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A NOTE ON INTEGRAL C-PARALLEL SUBMANIFOLDS IN $\mathbb{S}^7(c)$

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Abstract. We find the explicit parametric equations of the flat 3-dimensional integral C-parallel submanifolds in the sphere $\mathbb{S}^7$ endowed with the deformed Sasakian structure defined by Tanno.

1. Introduction

During the last three decades, in the geometry of Sasakian space forms, a special attention was paid to the study of integral submanifolds, and several classification results were obtained (see, for example, [1]-[4], [6]-[9]). These results were often illustrated by explicit examples obtained using the odd dimensional unit Euclidean spheres endowed with the canonical Sasakian structure $\mathbb{S}^{2n+1}(1)$, as the models of Sasakian space forms with constant φ-sectional curvature $c = 1$.

The study of integral submanifolds of Sasakian space forms have also been made under some natural supplementary conditions. These conditions were formulated in terms of the mean curvature vector field H or the second fundamental form B. The most studied were the minimal, i.e. $H = 0$, integral submanifolds (see, for example, [5, 8]), and then the submanifolds with H or B being C-parallel, which means that the covariant derivative of H or B, in the normal bundle, is parallel to the characteristic vector field (see [1, 4]).

Because of its peculiarities, the 7-sphere $\mathbb{S}^7(1)$ played an important role in most of the studies dedicated to integral submanifolds (see, for example, [3, 6, 9]).

In [4], the authors completely classified 3-dimensional integral C-parallel submanifolds of 7-dimensional Sasakian space forms, i.e. those integral submanifolds with C-parallel second fundamental form, and then they gave explicitly the flat integral C-parallel submanifolds in $\mathbb{S}^7(1)$.

The purpose of our paper is to go further and to obtain the explicit parametric equations of the flat integral C-parallel submanifolds in $\mathbb{S}^7$ endowed with the deformed Sasakian structure introduced by Tanno, $\mathbb{S}^7(c)$, seen as the model of the Sasakian space form with constant φ-sectional curvature $c > -3$ (see [10]).

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2. Preliminaries

A triple (φ, ξ, η) is called a *contact structure* on a manifold N^{2n+1}, where φ is a tensor field of type $(1,1)$ on N, ξ is a vector field and η is a 1-form, if

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

A Riemannian metric g on N is said to be an *associated metric*, and then $(N, \varphi, \xi, \eta, g)$ is a *contact metric manifold*, if

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X,Y), \quad \forall X, Y \in C^\infty(TN).$$

A contact metric structure (φ, ξ, η, g) is called *normal* if

$$N_\varphi + 2d\eta \otimes \xi = 0,$$

where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y], \quad \forall X, Y \in C^\infty(TN),$$

is the Nijenhuis tensor field of φ.

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is a *Sasakian manifold* if it is normal or, equivalently, if

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X, \quad \forall X, Y \in C^\infty(TN)$$

(see [5]). We note that on a Sasakian manifold we have $\nabla_X \xi = -\varphi X$.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a 2-plane generated by X and φX, where X is a unit vector orthogonal to ξ, is called *φ-sectional curvature* determined by X. A Sasakian manifold with constant φ-sectional curvature c is called a *Sasakian space form* and it is denoted by $N(c)$. The curvature tensor field of a Sasakian space form $N(c)$ is given by

$$\begin{aligned} R(X,Y)Z &= \tfrac{c+3}{4}\{g(Z,Y)X - g(Z,X)Y\} + \tfrac{c-1}{4}\{\eta(Z)\eta(X)Y \\ &- \eta(Z)\eta(Y)X + g(Z,X)\eta(Y)\xi - g(Z,Y)\eta(X)\xi \\ &+ g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\}. \end{aligned}$$

The classification of the complete, simply connected Sasakian space forms $N(c)$ was given in [10]. Thus, if $c = 1$ then $N(1)$ is isometric to the unit sphere $\mathbb{S}^{2n+1}$ endowed with its canonical Sasakian structure, and if $c > -3$ then $N(c)$ is isometric to $\mathbb{S}^{2n+1}$ endowed with the deformed Sasakian structure given by Tanno, which we present below.

Let $\mathbb{S}^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$ be the unit $2n+1$-dimensional sphere endowed with its standard metric field g_0. Consider the following structure tensor fields on $\mathbb{S}^{2n+1}$: $\xi_0 = -\mathcal{I}z$ for each $z \in \mathbb{S}^{2n+1}$, where $\mathcal{I}$ is the usual complex structure on $\mathbb{C}^{n+1}$ defined by

$$\mathcal{I}z = (-y^1, \dots, -y^{n+1}, x^1, \dots, x^{n+1}),$$

for $z = (z^1, \dots, z^{n+1}) = (x^1, \dots, x^{n+1}, y^1, \dots, y^{n+1})$, $z^k = x^k + \mathrm{i}y^k$, and $\varphi_0 = s \circ \mathcal{I}$, where $s : T_z\mathbb{C}^{n+1} \to T_z\mathbb{S}^{2n+1}$ denotes the orthogonal projection. Equipped with these tensors, $\mathbb{S}^{2n+1}$ becomes a Sasakian space form with φ_0-sectional curvature equal to 1, which is denoted by $\mathbb{S}^{2n+1}(1)$.

Now, consider the deformed Sasakian structure on $\mathbb{S}^{2n+1}$,

$$\eta = a\eta_0, \quad \xi = \frac{1}{a}\xi_0, \quad \varphi = \varphi_0, \quad g = ag_0 + a(a-1)\eta_0 \otimes \eta_0,$$

where a is a positive constant. The structure (φ, ξ, η, g) is still a Sasakian structure and $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant φ-sectional curvature $c = \frac{4}{a} - 3 > -3$ denoted by $\mathbb{S}^{2n+1}(c)$ (see also [5]).

A submanifold M^m of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$ is called an *integral submanifold* if $\eta(X) = 0$ for any vector field X tangent to M. We have $\varphi(TM) \subset NM$ and $m \leq n$, where TM and NM are the tangent bundle and the normal bundle of M, respectively. Moreover, for $m = n$, one gets $\varphi(NM) = TM$. If we denote by B the second fundamental form of M then, by a straightforward computation, one obtains the following relation which we shall use later in this paper

$$g(B(X,Y), \varphi Z) = g(B(X,Z), \varphi Y),$$

for any vector fields X, Y and Z tangent to M (see also [4, 8]).

If M^m, with $m \leq n$, is a submanifold of the sphere $\mathbb{S}^{2n+1}$ then M is integral with respect to its canonical Sasakian structure $(\varphi_0, \xi_0, \eta_0, g_0)$ if and only if it is integral with respect to the deformed one (φ, ξ, η, g), since $\eta_0(X) = 0$ if and only if $\eta(X) = 0$ for any vector field X tangent to M. Moreover, if M is an integral submanifold of $\mathbb{S}^{2n+1}$ then the normal bundle of M in $(\mathbb{S}^{2n+1}, g_0)$ coincides with the normal bundle of M in $(\mathbb{S}^{2n+1}, g)$, since for any $X \in T_pM$ and $Y \in T_p\mathbb{S}^{2n+1}$, where p is an arbitrary point in M, we have $g_0(X,Y) = 0$ if and only if $g(X,Y) = 0$.

Next, we consider M to be an integral submanifold of $\mathbb{S}^{2n+1}$, and denote by g_0^M and g^M the induced metrics on M by g_0 and g, respectively. Denote by $\dot{\nabla}^M$ and ∇^M their Levi-Civita connections. Then the identity map $\mathbf{1} : (M, g_0^M) \to (M, g^M)$ is an homothety and therefore $\dot{\nabla}^M = \nabla^M$.

The following Lemma holds.

Lemma 2.1. *Let M be an integral submanifold of $\mathbb{S}^{2n+1}$. If X and Y are vector fields tangent to M then*

$$\dot{\nabla}_X Y = \nabla_X Y \quad \text{and} \quad \dot{\nabla}_X \varphi Y = \nabla_X \varphi Y,$$

where $\dot{\nabla}$ and ∇ are the Levi-Civita connections on $(\mathbb{S}^{2n+1}, g_0)$ and $(\mathbb{S}^{2n+1}, g)$, respectively.

Proof. From the definition of the metric g we have, for any vector fields X, Y tangent to M and Z tangent to $\mathbb{S}^{2n+1}$,

$$g(\nabla_X Y, Z) = ag_0(\nabla_X Y, Z) + a(a-1)\eta_0(\nabla_X Y)\eta_0(Z).$$

But, since M is integral,

$$\eta_0(\nabla_X Y) = \frac{1}{a}\eta(\nabla_X Y) = \frac{1}{a}g(\nabla_X Y, \xi) = -\frac{1}{a}g(Y, \nabla_X \xi) = \frac{1}{a}g(Y, \varphi X) = 0,$$

and so

$$g(\nabla_X Y, Z) = ag_0(\nabla_X Y, Z).$$

On the other hand, applying the characterization of the Levi-Civita connection for ∇ and $\dot{\nabla}$, we obtain

$$g(\nabla_X Y, Z) = a g_0(\dot{\nabla}_X Y, Z).$$

From the last two relations we get

$$g_0(\nabla_X Y, Z) = g_0(\dot{\nabla}_X Y, Z)$$

and therefore $\dot{\nabla}_X Y = \nabla_X Y$ for any vector fields X and Y tangent to M.

For the second relation, we use $(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$ and $(\dot{\nabla}_X \varphi)Y = g_0(X,Y)\xi_0 - \eta_0(Y)X$ for vector fields X and Y tangent to M, and come to the conclusion. □

We shall end this section by recalling the notion of an integral C-parallel submanifold of a Sasakian manifold (see, for example, [4]). Let M^m be an integral submanifold of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$. Then M is said to be *integral C-parallel* if $\nabla^\perp B$ is parallel to the characteristic vector field ξ, where B is the second fundamental form of M and $\nabla^\perp B$ is given by

$$(\nabla^\perp B)(X,Y,Z) = \nabla^\perp_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for any vector fields X, Y, Z tangent to M, $\nabla^\perp$ and ∇ being the normal connection and the Levi-Civita connection on M, respectively. This means $(\nabla^\perp B)(X,Y,Z) = g(\varphi X, B(Y,Z))\xi$. If we denote $S(X,Y,Z) = g(\varphi X, B(Y,Z))$, then S is a totally symmetric tensor field of type $(0,3)$ on M.

It is easy to see that, if the dimension of an integral C-parallel submanifold M is maximal, i.e. it is equal to n, then the mean curvature $|H|$ of M is constant.

3. Main result

In [4] Baikoussis, Blair and Koufogiorgios classified the 3-dimensional integral C-parallel submanifolds in a Sasakian space form $(N^7(c), \varphi, \xi, \eta, g)$. In order to obtain the classification, they worked with a special local orthonormal basis (see also [6]). Here we shall briefly recall how this basis is constructed.

Let $\mathbf{i} : M^3 \to N^7(c)$ be an integral submanifold of constant mean curvature. Let p be an arbitrary point of M, and consider the function $f_p : U_pM \to \mathbb{R}$ given by

$$f_p(u) = g(B(u,u), \varphi u),$$

where $U_pM = \{u \in T_pM : g(u,u) = 1\}$ is the unit sphere in the tangent space T_pM. If $f_p(u) = 0$, for all $u \in U_pM$, then, for any $v_1, v_2 \in U_pM$ such that $g(v_1, v_2) = 0$ we have that

$$g(B(v_1,v_1), \varphi v_1) = 0 \quad \text{and} \quad g(B(v_1,v_1), \varphi v_2) = 0.$$

We obtain $B(v_1, v_1) = 0$, and then it follows that B vanishes at the point p.

Next, assume that the function f_p does not vanish identically. Since U_pM is compact, f_p attains an absolute maximum at a unit vector X_1. It follows that

$$\begin{cases} g(B(X_1,X_1), \varphi X_1) > 0, & g(B(X_1,X_1), \varphi X_1) \geq |g(B(w,w), \varphi w)| \\ g(B(X_1,X_1), \varphi w) = 0, & g(B(X_1,X_1), \varphi X_1) \geq 2g(B(w,w), \varphi X_1), \end{cases}$$

where w is a unit vector tangent to M at p and orthogonal to X_1. It is easy to see that X_1 is an eigenvector of the shape operator $A_1 = A_{\varphi X_1}$ with the corresponding eigenvalue λ_1. Then, since A_1 is symmetric, we consider X_2 and X_3 to be unit eigenvectors of A_1, orthogonal to each other and to X_1, with the corresponding eigenvalues λ_2 and λ_3. Further, we distinguish two cases.

If $\lambda_2 \neq \lambda_3$, we can choose X_2 and X_3 such that

$$\begin{cases} g(B(X_2, X_2), \varphi X_2) \geq 0, \quad g(B(X_3, X_3), \varphi X_3) \geq 0 \\ g(B(X_2, X_2), \varphi X_2) \geq g(B(X_3, X_3), \varphi X_3). \end{cases}$$

If $\lambda_2 = \lambda_3$, we consider $f_{1,p}$ the restriction of f_p to $\{w \in U_pM : g(w, X_1) = 0\}$, and we have two subcases:

(1) the function $f_{1,p}$ is identically zero. In this case, we have

$$\begin{cases} g(B(X_2, X_2), \varphi X_2) = 0, \quad g(B(X_2, X_2), \varphi X_3) = 0 \\ g(B(X_2, X_3), \varphi X_3) = 0, \quad g(B(X_3, X_3), \varphi X_3) = 0. \end{cases}$$

(2) the function $f_{1,p}$ does not vanish identically. Then we choose X_2 such that $f_{1,p}(X_2)$ is an absolute maximum. We have that

$$\begin{cases} g(B(X_2, X_2), \varphi X_2) > 0, \quad g(B(X_2, X_2), \varphi X_2) \geq g(B(X_3, X_3), \varphi X_3) \geq 0 \\ g(B(X_2, X_2), \varphi X_3) = 0, \quad g(B(X_2, X_2), \varphi X_2) \geq 2g(B(X_3, X_3), \varphi X_2). \end{cases}$$

Now, with respect to the orthonormal basis $\{X_1, X_2, X_3\}$, the shape operators A_1, $A_2 = A_{\varphi X_2}$ and $A_3 = A_{\varphi X_3}$, at p, can be written as follows

$$A_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \lambda_2 & 0 \\ \lambda_2 & \alpha & \beta \\ 0 & \beta & \gamma \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & \lambda_3 \\ 0 & \beta & \gamma \\ \lambda_3 & \gamma & \delta \end{pmatrix}. \tag{3.1}$$

We also have $A_0 = A_\xi = 0$. With these notations we have

$$\lambda_1 > 0, \quad \lambda_1 \geq |\alpha|, \quad \lambda_1 \geq |\delta|, \quad \lambda_1 \geq 2\lambda_2, \quad \lambda_1 \geq 2\lambda_3. \tag{3.2}$$

For $\lambda_2 \neq \lambda_3$ we get

$$\alpha \geq 0, \quad \delta \geq 0 \quad \text{and} \quad \alpha \geq \delta \tag{3.3}$$

and for $\lambda_2 = \lambda_3$ we obtain that

$$\alpha = \beta = \gamma = \delta = 0 \tag{3.4}$$

or

$$\alpha > 0, \quad \delta \geq 0, \quad \alpha \geq \delta, \quad \beta = 0 \quad \text{and} \quad \alpha \geq 2\gamma. \tag{3.5}$$

We can extend X_1 on a neighbourhood V_p of p such that $X_1(q)$ is a maximal point of $f_q : U_qM \to \mathbb{R}$, for any point q of V_p.

If the eigenvalues of A_1 have constant multiplicities, then the above basis $\{X_1, X_2, X_3\}$, defined at p, can be smoothly extended and we can work on the open dense subset of M defined by this property.

Using this basis, in [4], the authors proved that, when M is an integral C-parallel submanifold, the functions λ_i, $i = \overline{1,3}$, and α, β, γ, δ are constant on

V_p, and then classified all 3-dimensional integral C-parallel submanifolds in a 7-dimensional Sasakian space form.

According to that classification, if $c > -3$ then M is an integral C-parallel submanifold if and only if either:

Case I. M is totally geodesic, with the Gaussian curvature $K = \frac{c+3}{4}$.

Case II. M is flat, locally it is a product of three curves, which are helices of osculating orders $r \leq 4$, and $\lambda_1 = \frac{\lambda^2 - \frac{c+3}{4}}{\lambda} \neq 0$, $\lambda_2 = \lambda_3 = \lambda = \text{constant} \neq 0$, $\alpha = \text{constant}$, $\beta = 0$, $\gamma = \text{constant}$, $\delta = \text{constant}$, such that $-\frac{\sqrt{c+3}}{2} < \lambda < 0$, $0 < \alpha \leq \lambda_1$, $\alpha > 2\gamma$, $\alpha \geq \delta \geq 0$ and $\frac{c+3}{4} + \lambda^2 + \alpha\gamma - \gamma^2 = 0$.

Case III. M is locally isometric to a product $\Gamma \times \bar{M}^2$, where Γ is a curve and $\bar{M}^2$ is a C-parallel surface, and either

(1) $\lambda_1 = 2\lambda_2 = \frac{\sqrt{c+3}}{2\sqrt{2}}$, $\lambda_3 = -\frac{\sqrt{c+3}}{2\sqrt{2}}$, $\alpha = \gamma = \delta = 0$, $\beta = \pm\frac{\sqrt{3(c+3)}}{4\sqrt{2}}$. In this case Γ is a helix in N with curvatures $\kappa_1 = \frac{1}{\sqrt{2}}$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere of radius $\rho = \sqrt{\frac{8}{3(c+3)}}$.
or

(2) $\lambda_1 = \frac{\lambda^2 - \frac{c+3}{4}}{\lambda}$, $\lambda_2 = \lambda_3 = \lambda = \text{constant}$, $\alpha = \beta = \gamma = \delta = 0$, such that $-\frac{\sqrt{c+3}}{2} < \lambda < 0$. In this case Γ is a helix in N with curvatures $\kappa_1 = \lambda_1$ and $\kappa_2 = 1$, and $\bar{M}^2$ is the 2-dimensional Euclidean sphere of radius $\rho = \frac{1}{\sqrt{\frac{c+3}{4} + \lambda^2}}$.

In the same paper [4] one obtains the explicit parametric equation of the flat 3-dimensional integral C-parallel submanifolds in $\mathbb{S}^7(1)$. We shall prove, using the same techniques, the following result.

Theorem 3.1. *The position vector in the Euclidean space* $(\mathbb{R}^8, \langle,\rangle)$ *of a flat 3-dimensional integral C-parallel submanifold in* $\mathbb{S}^7(c)$, $c = \frac{4}{a} - 3 > -3$, *is*

$$\begin{aligned}
x(u,v,w) &= \frac{\lambda}{\sqrt{\lambda^2 + \frac{1}{a}}}\cos\left(\frac{1}{a\lambda}u\right)e_1 + \frac{1}{\sqrt{a(\gamma-\alpha)(2\gamma-\alpha)}}\cos(\lambda u - (\gamma-\alpha)v)e_2 \\
&+ \frac{1}{\sqrt{a\rho_1(\rho_1+\rho_2)}}\cos(\lambda u + \gamma v + \rho_1 w)e_3 \\
&+ \frac{1}{\sqrt{a\rho_2(\rho_1+\rho_2)}}\cos(\lambda u + \gamma v - \rho_2 w)e_4 \\
&+ \frac{\lambda}{\sqrt{\lambda^2 + \frac{1}{a}}}\sin\left(\frac{1}{a\lambda}u\right)\mathcal{I}e_1 - \frac{1}{\sqrt{a(\gamma-\alpha)(2\gamma-\alpha)}}\sin(\lambda u - (\gamma-\alpha)v)\mathcal{I}e_2 \\
&- \frac{1}{\sqrt{a\rho_1(\rho_1+\rho_2)}}\sin(\lambda u + \gamma v + \rho_1 w)\mathcal{I}e_3 \\
&- \frac{1}{\sqrt{a\rho_2(\rho_1+\rho_2)}}\sin(\lambda u + \gamma v - \rho_2 w)\mathcal{I}e_4,
\end{aligned} \tag{3.6}$$

where $\rho_{1,2} = \frac{1}{2}(\sqrt{4\gamma(2\gamma-\alpha)+\delta^2} \pm \delta)$ *and* $\lambda, \alpha, \gamma, \delta$ *are real constants such that* $-\frac{1}{\sqrt{a}} < \lambda < 0$, $0 < \alpha \leq \frac{\lambda^2 - \frac{1}{a}}{\lambda}$, $\alpha \geq \delta \geq 0$, $\alpha > 2\gamma$, $\frac{1}{a} + \lambda^2 + \alpha\gamma - \gamma^2 = 0$, *and* $\{e_i, \mathcal{I}e_j\}_{i,j=1}^4$ *are constant unit vectors orthogonal to one another.*

Proof. Let us denote by ∇, $\dot{\nabla}$ and by $\widetilde{\nabla}$ the Levi-Civita connections on $(\mathbb{S}^7, g)$, $(\mathbb{S}^7, g_0)$ and $(\mathbb{R}^8, \langle,\rangle)$, respectively, where g_0 is the canonical metric on $\mathbb{S}^7$ induced by the canonical inner product $\langle,\rangle$ from $\mathbb{R}^8$.

We denote by $\mathbf{i}$ the canonical inclusion of the submanifold $\mathbb{S}^7$ in $\mathbb{R}^8$. The map $\mathbf{i} : (\mathbb{S}^7, g_0) \to (\mathbb{R}^8, \langle,\rangle)$ is an isometric immersion, whilst the immersion $\mathbf{i} : (\mathbb{S}^7, g) \to (\mathbb{R}^8, \langle,\rangle)$ is not isometric.

Assume that M^3 is a flat integral C-parallel submanifold in $\mathbb{S}^7(c)$, i.e. it is given by the case II of the classification (see also Lemma 4.5 (ii) ([4])). Consider the orthonormal basis $\{X_1, X_2, X_3\}$ on M. We have $\nabla^M_{X_i} X_j = 0$, $i, j = 1, 2, 3$, where ∇^M is the Levi-Civita connection on M endowed with the metric g^M induced by g. It follows that $[X_i, X_j] = 0$ and therefore we can choose a local chart such that $x = x(u, v, w)$ with $x_u = X_1$, $x_v = X_2$ and $x_w = X_3$.

From (3.1) we see that the shape operators of M are given by

$$A_{\varphi X_1} = A_1 = \begin{pmatrix} \frac{\lambda^2 - \frac{1}{a}}{\lambda} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & \gamma \\ \lambda & \gamma & \delta \end{pmatrix},$$

and $A_\xi = 0$.

Now, we shall prove that $\widetilde{\nabla}_{X_1} X_1 = \frac{\lambda^2 - \frac{1}{a}}{\lambda} \varphi X_1 - \frac{1}{a} x$. Indeed, from the Gauss equation of M in $(\mathbb{S}^7, g)$ we have

$$\begin{aligned} \nabla_{X_1} X_1 &= \nabla^M_{X_1} X_1 + B(X_1, X_1) = B(X_1, X_1) \\ &= \textstyle\sum_{i=1}^3 g(A_i(X_1), X_1)\varphi X_i + g(A_\xi(X_1), X_1)\xi \\ &= g(A_1(X_1), X_1)\varphi X_1 = \frac{\lambda^2 - \frac{1}{a}}{\lambda} \varphi X_1. \end{aligned}$$

On the other hand, using Lemma 2.1 and the Gauss equation of $(\mathbb{S}^7, g_0)$ in $(\mathbb{R}^8, \langle,\rangle)$, we obtain

$$\nabla_{X_1} X_1 = \dot{\nabla}_{X_1} X_1 = \widetilde{\nabla}_{X_1} X_1 + \langle X_1, X_1 \rangle x = \widetilde{\nabla}_{X_1} X_1 + \frac{1}{a} x.$$

Next, we have

$$\nabla_{X_1} \varphi X_1 = \varphi \nabla_{X_1} X_1 + g(X_1, X_1)\xi = -\frac{\lambda^2 - \frac{1}{a}}{\lambda} X_1 + \xi = -\frac{\lambda^2 - \frac{1}{a}}{\lambda} X_1 + \frac{1}{a}\xi_0$$

and then, from Lemma 2.1 and the Gauss equation, it follows

$$\nabla_{X_1} \varphi X_1 = \dot{\nabla}_{X_1} \varphi X_1 = \widetilde{\nabla}_{X_1} \varphi X_1.$$

In the same way we get the following equations:

$$\begin{array}{rclrcl}
\widetilde{\nabla}_{X_1}X_1 &=& \frac{\lambda^2-\frac{1}{a}}{\lambda}\varphi X_1-\frac{1}{a}x & \widetilde{\nabla}_{X_2}\varphi X_2 &=& -\lambda X_1-\alpha X_2+\frac{1}{a}\xi_0\\
\widetilde{\nabla}_{X_1}\varphi X_1 &=& -\frac{\lambda^2-\frac{1}{a}}{\lambda}X_1+\frac{1}{a}\xi_0 & \widetilde{\nabla}_{X_2}X_3 &=& \widetilde{\nabla}_{X_3}X_2=\gamma\varphi X_3\\
\widetilde{\nabla}_{X_1}X_2 &=& \widetilde{\nabla}_{X_2}X_1=\lambda\varphi X_2 & \widetilde{\nabla}_{X_2}\varphi X_3 &=& \widetilde{\nabla}_{X_3}\varphi X_2=-\gamma X_3\\
\widetilde{\nabla}_{X_1}\varphi X_2 &=& \widetilde{\nabla}_{X_2}\varphi X_1=-\lambda X_2 & \widetilde{\nabla}_{X_2}\xi_0 &=& -\varphi X_2\\
\widetilde{\nabla}_{X_1}X_3 &=& \widetilde{\nabla}_{X_3}X_1=\lambda\varphi X_3 & \widetilde{\nabla}_{X_3}X_3 &=& \lambda\varphi X_1+\gamma\varphi X_2+\delta\varphi X_3-\frac{1}{a}x\\
\widetilde{\nabla}_{X_1}\varphi X_3 &=& \widetilde{\nabla}_{X_3}\varphi X_1=-\lambda X_3 & \widetilde{\nabla}_{X_3}\varphi X_3 &=& -\lambda X_1-\gamma X_2-\delta X_3+\frac{1}{a}\xi_0\\
\widetilde{\nabla}_{X_1}\xi_0 &=& -\varphi X_1 & \widetilde{\nabla}_{X_3}\xi_0 &=& -\varphi X_3\\
\widetilde{\nabla}_{X_2}X_2 &=& \lambda\varphi X_1+\alpha\varphi X_2-\frac{1}{a}x & & &
\end{array} \tag{3.7}$$

where we also used the fact that

$$\widetilde{\nabla}_X\xi_0=\dot{\nabla}_X\xi_0=-\varphi X$$

for all vector fields X tangent to $\mathbb{S}^7$ and orthogonal to ξ (we recall that X is orthogonal to ξ with respect to g if and only if it is orthogonal to ξ with respect to g_0).

From equations (3.7) we obtain:

$$\begin{cases}
x_{uuuu}+\Big(\lambda^2+\frac{1}{a^2\lambda^2}\Big)x_{uu}+\frac{1}{a^2}x=0\\
x_{uuv}+\lambda^2x_v=0,\quad x_{uuw}+\lambda^2x_w=0,\quad \lambda x_{vw}-\gamma x_{uw}=0\\
\lambda^2x_{uuu}-\Big(\lambda^2-\frac{1}{a}\Big)x_{uvv}+\frac{1}{a^2}x_u-\alpha\lambda\Big(\lambda^2-\frac{1}{a}\Big)x_v=0\\
\Big(\lambda^2-\frac{1}{a}\Big)x_{uvww}+\lambda^3\gamma x_{uu}+\gamma^2\Big(\lambda^2-\frac{1}{a}\Big)x_{uv}+\gamma\delta\Big(\lambda^2-\frac{1}{a}\Big)x_{uw}+\frac{\lambda\gamma}{a^2}x=0.
\end{cases} \tag{3.8}$$

From the first equation of (3.8) we get

$$\begin{aligned}
x(u,v,w) &= \cos(\tfrac{1}{a\lambda}u)v_1(v,w)+\sin(\tfrac{1}{a\lambda}u)v_2(v,w)+\cos(\lambda u)v_3(v,w)\\
&+ \sin(\lambda u)v_4(v,w),
\end{aligned}$$

where $v_1(v,w)$, $v_2(v,w)$, $v_3(v,w)$ and $v_4(v,w)$ are $\mathbb{R}^8$-valued functions of the variables v and w. By solving the following five equations of (3.8) one by one, we get

$$\begin{aligned}
x(u,v,w) &= \cos(\tfrac{1}{a\lambda}u)c_1+\cos(\lambda u-(\gamma-\alpha)v)c_2+\cos(\lambda u+\gamma v+\rho_1 w)c_3\\
&+ \cos(\lambda u+\gamma v-\rho_2 w)c_4+\sin(\tfrac{1}{a\lambda}u)c_5+\sin(\lambda u-(\gamma-\alpha)v)c_6\\
&+ \sin(\lambda u+\gamma v+\rho_1 w)c_7+\sin(\lambda u+\gamma v-\rho_2 w)c_8,
\end{aligned} \tag{3.9}$$

where $\rho_{1,2} = \frac{1}{2}(\sqrt{4\gamma(2\gamma-\alpha)+\delta^2} \pm \delta)$ and $\{c_i\}$ are constant vectors in $\mathbb{R}^8$.

The next step is to determine the conditions which must be satisfied by the vectors $\{c_i\}$. For this purpose we shall denote $c_{ij} = \langle c_i, c_j \rangle$.

In the expression of x_w, obtained from (3.9), we take $\lambda u + \gamma v = \rho_2 w$ and get

$$x_w = -\rho_1 \sin((\rho_1+\rho_2)w)c_3 + \rho_1 \cos((\rho_1+\rho_2)w)c_7 - \rho_2 c_8$$

Then, computing $\langle x_w, x_w \rangle = \frac{1}{a}$ in $w = 0$, $w = \frac{\pi}{\rho_1+\rho_2}$, $w = \frac{\pi}{2(\rho_1+\rho_2)}$ and in $w = -\frac{\pi}{2(\rho_1+\rho_2)}$ we easily get

$$\begin{cases} \rho_1^2 c_{77} + \rho_2^2 c_{88} - 2\rho_1\rho_2 c_{78} = \frac{1}{a}, & \rho_1^2 c_{77} + \rho_2^2 c_{88} + 2\rho_1\rho_2 c_{78} = \frac{1}{a} \\ \rho_1^2 c_{33} + \rho_2^2 c_{88} + 2\rho_1\rho_2 c_{38} = \frac{1}{a}, & \rho_1^2 c_{33} + \rho_2^2 c_{88} - 2\rho_1\rho_2 c_{38} = \frac{1}{a} \end{cases}$$

and it follows that $c_{38} = c_{78} = 0$, $c_{33} = c_{77}$ and

$$\rho_1^2 c_{77} + \rho_2^2 c_{88} = \frac{1}{a}. \tag{3.10}$$

In the same way, by taking $\lambda u + \gamma v = -\rho_1 w$, we obtain $c_{47} = c_{48} = 0$ and $c_{44} = c_{88}$. Since $\langle x_w, x_w \rangle = \frac{1}{a}$ at any triple (u, v, w), for $\lambda u + \gamma v = \frac{\pi}{2}$ and $w = 0$, we have $c_{34} = 0$, and from $\langle x_w, x_{ww} \rangle = 0$, it results $c_{37} = 0$, when $u = v = w = 0$.
Now, computing

$$\langle x_{ww}, x_{ww} \rangle = \frac{\lambda^2+\gamma^2+\delta^2}{a} + \frac{1}{a^2} = \frac{\rho_1^2+\rho_2^2-\rho_1\rho_2}{a}$$

in $u = v = w = 0$, we have

$$\rho_1^4 c_{33} + \rho_2^4 c_{44} = \frac{\rho_1^2+\rho_2^2-\rho_1\rho_2}{a}. \tag{3.11}$$

Since $c_{33} = c_{77}$ and $c_{44} = c_{88}$, from (3.10) and (3.11), one obtains

$$c_{33} = c_{77} = \frac{1}{a\rho_1(\rho_1+\rho_2)} \quad \text{and} \quad c_{44} = c_{88} = \frac{1}{a\rho_2(\rho_1+\rho_2)}.$$

We have just proved that

$$c_3 \perp c_4 \perp c_7 \perp c_8 \perp c_3$$

and

$$|c_3|^2 = |c_7|^2 = \frac{1}{a\rho_1(\rho_1+\rho_2)}, \quad |c_4|^2 = |c_8|^2 = \frac{1}{a\rho_2(\rho_1+\rho_2)},$$

where $c_i \perp c_j$ means $\langle c_i, c_j \rangle = 0$ and $|c_i|^2 = \langle c_i, c_i \rangle$.

In order to calculate c_{2j} and c_{6j}, for $j \in \{2,3,4,6,7,8\}$, we shall take first $\lambda u = (\gamma-\alpha)v$ and $w = 0$ in the expression of x_v and, from $\langle x_v, x_v \rangle = \frac{1}{a}$, we obtain

$$\begin{aligned} f_1(v) &= \langle x_v, x_v \rangle \\ &= (\gamma-\alpha)^2 c_{66} + \gamma^2(c_{33}+c_{44}) + 2\gamma(\gamma-\alpha)\sin((2\gamma-\alpha)v)(c_{36}+c_{46}) \\ &\quad -2\gamma(\gamma-\alpha)\cos((2\gamma-\alpha)v)(c_{67}+c_{68}) \\ &= \tfrac{1}{a}. \end{aligned}$$

As $f_1'(0)=0$ and $f_1'(\frac{\pi}{2(2\gamma-\alpha)})=0$ it follows

$$c_{36}+c_{46}=0 \quad \text{and} \quad c_{67}+c_{68}=0. \tag{3.12}$$

Next, consider $\lambda u=(\gamma-\alpha)v+\frac{\pi}{2}$ and $w=0$ in the expression of x_v and, in the same way as above, we get

$$c_{23}+c_{24}=0 \quad \text{and} \quad c_{27}+c_{28}=0. \tag{3.13}$$

Now, consider

$$\begin{aligned} f_2(w) &= \langle x_v(0,0,w), x_v(0,0,w)\rangle \\ &= (\gamma-\alpha)^2 c_{66}+\gamma^2(c_{33}+c_{88}) \\ &\quad +2\gamma(\gamma-\alpha)\sin(\rho_1 w)c_{36}-2\gamma(\gamma-\alpha)\cos(\rho_1 w)c_{67} \\ &\quad -2\gamma(\gamma-\alpha)\sin(\rho_2 w)c_{46}-2\gamma(\gamma-\alpha)\cos(\rho_2 w)c_{68} \\ &= \frac{1}{a} \end{aligned}$$

and, from $f_2'(0)=0$ and $f_2''(0)=0$, we have

$$\rho_1 c_{36}-\rho_2 c_{46}=0 \quad \text{and} \quad \rho_1^2 c_{67}-\rho_2^2 c_{68}=0,$$

which together with (3.12) give $c_{36}=c_{46}=c_{67}=c_{68}=0$. Replacing in $f_1(v)=\frac{1}{a}$ we also obtain

$$c_{66}=\frac{1}{a(\gamma-\alpha)(2\gamma-\alpha)}.$$

Next, using $\langle x_v(\frac{\pi}{2\lambda},0,0), x_v(\frac{\pi}{2\lambda},0,0)\rangle=\frac{1}{a}$, it results

$$c_{22}=\frac{1}{a(\gamma-\alpha)(2\gamma-\alpha)}$$

and then, from (3.13), $\langle x_v(\frac{\pi}{2\lambda},0,0), x_w(\frac{\pi}{2\lambda},0,0)\rangle=0$ and $\langle x_v(0,0,0), x_w(0,0,0)\rangle=0$, we have $c_{23}=c_{24}=c_{27}=c_{28}=0$.
With all values of c_{ij} obtained so far in mind, from $\langle x_v(\frac{\pi}{4\lambda},0,0), x_v(\frac{\pi}{4\lambda},0,0)\rangle=\frac{1}{a}$, we obtain $c_{26}=0$ and thus

$$c_2\perp c_3\perp c_4\perp c_6\perp c_7\perp c_8\perp c_2.$$

We have also proved that

$$|c_2|^2=|c_6|^2=\frac{1}{a(\gamma-\alpha)(2\gamma-\alpha)}.$$

Now we only have to calculate c_{1j} and c_{5j}, for $j = \{1, 2, \dots, 8\}$. In order to do this, we consider

$$\begin{aligned} f_3(w) &= \langle x(0,0,w), x(0,0,w)\rangle \\ &= c_{11} + c_{22} + 2c_{12} + c_{33} + c_{44} + 2\cos(\rho_1 w)c_{13} + 2\cos(\rho_2 w)c_{14} \\ &\quad +2\sin(\rho_1 w)c_{17} - 2\sin(\rho_2 w)c_{18} \\ &= 1 \end{aligned}$$

and, since $f_3'(0) = 0$ and $f_3''(0) = 0$, we have

$$\rho_1 c_{17} - \rho_2 c_{18} = 0 \quad \text{and} \quad \rho_1^3 c_{17} - \rho_2^3 c_{18} = 0,$$

which give $c_{17} = c_{18} = 0$. Replacing in $f_3'(w) = 0$ we also obtain $c_{13} = c_{14} = 0$. Next, as $\langle x(0,0,0), x_v(0,0,0)\rangle = 0$ and

$$\langle x(0,0,0), x_{vv}(0,0,0)\rangle = -\langle x_v(0,0,0), x_v(0,0,0)\rangle = -\frac{1}{a},$$

we easily get $c_{16} = 0$ and $c_{12} = 0$. Thus, from $f_3(w) = 1$, it results

$$c_{11} + c_{22} + c_{33} + c_{44} = 1,$$

which means that

$$c_{11} = \frac{\lambda^2}{\lambda^2 + \frac{1}{a}}.$$

Now, consider

$$\begin{aligned} f_4(w) &= \langle x_u(0,0,w), x_u(0,0,w)\rangle \\ &= \tfrac{1}{a^2\lambda^2} c_{11} + \lambda^2(c_{66} + c_{33} + c_{44}) + \tfrac{2}{a} c_{56} - \tfrac{2}{a}\sin(\rho_1 w)c_{35} \\ &\quad +\tfrac{2}{a}\sin(\rho_2 w)c_{45} + \tfrac{2}{a}\cos(\rho_1 w)c_{57} + \tfrac{2}{a}\cos(\rho_2 w)c_{58} \\ &= \tfrac{1}{a} \end{aligned}$$

and, from $f_4'(0) = 0$, $f_4''(0) = 0$, $f_4'''(0) = 0$ and $f_4^{(iv)}(0) = 0$, we have the following equations

$$\begin{cases} \rho_1 c_{35} + \rho_2 c_{45} = 0, & \rho_1^2 c_{57} + \rho_2^2 c_{58} = 0 \\ \rho_1^3 c_{35} + \rho_2^3 c_{45} = 0, & \rho_1^4 c_{57} + \rho_2^4 c_{58} = 0 \end{cases}$$

with solutions $c_{35} = c_{45} = c_{57} = c_{58} = 0$.
From

$$\langle x_u(0,0,0), x_v(0,0,0)\rangle = 0, \langle x_u(0,0,0), x_{vv}(0,0,0)\rangle = 0, \langle x(0,0,0), x_u(0,0,0)\rangle = 0$$

it results $c_{56} = c_{25} = c_{15} = 0$ and, from $f_4(w) = \frac{1}{a}$, we have

$$c_{55} = \frac{\lambda^2}{\lambda^2 + \frac{1}{a}}.$$

So far we proved that $c_i \perp c_j$ for every $i,j \in \{1,2,\dots,8\}$ and $|c_1|^2 = |c_5|^2 = \frac{\lambda^2}{\lambda^2+\frac{1}{a}}$, $|c_2|^2 = |c_6|^2 = \frac{1}{a(\gamma-\alpha)(2\gamma-\alpha)}$, $|c_3|^2 = |c_7|^2 = \frac{1}{a\rho_1(\rho_1+\rho_2)}$, $|c_4|^2 = |c_8|^2 = \frac{1}{a\rho_2(\rho_1+\rho_2)}$.

Finally, imposing M to be an integral submanifold we conclude that its position vector in $\mathbb{R}^8$ is given by equation (3.6). □

Remark 3.2. Using complex coordinates, (3.6) can be rewritten as

$$\begin{aligned} x(u,v,w) &= \frac{\lambda}{\sqrt{\lambda^2+\frac{1}{a}}}\exp(\mathrm{i}(\tfrac{1}{a\lambda}u))E_1 + \frac{1}{\sqrt{a(\gamma-\alpha)(2\gamma-\alpha)}}\exp(-\mathrm{i}(\lambda u-(\gamma-\alpha)v))E_2 \\ &+ \frac{1}{\sqrt{a\rho_1(\rho_1+\rho_2)}}\exp(-\mathrm{i}(\lambda u+\gamma v+\rho_1 w))E_3 \\ &+ \frac{1}{\sqrt{a\rho_2(\rho_1+\rho_2)}}\exp(-\mathrm{i}(\lambda u+\gamma v-\rho_2 w))E_4, \end{aligned}$$

where $\{E_i\}_{i=1}^4$ is an orthonormal basis of $\mathbb{C}^4$ with respect to the usual Hermitian inner product.

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