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A NEW APPLICATION OF POWER INCREASING SEQUENCES

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Abstract. In the present paper, we have proved a general summability factor theorem by using a general summability method. This theorem also includes several new results.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We denote by $\mathcal{BV}_{\mathcal{O}}$ the expression $\mathcal{BV} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}}$ and $\mathcal{BV}$ are the set of all null sequences and the set of all sequences with bounded variation, respectively. A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \tag{1}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse is not true for $\beta > 0$. Moreover, for any positive β there exists a quasi β - power increasing sequence tending to infinity, but it is not almost increasing. In fact, if we take (γ_n) is an almost increasing, that is, if

$$Ac_n \leq \gamma_n \leq Bc_n, \tag{2}$$

holds for all n with an increasing sequence (c_n) , then for any $n \geq m \geq 1$

$$\gamma_m \leq Bc_m \leq Bc_n \leq \frac{B}{A}\gamma_n \tag{3}$$

also holds, whence (1) follows obviously for any $\beta \geq 0$ with $K = \frac{B}{A}$. Thus any almost increasing sequence is quasi β-power increasing for any $\beta \geq 0$. We can show that the converse is not true. For this, if we take $\gamma_n = n^{-\beta}$ for $\beta > 0$, then $\gamma_n \to 0$. Thus it is obviously not an almost increasing sequence (see [11] for extra details). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\sigma}$ and $t_n^{\alpha,\sigma}$ the n-th Cesàro means of order (α, σ), with $\alpha + \sigma > -1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [7])

$$u_n^{\alpha,\sigma} = \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^\sigma s_v \tag{4}$$

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$$t_n^{\alpha,\sigma} = \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\sigma} v a_v, \tag{5}$$

where

$$A_n^{\alpha+\sigma} = O(n^{\alpha+\sigma}), \quad \alpha+\sigma > -1, \quad A_0^{\alpha+\sigma} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\sigma} = 0 \quad \text{for} \quad n > 0. \tag{6}$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \alpha, \sigma \mid_k$, $k \geq 1$ and $\alpha + \sigma > -1$, if (see [6])

$$\sum_{n=1}^{\infty} \mid \varphi_n (u_n^{\alpha,\sigma} - u_{n-1}^{\alpha,\sigma}) \mid^k < \infty. \tag{7}$$

But, since $t_n^{\alpha,\sigma} = n(u_n^{\alpha,\sigma} - u_{n-1}^{\alpha,\sigma})$ (see [8]) condition (7) can also be written as

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n t_n^{\alpha,\sigma} \mid^k < \infty. \tag{8}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$, $\varphi - \mid C, \alpha, \sigma \mid_k$ summability is the same as $\mid C, \alpha, \sigma \mid_k$ (see [8]) summability. Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then $\varphi - \mid C, \alpha, \sigma \mid_k$ summability reduces to $\mid C, \alpha, \sigma; \delta \mid_k$ summability. If we take $\sigma = 0$, then we have $\varphi - \mid C, \alpha \mid_k$ summability (see [2]). If we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\sigma = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [9]). Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$ and $\sigma = 0$, then we obtain $\mid C, \alpha; \delta \mid_k$ (see [10]) summability. Quite recently, Bor and Seyhan [4] have proved the following theorem for $\varphi - \mid C, \alpha \mid_k$ summability.

Theorem A. Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$ and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{9}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{11}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{12}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and if the sequence (θ_n^α) defined by (see [12])

$$\theta_n^\alpha = \mid t_n^\alpha \mid, \quad \alpha = 1 \tag{13}$$

$$\theta_n^\alpha = \max_{1 \leq v \leq n} \mid t_v^\alpha \mid, \quad 0 < \alpha < 1 \tag{14}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k} (\mid \varphi_n \mid \theta_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{15}$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - \mid C, \alpha \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$ and $k\alpha + \epsilon > 1$.

2. A generalization

The aim of this paper is to generalize Theorem A for $\varphi- \mid C,\alpha,\sigma \mid_k$ summability.
Now, we shall prove the following more general theorem.

Theorem B. Let $(\lambda_n) \in \mathcal{BV_O}$ and (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$ and the sequences (λ_n) and (β_n) such that conditions (9)-(12) of Theorem A are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and the sequence $(\theta_n^{\alpha,\sigma})$ defined by

$$\theta_n^{\alpha,\sigma} = \mid t_n^{\alpha,\sigma} \mid, \quad \alpha = 1 \ , \ \sigma > -1 \tag{16}$$

$$\theta_n^{\alpha,\sigma} = \max_{1\leq v\leq n} \mid t_v^{\alpha,\sigma} \mid, \quad 0 < \alpha < 1 \ , \ \sigma > -1 \tag{17}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(\mid \varphi_n \mid \theta_n^{\alpha,\sigma})^k = O(X_m) \quad as \quad m \to \infty, \tag{18}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi- \mid C,\alpha,\sigma \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\sigma > -1$ and $(\alpha+\sigma)k + \epsilon > 1$.
We need the following lemmas for the proof of our theorem.

Lemma 1. ([5]). If $0 < \alpha \leq 1$, $\sigma > -1$ and $1 \leq v \leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\sigma} a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\sigma} a_p \mid . \tag{19}$$

Lemma 2. ([11]). Except for the condition $(\lambda_n) \in \mathcal{BV_O}$, under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of Theorem B, the following conditions hold:

$$nX_n\beta_n = O(1), \tag{20}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{21}$$

3. Proof of Theorem B

Let $(T_n^{\alpha,\sigma})$ be the n-th (C,α,σ) mean of the sequence $(na_n\lambda_n)$. Then, by means of (5) we have

$$T_n^{\alpha,\sigma} = \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\sigma} v a_v \lambda_v.$$

First applying Abel's transformation and then using Lemma 1, we have that

$$T_n^{\alpha,\sigma} = \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\sigma} p a_p + \frac{\lambda_n}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\sigma} v a_v,$$

$$
\begin{aligned}
| T_n^{\alpha,\sigma} | &\leq \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n-1} | \Delta\lambda_v || \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\sigma} p a_p | + \frac{| \lambda_n |}{A_n^{\alpha+\sigma}} | \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\sigma} v a_v | \\
&\leq \frac{1}{A_n^{\alpha+\sigma}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\sigma} \theta_v^{\alpha,\sigma} | \Delta\lambda_v | + | \lambda_n | \theta_n^{\alpha,\sigma} \\
&= T_{n,1}^{\alpha,\sigma} + T_{n,2}^{\alpha,\sigma}, \quad \text{say.}
\end{aligned}
$$

Since

$$
| T_{n,1}^{\alpha,\sigma} + T_{n,2}^{\alpha,\sigma} |^k \leq 2^k (| T_{n,1}^{\alpha,\sigma} |^k + | T_{n,2}^{\alpha,\sigma} |^k),
$$

to complete the proof of Theorem B, by using (8) it is sufficient to show that

$$
\sum_{n=1}^{\infty} n^{-k} | \varphi_n T_{n,r}^{\alpha,\sigma} |^k < \infty \quad \text{for} \quad r = 1, 2.
$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$
\begin{aligned}
\sum_{n=2}^{m+2} n^{-k} | \varphi_n T_{n,1}^{\alpha,\sigma} |^k &\leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha+\sigma})^{-k} | \varphi_n |^k \{ \sum_{v=1}^{n-1} A_v^{\alpha+\sigma} \theta_v^{\alpha,\sigma} | \Delta\lambda_v | \}^k \\
&\leq \sum_{n=2}^{m+1} n^{-k} n^{-(\alpha+\sigma)k} | \varphi_n |^k \{ \sum_{v=1}^{n-1} v^{(\alpha+\sigma)k} (\theta_v^{\alpha,\sigma})^k | \Delta\lambda_v | \} \\
&\times \{ \sum_{v=1}^{n-1} | \Delta\lambda_v | \}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\sigma)^k} (\theta_v^{\alpha,\sigma})^k \beta_v \sum_{n=v+1}^{m+1} \frac{n^{-k} | \varphi_n |^k}{n^{(\alpha+\sigma)k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\sigma)k} (\theta_v^{\alpha,\sigma})^k \beta_v \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} | \varphi_n |^k}{n^{(\alpha+\sigma)k+\epsilon}}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\sigma)k}(\theta_v^{\alpha,\sigma})^k\beta_v v^{\epsilon-k} \mid \varphi_v \mid^k \sum_{n=v+1}^{m+1}\frac{1}{n^{(\alpha+\sigma)k+\epsilon}}\\
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\sigma)k}(\theta_v^{\alpha,\sigma})^k\beta_v v^{\epsilon-k} \mid \varphi_v \mid^k \int_v^{\infty}\frac{dx}{x^{(\alpha+\sigma)k+\epsilon}}\\
&= O(1)\sum_{v=1}^{m} v\beta_v v^{-k}(\theta_v^{\alpha,\sigma} \mid \varphi_v \mid)^k\\
&= O(1)\sum_{v=1}^{m-1} \Delta(v\beta_v)\sum_{r=1}^{v} r^{-k}(\theta_r^{\alpha,\sigma} \mid \varphi_r \mid)^k + O(1)m\beta_m\sum_{v=1}^{m} v^{-k}(\theta_v^{\alpha,\sigma} \mid \varphi_v \mid)^k\\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1} \mid \beta_v \mid X_v + O(1)m\beta_m X_m\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem B and Lemma 2. Finally, we have that

$$
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\sigma} \mid^k \quad = \quad O(1)\sum_{n=1}^{m} \mid \lambda_n \mid n^{-k}(\theta_n^{\alpha,\sigma} \mid \varphi_n \mid)^k
$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{-k}(\theta_v^{\alpha,\sigma} \mid \varphi_v \mid)^k + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{-k}(\theta_n^{\alpha,\sigma} \mid \varphi_n \mid)^k\\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m\\
&= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem B and Lemma 2. Therefore, we get that

$$
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,r}^{\alpha,\sigma} \mid^k = O(1) \quad \text{as} \quad m\to\infty, \quad \text{for} \quad r=1,2.
$$

This completes the proof of Theorem B. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result for $\mid C,\alpha,\sigma \mid_k$ summability. Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we obtain another new result for $\mid C,\alpha,\sigma;\delta \mid_k$ summability. Furthermore, if we take $\sigma = 0$, then we obtain Theorem A. If we take (X_n) as an almost increasing sequence, $\epsilon = 1$, $\sigma = 0$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we get a known result dealing with $\mid C, \alpha \mid_k$ summability (see [3]), in this case the condition $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ is not needed. Finally, if we take $\epsilon = 1$, $\sigma = 0$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we get a result for $\mid C, \alpha; \delta \mid_k$ summability.

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