parameters (a,b), $a>0$, $b>0$, denoted as $X \sim \mathrm{B1}(a,b)$, if its p.d.f. is given by

$$\{B(a,b)\}^{-1}x^{a-1}(1-x)^{b-1}, \quad 0<x<1,$$

where $B(a,b)$ is the beta function given by

$$B(a,b) = \Gamma(a)\Gamma(b)\{\Gamma(a+b)\}^{-1}.$$

Definition 2.3. The random variable X is said to have a beta type 2 distribution with parameters (a, b), denoted as $X \sim \mathrm{B2}(a, b)$, $a > 0$, $b > 0$, if its p.d.f. is given by

$$\{B(a,b)\}^{-1} x^{a-1}(1+x)^{-(a+b)}, \quad x > 0.$$

Several univariate generalizations of these distributions are given in Gordy [2], Ng and Kotz [17], Nagar and Zarrazola [15] and McDonald and Xu [11]. The matrix variate generalizations of beta type 1 and beta type 2 distributions have also been defined and studied extensively. For example, see Gupta and Nagar [4].

Theorem 2.4. *Let X_1, X_2 and X_3 be independent, $X_i \sim \mathrm{Ga}(\kappa_i)$, $i = 1, 2$ and $X_3 \sim \mathrm{B1}(a, b)$. Then, $(X_1/X_3, X_2/X_3) \sim \mathrm{CH}(\kappa_1, \kappa_2, a + \kappa_1 + \kappa_2, a + b + \kappa_1 + \kappa_2, \text{kind } 1)$.*

Proof. Using independence, the joint density of X_1, X_2 and X_3 is given

$$\frac{\exp[-(x_1+x_2)] x_1^{\kappa_1-1} x_2^{\kappa_2-1} x_3^{a-1}(1-x_3)^{b-1}}{\Gamma(\kappa_1)\Gamma(\kappa_2)B(a,b)}. \tag{2.3}$$

Now, transforming $Z_1 = X_1/X_3, Z_2 = X_2/X_3$ with the Jacobian $J(x_1, x_2 \to z_1, z_2) = x_3^2$ in (2.3), the joint density of Z_1, Z_2 and X_3 is obtained as

$$\frac{\exp[-(z_1+z_2)x_3] z_1^{\kappa_1-1} z_2^{\kappa_2-1} x_3^{\kappa_1+\kappa_2+a-1}(1-x_3)^{b-1}}{\Gamma(\kappa_1)\Gamma(\kappa_2)B(a,b)},$$

where $z_1 > 0$, $z_2 > 0$ and $x_3 > 0$. Now, the result follows by using (1.2). □

Theorem 2.5. *Let X_1, X_2 and X_3 be independent, $X_i \sim \mathrm{Ga}(\kappa_i)$, $i = 1, 2$ and $X_3 \sim \mathrm{B1}(a, b)$. Then, $(X_1/(1-X_3), X_2/(1-X_3)) \sim \mathrm{CH}(\kappa_1, \kappa_2, b+\kappa_1+\kappa_2, a+b+\kappa_1+\kappa_2, \text{kind } 1)$.*

Proof. Noting that $1-X_3 \sim \mathrm{B1}(b, a)$ and using Theorem 2.4, we get the result. □

Theorem 2.6. *Let X_1, X_2 and X_3 be independent, $X_i \sim \mathrm{Ga}(\kappa_i)$, $i = 1, 2$ and $X_3 \sim \mathrm{B2}(a, b)$. Then, $(X_1(1+X_3), X_2(1+X_3)) \sim \mathrm{CH}(\kappa_1, \kappa_2, b+\kappa_1+\kappa_2, a+b+\kappa_1+\kappa_2, \text{kind } 1)$.*

Proof. The desired result is obtained by observing that $1/(1+X_3) \sim \mathrm{B1}(b, a)$ and using Theorem 2.4. □

Theorem 2.7. *Let X_1, X_2 and X_3 be independent, $X_i \sim \mathrm{Ga}(\kappa_i)$, $i = 1, 2$ and $X_3 \sim \mathrm{B2}(a, b)$. Then, $(X_1(1+X_3)/X_3, X_2(1+X_3)/X_3) \sim \mathrm{CH}(\kappa_1, \kappa_2, a+\kappa_1+\kappa_2, a+b+\kappa_1+\kappa_2, \text{kind } 1)$.*

Proof. Noting that $X_3/(1+X_3) \sim \mathrm{B1}(a, b)$ and using Theorem 2.4, we get the result. □

3. Distributions of Sum and Quotients

In statistical distribution theory it is well known that if X_1 and X_2 are independent, $X_1 \sim \mathrm{Ga}(\nu_1)$ and $X_2 \sim \mathrm{Ga}(\nu_2)$, then (i) $X_1/X_2 \sim \mathrm{B2}(\nu_1, \nu_2)$, (ii) $X_1/(X_1+X_2) \sim \mathrm{B1}(\nu_1, \nu_2)$, (iii) $X_1 + X_2 \sim \mathrm{Ga}(\nu_1+\nu_2)$ and (iv) $2\sqrt{X_1 X_2} \sim \mathrm{Ga}(2\nu_1)$ if $\nu_2 = \nu_1 + 1/2$. Recently, Nagar and Sepúlveda-Murillo [16] have derived distributions of X_1/X_2, $X_1/(X_1+X_2)$ and X_1+X_2 when X_1 and X_2 are independent confluent hypergeometric function kind 1 and gamma variables, respectively. In this section we derive similar results when X_1 and X_2 have a bivariate generalization of the confluent hypergeometric function kind 1 distribution.

Theorem 3.1. *Let* $(X_1, X_2) \sim \mathrm{CH}(\nu_1, \nu_2, \alpha, \beta, \text{kind } 1)$. *Then,* $Z = X_1/(X_1 + X_2)$ *and* $S = X_1 + X_2$ *are independent,* $Z \sim \mathrm{B1}(\nu_1, \nu_2)$ *and* $S \sim \mathrm{CH}(\nu_1 + \nu_2, \alpha, \beta, \text{kind } 1)$.

Proof. Substituting $Z = X_1/(X_1 + X_2)$ and $S = X_1 + X_2$ with the Jacobian of transformation $J(x_1, x_2 \to z, s) = s$ in (1.5) we obtain the joint p.d.f. of Z and S as

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} z^{\nu_1-1}(1-z)^{\nu_2-1} s^{\nu_1+\nu_2-1} {}_1F_1(\alpha;\beta;-s),$$

where $0 < z < 1$ and $x_2 > 0$. Now, from the above factorization it is clear that Z and S are independent, $Z \sim \mathrm{B1}(\nu_1, \nu_2)$ and $S \sim \mathrm{CH}(\nu_1 + \nu_2, \alpha, \beta, \text{kind } 1)$. □

Corollary 3.1.1. *Let* $(X_1, X_2) \sim \mathrm{CH}(\nu_1, \nu_2, \alpha, \beta, \text{kind } 1)$. *Then,* $X_1/X_2 \sim \mathrm{B2}(\nu_1, \nu_2)$ *and is independent of* $X_1 + X_2$.

Corollary 3.1.2. *Let* X_1 *and* X_2 *be independent random variables,* $X_1 \sim \mathrm{Ga}(\nu_1)$ *and* $X_2 \sim \mathrm{Ga}(\nu_2)$. *Then,* $X_1 + X_2$ *is independent of* $X_1/(X_1 + X_2)$ *and* X_1/X_2. *Further,* $X_1 + X_2 \sim \mathrm{Ga}(\nu_1 + \nu_2)$, $X_1/(X_1 + X_2) \sim \mathrm{B1}(\nu_1, \nu_2)$ *and* $X_1/X_2 \sim \mathrm{B2}(\nu_1, \nu_2)$.

Theorem 3.2. *Let* $(X_1, X_2) \sim \mathrm{CH}(\nu_1, \nu_2, \alpha, \beta, \text{kind } 1)$. *Then, the p.d.f. of* $Y = 2\sqrt{X_1 X_2}$ *is given by*

$$\begin{aligned} &\frac{2\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\alpha-\nu_1-\nu_2)\Gamma(\beta-\alpha)} \left(\frac{y}{2}\right)^{\nu_1+\nu_2-1} \\ &\times \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1} K_{\nu_1-\nu_2}(yt)\,\mathrm{d}t, \quad y > 0. \end{aligned} \tag{3.1}$$

Proof. Transforming $Y = 2\sqrt{X_1 X_2}$, $X_1 = X_1$ with the Jacobian $J(x_1, x_2 \to x_1, y) = y/2x_1$ in (1.5), we obtain the joint p.d.f. of X_1 and Y as

$$\begin{aligned} &\frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{2^{2\nu_2-1}\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} \\ &\times y^{2\nu_2-1} x_1^{\nu_1-\nu_2-1} {}_1F_1\left(\alpha;\beta;-x_1-\frac{y^2}{4x_1}\right), \quad x_1 > 0, \quad y > 0, \end{aligned} \tag{3.2}$$

To find the marginal p.d.f. of Z, we integrate (3.2) with respect to x_1 to get

$$\begin{aligned} &\frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{2^{2\nu_2-1}\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} \\ &\times y^{2\nu_2-1} \int_0^\infty x_1^{\nu_1-\nu_2-1} {}_1F_1\left(\alpha;\beta;-x_1-\frac{y^2}{4x_1}\right) \mathrm{d}x_1. \end{aligned} \tag{3.3}$$

Replacing ${}_1F_1\left(\alpha;\beta;-x_1-y^2/4x_1\right)$ by its equivalent integral represenation, namely,

$$
\begin{aligned}
&{}_1F_1\left(\alpha;\beta;-x_1-\frac{y^2}{4x_1}\right) \qquad (3.4)\\
&=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left[-\Big(x_1+\frac{y^2}{4x_1}\Big)t\right]\,\mathrm{d}t,\\
&\qquad \operatorname{Re}(\beta)>\operatorname{Re}(\alpha)>0,
\end{aligned}
$$

and changing the order of integration, the above density is rewritten as

$$
\begin{aligned}
&\frac{\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\alpha-\nu_1-\nu_2)\Gamma(\beta-\alpha)}\left(\frac{y}{2}\right)^{2\nu_2-1}\\
&\times\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\int_0^\infty x_1^{\nu_1-\nu_2-1}\exp\left[-\Big(x_1+\frac{y^2}{4x_1}\Big)t\right]\,\mathrm{d}x_1\,\mathrm{d}t.
\end{aligned}
$$

Now, using the integral (Gradshteyn and Ryzhik [3, Eq. 3.471.9])

$$
\int_0^\infty \exp\left(-\alpha t-\frac{\beta}{t}\right)t^{\nu-1}\,\mathrm{d}t=2\left(\frac{\beta}{\alpha}\right)^{\nu/2}K_\nu(2\sqrt{\alpha\beta}),\ \operatorname{Re}(\alpha)>0,\ \operatorname{Re}(\beta)>0,
$$

where K_ν is the modified Bessel function of the second kind, we obtain the desired result. □

For a non-negative integer n, we use the result (Erdélyi, Magnus, Oberhettinger and Tricomi [1, Eq. 7.2.6.40]),

$$
K_{n+1/2}(z)=\sqrt{\frac{\pi}{2z}}\exp(-z)\sum_{m=0}^{n}(2z)^{-m}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)\,m!}
$$

in the above theorem to get the following corollary.

Corollary 3.2.1. *Let* $(X_1,X_2)\sim \mathrm{CH}(\nu,\nu+n+1/2,\alpha,\beta,\text{kind }1)$, *where* n *is a non-negative integer. Then, the p.d.f. of* $Y=2\sqrt{X_1X_2}$ *is given by*

$$
\begin{aligned}
&\frac{\sqrt{\pi}\,\Gamma(\beta-2\nu-n-1/2)}{\Gamma(\nu)\Gamma(\nu+n+1/2)\Gamma(\alpha-2\nu-n-1/2)}\left(\frac{y}{2}\right)^{2\nu+n-1}\\
&\times\sum_{m=0}^{n}(2y)^{-m}\frac{\Gamma(n+m+1)\Gamma(\alpha-m-1/2)}{\Gamma(n-m+1)\Gamma(\beta-m-1/2)\,m!}\\
&\times{}_1F_1\left(\alpha-m-\frac{1}{2};\beta-m-\frac{1}{2};-y\right),\quad y>0.
\end{aligned}
$$

Corollary 3.2.2. *Let* $(X_1,X_2)\sim\mathrm{CH}(\nu,\nu+1/2,\alpha,\beta,\text{kind }1)$. *Then,* $2\sqrt{X_1X_2}\sim \mathrm{CH}(2\nu,\alpha-1/2,\beta-1/2,\text{kind }1)$.

Corollary 3.2.3. *Let the random variables* X_1 *and* X_2 *be independent,* $X_1\sim \mathrm{Ga}(\nu)$ *and* $X_2\sim\mathrm{Ga}(\nu+n+1/2)$, *where* n *is a non-negative integer. Then, the p.d.f. of* $Y=2\sqrt{X_1X_2}$ *is given by*

$$
\frac{\sqrt{\pi}\,(y/2)^{2\nu+n-1}\exp(-y)}{\Gamma(\nu)\Gamma(\nu+n+1/2)}\sum_{m=0}^{n}(2y)^{-m}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)\,m!},\quad y>0.
$$

Corollary 3.2.4. *Let the random variables* X_1 *and* X_2 *be independent,* $X_1 \sim \mathrm{Ga}(\nu)$ *and* $X_2 \sim \mathrm{Ga}(\nu + 1/2)$*. Then,* $2\sqrt{X_1 X_2} \sim \mathrm{Ga}(2\nu)$*.*

The above corollary was derived by Malik [9].

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