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ERGODIC PROPERTIES OF LINEAR OPERATORS

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Abstract. Let T be a bounded linear operator on a Banach space X. We prove some properties of $X_1 = \{z \in X \ : \ \lim_n \sum_{k=1}^{n} \frac{T^k z}{k}$ exists$\}$ and we construct an operator T such that $\lim_n \|T^n/n\| = 0$, but $(I-T)X$ is not included in X_1.

1. Introduction

Let X be a Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators from X to itself. An operator $T \in \mathcal{L}(X)$ is called *uniformly ergodic* if the averages $A_n(T) = n^{-1} \sum_{k=1}^{n} T^k$ converge in the uniform operator topology.

M. Lin [3] showed that when $\lim_n \|T^n/n\| = 0$, T is uniformly ergodic if and only if $(I-T)X$ is closed.

In [2], S. Grabiner and J. Zemánek give the following generalization of Lin's theorem. Under the hypothesis of boundedness of $A_n(T)$ or convergence to zero of T^n/n in some operator topology, they prove that if $(I-T)^n X$ is closed for some $n \geq 2$ ($n \geq 1$ if T^n/n converges to zero in the uniform operator topology) or if $(I-T)X + \mathrm{Ker}(I-T)$ is closed for some $n \geq 1$, then X is the direct sum of the closed subspaces $(I-T)X$ and $\mathrm{Ker}(I-T)$. In this case the sequence $A_n(T)$ converges in some operator topology if and only if T^n/n converges to zero in the same operator topology.

In [1] we proved that if $\lim_n \|T^n/n\| = 0$, and $(I-T)X \subseteq X_1$ then T is uniformly ergodic if and only if X_1 is closed.

In this paper we prove the following result.

Theorem 1.1. *There exists an operator $T \in \mathcal{L}(X)$ with $\lim_n \|T^n/n\| = 0$, for which $(I-T)X$ is not included in X_1. Moreover, this operator is not uniformly ergodic.*

We remark that if T is Cesàro bounded (i.e. $\sup_n \|A_n(T)\| < \infty$) and if $\lim_n \|T^n/n\| = 0$, then $(I-T)X \subseteq X_1$. (See Proposition 2.2).

2. The Results

Proposition 2.1. *Let $T \in \mathcal{L}(X)$. Then*

$$X_1 \subseteq \left\{ x \in X : \lim_n \ A_n(T)x = 0 \right\} \subseteq cl(I-T)X$$

Proof. Let $x \in X_1$. For each positive integer n, put $S_n(x) = \sum_{k=1}^{n} \frac{T^k x}{k}$. Then the first inclusion follows from

$$A_n(T)x = S_n(x) - \frac{1}{n} \sum_{k=1}^{n-1} S_k(x).$$

Let $x \in X$. The fact that $x - A_n(T)x$ belongs to $(I - T)(X)$ for each positive integer n implies the well-known second inclusion. □

Proposition 2.2. *If T is Cesàro bounded and $\lim_n \|T^n(x)/n\| = 0$ for each $x \in X$, then $(I - T)X \subseteq X_1$.*

Proof. Let $z \in (I - T)X$. Then $z = (I - T)x$. We have

$$\sum_{k=1}^{n} \frac{T^k z}{k} = Tx - \frac{T^{n+1}x}{n} - \sum_{k=2}^{n} \frac{T^k x}{k(k-1)} \ . \tag{1}$$

Thus, it is enough to prove that $\sum_{k=2}^{n} \frac{T^k x}{k(k-1)}$ converges for each $x \in X$. Let $x \in X$. By writing $T^k x = kA_k(T)x - (k-1)A_{k-1}(T)x$ and making use of the partial summation formula of Abel, we obtain

$$\sum_{k=2}^{n} \frac{T^k x}{k(k-1)} = -\frac{Tx}{2} + \frac{A_n(T)x}{n-1} + \sum_{k=2}^{n-1} \frac{2A_k(T)x}{(k-1)(k+1)} \ .$$

Since T is Cesàro bounded, the proposition is proved.

□

Corollary. *Let $T \in \mathcal{L}(X)$ uniformly ergodic. Then X_1 is closed.*

Proof. It follows from Remark 2 of [1]. □

The following example provides a proof of Theorem 1.1.

The Example.

Let $(a_j)_{j \geq 1}$ be any sequence of positive real numbers such that:

(1) $\sum_{j=1}^{\infty} \frac{a_j}{j^2}$ diverges.
(2) $\lim_{j \to \infty} \frac{a_j}{j} = 0$.
(3) There exists $c > 0$ such that $a_{j+k} \leq c a_j a_k$, $j, k \in \mathbb{N}$.

We can take, for example, $a_j = \frac{j+1}{ln(j+1)}$.

Now, let $X = l^1(\mathbb{N})$ and let T be the unilateral weighted shift defined by

$$(Tx)_n = \begin{cases} 0, & \text{if } n = 1; \\ \frac{a_n}{a_{n-1}} x_{n-1}, & \text{if } n \geq 2. \end{cases}$$

By property 3, $T \in \mathcal{L}(X)$.

Lemma 2.3. *There are positive constants c_1 and c_2 such that*

$$c_1 a_{k+1} \leq \|T^k\| \leq c_2 a_k, \quad k \in \mathbb{N}.$$

Proof. Let $x = (x_n)_{n\geq 1} \in l^1(\mathbb{N})$. We have

$$(T^k x)_n = \begin{cases} 0, & \text{if } 1 \leq n \leq k; \\ \frac{a_n}{a_{n-k}} x_{n-k}, & \text{if } n > k. \end{cases}$$

It follows that

$$\left\|T^k\right\| \leq \sup_{j\geq 1} \frac{a_{k+j}}{a_j}.$$

Therefore, $\left\|T^k\right\| \leq ca_k$. We also see that $T^k e_1 = \frac{a_{k+1}}{a_1} e_{k+1}$, where $(e_k)_n = \delta_{k,n}$. Thus $\left\|T^k\right\| \geq \frac{a_{k+1}}{a_1}$.

This completes the proof of the lemma.

□

Corollary. *T satisfies $\lim_{k\to\infty} \frac{\left\|T^k\right\|}{k} = 0$. Moreover, we can take $(a_j)_{j\geq 1}$ such that $\lim_{k\to\infty} \frac{\left\|T^k\right\|}{k^w} = \infty$, for $0 \leq w < 1$.*

Next we prove that $(I - T)e_1$ is not in X_1. By formula (1) stated in the proof of Proposition 2.2, we see that $(I - T)e_1 \in X_1$ if and only if $\sum_{k=2}^{n} \frac{T^k e_1}{k(k-1)}$ converges.

Fix $j \in \mathbb{N}$, $j > 2$. For $n \geq j - 1$, we have $\left(\sum_{k=2}^{n} \frac{T^k e_1}{k(k-1)}\right)_j = \frac{a_j}{a_1(j-1)(j-2)}$.

Since the convergence in $l^1(\mathbb{N})$ of the sequence $\left\{\sum_{k=2}^{n} \frac{T^k e_1}{k(k-1)}\right\}_n$ implies the convergence of the series $\sum_{j=3}^{\infty} \frac{a_j}{(j-1)(j-2)}$, we conclude that $(I - T)e_1$ is not in X_1.

Remark. By Proposition 2.2, T cannot be Cesàro bounded. Therefore T is not uniformly ergodic.

References

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