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GLOBAL L^p ESTIMATES FOR DEGENERATE ORNSTEIN-UHLENBECK OPERATORS: A GENERAL APPROACH

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Abstract. We present a new approach to prove *global* L^p estimates for degenerate Ornstein-Uhlenbeck operators in $\mathbb{R}^N$. We then show how to pave the way to extend such a technique to classes of general Hörmander operators. Several historical notes related to Calderón-Zygmund's singular integrals theory in Euclidean and in non-Euclidean settings are also provided.

1. Some historical notes

In a paper dated 1906, Beppo Levi [Le] solved the Dirichlet problem

$$\Delta u = f, \text{ in } \Omega \subset \mathbb{R}^2, \quad u|_{\partial\Omega} = 0$$

by minimizing the Dirichlet *energy Integral* in a new functional space that fifty years later J. Deny and J.L. Lions [DeL] generalized to higher dimensions, and called it of *Beppo Levi-type*. This space can be identified with what we call today the *Sobolev space* $W_0^{1,2}$.

The theory of boundary value problems in Sobolev spaces is one of the most important parts of the modern theory of PDE's. A major breakthrough was done by Calderón and Zygmund in 1952, with their celebrated theorem on *L^p-continuity of singular integrals*. Let $u \in C_0^\infty(\mathbb{R}^N)$ and f be such that

$$\Delta u = f.$$

Then, if $N \geq 3$,

$$u(x) = -f \star \Gamma(x) = -\int_{\mathbb{R}^N} f(y)\,\Gamma(x-y)\,dy, \;\; \Gamma(z) = C_N \frac{1}{|z|^{N-2}}.$$

Hence

$$\partial_{i,j} u(x) = \int_{\mathbb{R}^N} f(y)\,\Gamma_{i,j}(x-y)dy, \quad i,j = 1,\dots,N,$$

where

$$\Gamma_{i,j}(z) = \partial_{i,j}\Gamma(z) = \omega_{i,j}(z)\frac{1}{|z|^N} \qquad \text{with} \quad \int_{|z|=1} \omega_{i,j}\,d\sigma = 0.$$

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Since $\Gamma_{i,j}$ is not in $L^1_{loc}(\mathbb{R}^N)$, the last convolution needs to be understood in a weak sense:

$$\text{P.V.} \int_{\mathbb{R}^N} f(y)\,\Gamma_{i,j}(x-y)\,dy := \lim_{\varepsilon\to 0} \int_{\varepsilon<|x-y|<\frac{1}{\varepsilon}} f(y)\,\Gamma(x-y)\,d\,y.$$

Calderón and Zygmund proved that this limit exists in the topology of $L^p(\mathbb{R}^N)$, $f \in L^p(\mathbb{R}^N)$, $1 < p < \infty$, [CaZ]. As a consequence:

$$\|\partial_{x_i x_j} u\|_{L^p(\mathbb{R}^N)} \le C_{p,N}\,\|\Delta u\|_{L^p(\mathbb{R}^N)} \;\; i,j = 1,\dots,N, \tag{1}$$

a fundamental inequality for studying the Laplace equation in Sobolev spaces. In their work Calderón and Zygmund used a celebrated *covering lemma* by mean of *cubes* in $\mathbb{R}^N$. The cubes are crucial for (1) since they reflect the shape of the level sets of Γ. The Calderón-Zygmund covering lemma basically exploits the following property: *every cube of side r can be covered by at most 2^N non overlapping cubes of side $\frac{r}{2}$.*

At the beginning of the '70s, Coifman and Weiss [CoW] extended the Calderón-Zygmund covering lemma to general *quasimetric spaces (X,d) of homogeneous type.* Let X be a non empty set and let $d : X \times X \to [0,\infty[$. On says that (X,d) is a quasimetric space if

(i) $d(x,y) > 0$ if $x \neq y$,
(ii) $d(x,y) \le C(d(x,z) + d(z,y))$ and $d(x,y) \le Cd(y,x)$, for a suitable $C > 0$ independent of x and y.

The quasimetric space (X,d) is said to be of homogeneous type if, moreover,

(iii) there exists $M > 0$ such that every d-ball of radius r contains at most M d balls of radius $\frac{r}{2}$, M independent of r.

Important examples of spaces of homogeneous type are the *doubling quasi metric spaces.* Precisely, (X,d,μ) is a doubling quasi metric space if (X,d) satisfies the previous properties $(i),(ii)$ and if μ is a nonnegative measure such that the open d-balls are μ-measurable and satisfy the following *doubling property*:

$$0 < \mu(B(x,2r) \;\le A\;\mu(B(x,r)),$$

for every $x \in X$ and $r > 0$, $A > 0$ independent of x and r.

One can expect that Calderón-Zygmund's technique applies to "any" linear second order PDE endowed with a fundamental solution having level sets shaped as the balls of a doubling quasi metric space.

In 1963, before the work of Coifman and Weiss, Calderón-Zygmund's theory was extended by B.F. Jones [Jo] to the heat operator in $\mathbb{R}^{N+1}$

$$H := \Delta - \partial_t.$$

It is well known that the Gauss-Weierstrass kernel

$$\Gamma(x,t) := \begin{cases} 0, & \text{if } t \le 0 \\ (4\pi)^{-\frac{N}{2}} \exp(-\frac{|x|^2}{4t}), & \text{if } t > 0 \end{cases}$$

is the fundamental solution of H with pole at the origin. Moreover, the function

$$d((x,t),(x',t')) = ((x-x')^4 + (t-t')^2)^{\frac{1}{4}}$$

is a metric in $\mathbb{R}^{N+1}$ which is *translation invariant* and *homogeneous of degree two* with respect to the anisotropic dilations

$$\delta_r(x,t) \;=\; (r\,x, r^2\,t).$$

The heat operator has similar properties: this implies that the level sets of its fundamental solution $\Gamma(x-x', t-t')$ have a shape comparable with the one of the d-balls.

In 1966, E.B. Fabes and N. Rivière [FabR] extended Jones' work: they studied singular integrals involving kernels of the kind $K(x-y), x, y \in \mathbb{R}^N$, with K homogeneous with respect to general anisotropic dilations

$$\delta_r\,(x_1,\ldots,x_N) = (r^{\sigma_1}\,x_1,\ldots,r^{\sigma_N}\,x_N).$$

Calderón-Zygmund's covering lemma, and its applications to PDE's and Harmonic Analysis, have been extended in hundreds of directions: we directly refer to Stein's monograph [St] for a wide and deep exposition of the subject, and a huge list of references. For our purposes, here we only want to mention the 1974 work by Folland [Fo], dealing with the application of the theory of singular integrals to the study of linear second order PDO's with underlying *homogeneous Lie group structures*.

To put Folland's results in the right perspective, we need to recall the celebrated notion of Hörmander's operator in $\mathbb{R}^N$. The partial differential operator

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0$$

is called a Hörmander operator if

$$X_j = \sum_{k=1}^{N} a_j^k\,\partial_{x_k} \;\;\equiv\;\; (a_j^1,\ldots,a_j^N)^T,\; j=0,1,\ldots,m,$$

with $a_j^k \in C^\infty(\mathbb{R}^N)$ and

$$\dim(\mathcal{L}\mathrm{ie}\{X_0,X_1,\ldots,X_m\}(x)) = N, \quad \forall\, x\in\mathbb{R}^N. \tag{H}$$

This is the well known Hörmander's rank condition: it implies the *hypoellipticity* of $\mathcal{L}$, see [H]. A considerable amount of work has been devoted to Hörmander's operators, but apart from the fundamental papers by Folland [Fo] and by Rothschild and Stein [RS], mainly to the *sum of squares*

$$\sum_{j=1}^{m} X_j^2, \tag{2}$$

and to their *heat-type counterpart*

$$\sum_{j=1}^{m} X_j^2 - \partial_t. \tag{3}$$

Prototypical examples are, respectively, the *sub-Laplacians* and the *heat operators* on stratified Lie groups. Among the main contributors to the theory of operators of types (2) and (3) we want to mention Sánchez-Calle [Sa], Fefferman and Sánchez-Calle [FeS], Jerison and Sánchez-Calle [JeS], Kusuoka and Strook [KuS1][KuS2][KuS3], Varopoulos, Saloff-Coste and Coulhon [VSC].

As a matter of fact, Hörmander's operators in the general form

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0$$

with the drift term X_0, which is not merely a derivative along a constant direction, are of great interest. Indeed, they appear in several settings, both theoretical and applied, for example, in the following instances:

(a) as Kolmogorov operator of stochastic equations (see Da Prato and Zabczyk [DaZ1],[DaZ2]);
(b) in PDE's models in Diffusion Theory and in Finance (see Lanconelli, Pascucci and Polidoro [LaPP], and Pascucci [Pa]);
(c) in Computer Vision (see Mumford [Mu]);
(d) Curvature Brownian motion (see Wung, Zhou, Masle and Chirikjian [WZMC]);
(e) Phase-noise Fokker-Planck equations (see August and Zucker [AZ]).

A few examples of *complete* Hörmander operators are the following.

EXAMPLE 1. *Degenerate Ornstein-Uhlenbeck operator.* It is well known, the classical Ornstein-Uhlenbeck operator in $\mathbb{R}^N$ given by $\Delta + x \cdot \nabla$ (see [OU]). Now, let us consider in $\mathbb{R}^{2n}$ the linear second order PDO

$$\mathcal{A} = \Delta_{\mathbb{R}^n} + \sum_{j=1}^{n} (x_{n+j}\partial_{x_j} + x_j \partial_{x_{n+j}}) =: \Delta_{\mathbb{R}^n} + Y_0.$$

We call $\mathcal{A}$ a *degenerate Ornstein-Uhlenbeck operator* which is the "stationary" counterpart of the Kolmogorov-Fokker-Planck operator in $\mathbb{R}^{2n} \times \mathbb{R}$

$$\mathcal{L} := \mathcal{A} - \partial_t.$$

$\mathcal{L}$ is a prototypical operator of the ones introduced by Kolmogorov in 1934 in studying diffusion processes from a probabilistic point of view (see [Kol]). If we let $X_j := \partial_{x_j}, j = 1, \dots n$, and

$$Y := Y_0 - \partial_t = \sum_{j=1}^{n} (x_{n+j}\partial_{x_j} + x_j \partial_{x_{n+j}}) - \partial_t,$$

then we can write

$$\mathcal{A} = \sum_{j=1}^{n} X_j^2 + Y_0, \quad \text{and} \quad \mathcal{L} = \sum_{j=1}^{n} X_j^2 + Y.$$

Moreover, since $[X_j, Y_0] = [X_j, Y] = \partial_{x_{n+j}}$, for $j = 1, \dots, n$, we have

$$\text{Lie}\,\{X_1, X_2, ..., X_n, Y_0\} = \text{span}\{\partial_{x_1}, \dots, \partial_{x_n}, \partial_{x_{n+1}}, \dots, \partial_{x_{2n}}, Y_0\}$$

so that,

$$\text{Lie}\{X_1, X_2, ..., X_n, Y\} = \text{span}\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{x_{n+1}}, \ldots, \partial_{x_{2n}}, Y\}.$$

Then, $\mathcal{A}$ and $\mathcal{L}$ are Hörmander operators in $\mathbb{R}^{2n}$ and in $\mathbb{R}^{2n+1}$, respectively.

EXAMPLE 2 (Forward and backward Mumford operators). In 1994, D. Mumford [Mu] introduced the following partial differential operators of Fokker-Planck type:

$$\mathcal{M}_f = \partial^2_{x_1} + \cos x_1 \, \partial_{x_2} + \sin x_1 \, \partial_{x_3} - \partial_t, \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R},$$
$$\mathcal{M}_b = (x_2\partial_{x_1} - x_1\partial_{x_2} + \partial_{x_3})^2 + \partial_{x_1} + \partial_t \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}.$$

$\mathcal{M}_f$ and $\mathcal{M}_b$ are called *forward* and *backward* Mumford operators, respectively. They appear in a mathematical model created by Mumford in Computer Vision [M]. It is quite easy to verify that $\mathcal{M}_f$ and $\mathcal{M}_b$, which are of the type *sum of squares plus drift*, satisfy the Hörmander rank condition. Partial differential operators of Fokker-Planck type, very similar to the Mumford ones, have been also considered in the papers by Wung, Zhou, Masle and Chirikjian [WZMC], and by August and Zucker [AZ], quoted above.

EXAMPLE 3 (Non-autonomous Kolmogorov operators). Da Prato and Lunardi recently studied in [DaL2] a class of non-autonomous Kolmogorov operators with periodic coefficients containing, in particular, the following one:

$$\mathcal{L} = \partial^2_{x_1} + (\cos t \, \partial_{x_2} + \sin t \, \partial_{x_3})^2 - (\partial_t + x_1\partial_{x_4}) \quad \text{in} \quad \mathbb{R}^4 \times \mathbb{R}.$$

It is easily seen that $\mathcal{L}$ falls into the class of general Hörmander operators.

Let us now come back to the PDO operators studied by Folland in [Fo]. They are operators of the Hörmander form:

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0, \quad \text{in} \quad \mathbb{R}^N$$

and such that

(i) There exists a Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ such that the X_j's are left translation invariant on $\mathbb{G}$,

(ii) There exists a family of anisotropic dilations in $\mathbb{R}^N$, like the ones already considered by Fabes and Rivière,

$$\delta_r\,(x_1, \ldots, x_N) = (r^{\sigma_1}\, x_1, \ldots, r^{\sigma_N}\, x_N),$$

compatible with $\circ$, and such that $X_1, \ldots, X_m$ are δ_r-homogeneous of degree one while X_0 is δ_r homogeneous of degree two.

When we say that the family $(\delta_r)_{r>0}$ is compatible with $\circ$ we mean that

$$\delta_r(x \circ y) = \delta_r(x) \circ \delta_r(y) \quad \text{and} \quad \delta_r(x^{-1}) = (\delta_r(x))^{-1},$$

and in such case one says that $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_r)$ is a *homogeneous Lie group*.

We stress that the classical Δ and $H = \Delta - \partial_t$ belong to the Folland class. The Lie group underlying these operators is the Euclidean one. The relevant dilations are $\delta_r(x) = r\,x$ and $\delta_r(x,t) = (r\,x,\, r^2 t)$, respectively.

Using the theory of singular integrals in homogeneous Lie groups, Folland [Fo] proved the following L^p estimates:

$$\|X_i\, X_j\, u\|_{L^p(\mathbb{R}^N)} \le \|\mathcal{L}u\|_{L^p(\mathbb{R}^N)}, \quad u \in C_0^\infty(\mathbb{R}^N),$$

$i, j = 1, \dots, m,\ \ 1 < p < \infty.$

We emphasize that, in this general context, the usual Euclidean *norm* and *distance* are replaced respectively by

$$\|(x_1, \dots, x_N)\| \;=\; \sum_{j=1}^{N} |x_j|^{\frac{1}{\sigma_j}}, \;\text{ and }\; d(x,y) \;=\; \|y^{-1} \circ x\|.$$

We close this section by stressing that the degenerate Ornstein-Uhlenbeck operator

$$\mathcal{A} = \Delta_{\mathbb{R}^n} + \sum_{j=1}^{n} (x_{n+j}\partial_{x_j} + x_j \partial_{x_{n+j}})$$

in not contained in the Folland class. Indeed, there are no dilations in $\mathbb{R}^N$ making $\mathcal{A}$ homogenous of some positive degree. On the other hand, L^p a priori estimates similar to the previous ones, like

$$\|\partial_{x_i,x_j}\, u\|_{L^p(\mathbb{R}^{2n})} \le c\{\|\mathcal{A}u\|_{L^p(\mathbb{R}^{2n})} \;+\; \|u\|_{L^p(\mathbb{R}^{2n})}\}, \quad i, j \;= 1, \dots, n,$$

are crucial in studying the initial value problem for the Kolmogorov-Fokker-Planck equation

$$\mathcal{L} := \mathcal{A} - \partial_t \;=\; \Delta_{\mathbb{R}^n} + \sum_{j=1}^{n} (x_{n+j}\partial_{x_j} + x_j \partial_{x_{n+j}}) \;-\; \partial_t.$$

With Bramanti, Cupini and Priola, we proved such estimates using a new technique, that seems to work also for more general classes of complete Hörmander operators.

2. A general class of Ornstein-Uhlenbeck operators

In a paper with S. Polidoro [LaP] we studied general Ornstein-Uhlenbeck operators of the following form:

$$\mathcal{A} = div\,(A\nabla) + \langle x, B\nabla\rangle\,, \tag{4}$$

where A and B are constant $N \times N$ matrices, with $A \ge 0$, and $\langle\, ,\, \rangle$, ∇ denote, respectively, the inner product and the usual gradient in $\mathbb{R}^N$. When $A = B = I_N$, the identity $N \times N$ matrix, $\mathcal{A}$ becomes the classical Ornstein-Uhlenbeck operator. In this general form $\mathcal{A}$ is degenerate elliptic. Its regularity properties are determined by the matrix

$$C(t) = \int_0^t E(s) A E^T(s) ds, \quad E(s) = \exp(-sB^T).$$

In [LaP] we proved that the following conditions are equivalent:

(i) $C(t) > 0$ for any $t > 0$;

(ii) $\operatorname{rank}\operatorname{Lie}(X_1, X_2, ..., X_N, Y_0) = N$, at any $x \in \mathbb{R}^N$,

where $Y_0 = \langle x, B\nabla\rangle$ while, if $A = (a_{i,j})_{i,j=1}^N$, $X_j = \sum_{j=1}^N a_{ij}\partial_{x_j}$. We also proved that $C(t) > 0$ for any $t > 0$ if and only if, with respect to a suitable basis of $\mathbb{R}^N$, the matrices A and B take the following form

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{bmatrix}, \tag{5}$$

where $A_0 = (a_{ij})_{i,j=1}^{p_0}$ is a $p_0 \times p_0$ symmetric and positive definite matrix, with $p_0 \leq N$. Moreover, for every $j = 1, \dots, r$ the block B_j has dimension $p_{j-1} \times p_j$ and rank p_j, $j = 1, 2, \dots, r$. Finally: $p_0 \geq p_1 \geq \dots \geq p_r \geq 1$ and $p_0 + p_1 + \dots + p_r = N$.

We notice that the prototypical Ornstein-Uhlenbeck

$$\mathcal{A} = \Delta_{\mathbb{R}^n} + \sum_{j=1}^{n} (x_{n+j}\partial_{x_j} + x_j\partial_{x_{n+j}})$$

can be written as in (4) by taking $N = 2n$ and

$$A = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

In the very recent work [BrCLP], with M. Bramanti, G. Cupini and E. Priola, we proved the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be the operator in (4) and $p \in (1, \infty)$. Then, for every $u \in C_0^\infty(\mathbb{R}^N)$:*

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\} \text{ for } i, j = 1, 2, ..., p_0 \tag{6}$$

$$\|Y_0 u\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\}.$$

Moreover, for every $\alpha > 0$,

$$\left| \left\{ x \in \mathbb{R}^N : \left| \partial^2_{x_i x_j} u(x) \right| > \alpha \right\} \right| \leq \frac{c}{\alpha} \left\{ \|\mathcal{A}u\|_{L^1(\mathbb{R}^N)} + \|u\|_{L^1(\mathbb{R}^N)} \right\}$$

$$\left| \left\{ x \in \mathbb{R}^N : |Y_0 u(x)| > \alpha \right\} \right| \leq \frac{c}{\alpha} \left\{ \|\mathcal{A}u\|_{L^1(\mathbb{R}^N)} + \|u\|_{L^1(\mathbb{R}^N)} \right\}$$

The constant c is independent of u.

Before proceeding, some motivational and bibliographical remarks are in order.

REMARK 1. The operator $\mathcal{A}$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup, the Markov semigroup associated to the stochastic differential equation:

$$d\xi(t) = B^T\xi(t)dt + \sqrt{2}\,A_0^{1/2}dW(t), \quad t > 0, \quad \xi(0) = x,$$

where $W(t)$ is a standard Brownian motion taking values in $\mathbb{R}^{p_0}$.

It describes the random motion of a particle in a fluid. $\mathcal{A}$ and its parabolic counterpart $\mathcal{A} - \partial_t$ have several interpretations, in physics and finance: we directly refer to the survey papers [LaPP] and [P] for a presentation and a bibliography on this subjects.

REMARK 2. Our result is a first step towards existence and uniqueness for the Cauchy problem for $\mathcal{A}$, as well as towards the characterization of the domain of $\mathcal{A}$ in L^p spaces.

REMARK 3. Global estimates for $\mathcal{A}$ in Hölder spaces where proved by Da Prato and Lunardi, in the non degenerate case ($p_0 = N$), and by Lunardi in the degenerate case, see [DaL1] and [Lu1]. Metfune, Prüss, Rhandi and Schnaubelt proved global L^p estimates, for the non degenerate Ornstein-Uhlenbeck operator [MePRS]. Global estimates in L^2 with respect to invariant Gaussian measure where proved by Lunardi and by Farkas and Lunardi, in the non degenerate and in the degenerate case, respectively, see [Lu2], [FaL]. In all these papers, a semigroups approach is used. More recently, Di Francesco and Polidoro [DiP] proved local Hölder estimates for $\mathcal{A}$ as a consequence of analogous estimates for its parabolic counterpart $\mathcal{L} =: \mathcal{A} - \partial_t$. They used a direct approach based on the properties of a *Lie group structure* underlying the operator $\mathcal{L}$.

Our approach to the L^p estimates is closer to the one used by Di Francesco and Polidoro. Indeed, our starting idea for proving (6), is to look at $\mathcal{A}$ as the stationary counterpart of the Kolmogorov-Fokker-Planck operator in $\mathbb{R}^N \times \mathbb{R}$

$$\mathcal{L} := \mathcal{A} - \partial_t = div\,(A\nabla) + \langle x, B\nabla\rangle - \partial_t = div\,(A\nabla) + Y$$

where $Y = \langle x, B\nabla\rangle - \partial_t$. For this operator we proved the estimate

$$\left\|\partial^2_{x_i x_j} u\right\|_{L^p(S)} \leq c\,\|\mathcal{L}u\|_{L^p(S)} \quad \text{for } i,j = 1,2,...,p_0 \tag{7}$$

for any $u \in C_0^\infty(S)$, where S is the strip $\mathbb{R}^{2n} \times]-1,1[$. From this inequality the L^p estimate (6) for $\mathcal{A}$ trivially follows.

The crucial properties we used to prove (7) are the following ones:

(I) There exists a composition law in in $\mathbb{R}^N \times \mathbb{R}$ making $\mathbb{K} = (\mathbb{R}^N \times \mathbb{R}, \circ)$ a Lie group such that $\mathcal{L}$ is *left translation invariant* on $\mathbb{K}$. The composition law is given by

$$(x,t) \circ (x',t') = (x' + E(t')x, t + t')$$

and first explicitly appeared in [LaP].

(II) $\mathcal{L}$ has a global fundamental solution taking the form

$$\Gamma(z,\zeta) = \gamma(\zeta^{-1} \circ z),$$

where γ is nonnegative, smooth out of the origin, and summable in a neighborhood of infinity in the strip S, together with its second derivatives $\partial^2_{x_i x_j}\gamma, \quad i,j =$

$1, \dots p_0$. The function is explicitly given by

$$\gamma(z) = \begin{cases} 0 \text{ for } & t \le 0 \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\, x, x\right\rangle - t\mathrm{Tr}B\right) & \text{for } t > 0. \end{cases}$$

These ingredients allow to get the following representation formula

$$\partial^2_{x_i x_j} u \;=\; -(PV)(\partial^2_{x_i x_j}\gamma \;*\; \mathcal{L}u) \;+\; c_{i.j}\,\mathcal{L}u$$

where

(i) $*$ is convolution on $\mathbb{K}$ and
(ii) $c_{i.j}$ are suitable real constants.

We also used another crucial fact: the strip S can be endowed with a structure of what we called *local quasimetric space* whose balls "fit" the level sets of $\partial^2_{x_i x_j}\gamma$.

Given a nonempty set X and a function $d : X \times X \to [0, \infty[$ we say that (X, d) is a local quasi metric space if

(a) $d(x, y) > 0$ if $x \neq y$
(b) $d(z, \zeta) \le C\, d(\zeta, z)$ if $d(z, \zeta) \le 1$
(c) $d(z, \zeta) \le C\,(\,d(z, w) + d(w, \zeta))$ if $d(z, w)\,,\, d(w, \zeta) \le 1$.

The starting idea to endow S with a metric whose balls fit the level set of the fundamental solution Γ is to link the properties of $\mathcal{L} = \mathcal{A} - \partial_t$ with the ones of its *principal part* $\mathcal{L}_0 = \mathcal{A}_0 - \partial_t$, with $\mathcal{A}_0 := div(A\,\nabla) + \langle x, B_0\,\nabla\rangle$ and B_0 is obtained by annihilating the $\star$ blocks in B:

$$B_0 = \begin{bmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

L_0 is homogeneous of degree 2 w.r. to the group of dilations in $\mathbb{R}^{N+1}$

$$\delta_r := \mathrm{diag}\,(r\, I_{p_0}, r^3\, Ip_1, \dots, r^{2r+1}\, I_{p_r}, r^2).$$

Then, if $\|\cdot\|$ is any norm δ_r-homogeneous of degree 1, then we let

$$d(z, \zeta) := \|\, \zeta^{-1} \circ z\, \|$$

where $\zeta^{-1} \circ z$ is the operation on $\mathbb{K}$, the Lie group related to $\mathcal{L}$. We proved that (S, d) is a local quasi metric space. Moreover, by exploiting the Lie left invariance of d and the homogeneity of $\|\cdot\|$ with respect to δ_r, we easily recognized that

$$|B_d(z_0, r)| = r^Q |B_d(0, 1)|, \quad \text{where} \quad Q = p_0 + 3p_1 + \dots + (2r+1)p_r + 2.$$

Hereafter, if $B \subset S$, $|B|$ stands for the Lebesgue measure of B. Moreover B_d denotes the d-ball. From these last identities we then obtained the following doubling property

$$|B_d(z_0, 2r) \cap S| \;\le\; A\,|B_d(z_0, r) \cap S|, \quad \text{for every} \quad z_0 \in S \text{ and } r > 0.$$

All these facts allow us to use a very recent result by M. Bramanti on singular integrals on nonhomogeneous local quasimetric spaces (see [Br1]), to obtain a "local version" of the L^p estimate (7). Then, again exploiting the left invariance of $\mathcal{L}$ on the Lie group $\mathbb{K}$, together with a suitable *covering lemma* in local quasi metric spaces endowed with a doubling measure, we finally get the complete proof of (7).

We would like to mention that the result of M. Bramanti on which our L^p estimates rest, uses several ideas and results from the deep paper [NTV] by F. Nazarov, S. Treil, A. Volberg, on the Calderón-Zygmund theory in nonhomogeneous metric spaces. In his extension of the results in [NTV], Bramanti also used some ideas from a paper by E. Fabes, I .Mitrea, and M. Mitrea [FaMM].

We would like to close this section with a comparison remark. The operators $\mathcal{A}$ and $\mathcal{L}$ are contained in the general class of the Hörmander operators. However the previous L^p estimates cannot be derived from Folland's and Rothschild and Stein's works. Indeed, Folland considered operators with underlying structures of *homogeneous* Lie groups. Our group $\mathbb{K}$ is non-nilpotent, hence nonhomogeneous.

Rothschild and Stein's work [RS] only contains *local estimates*. Our estimates are global (on $\mathbb{R}^N$) or related to unbounded domains (the strip S).

3. General Hörmander operators: construction of Lie groups and existence of global fundamental solutions

To get L^p a priori estimates for general Hörmander operators in $\mathbb{R}^N$ by overcoming the restriction of dealing with homogeneous groups (as in Folland's work) or with local estimates (as in Rothschild and Stein's work) one can try to use the techniques we introduced for degenerate Ornstein-Uhlenbeck operators.

This would require to deal with operators

(i) left translation invariant on some Lie group in $\mathbb{R}^N$ and

(ii) equipped with a global fundamental solution left invariant on the group.

For these reasons, we addressed the following two questions, of some interest on their own.

Problem (P1) Given a system $\{X_0, X_1, \dots, X_m\}$ of vector fields in $\mathbb{R}^N$ satisfying the Hörmander rank condition

$$\dim(\mathcal{L}\text{ie}\{X_0, X_1, \dots, X_m\}(x)) = N, \quad \forall\, x \in \mathbb{R}^N \tag{H}$$

is there a Lie Group $\mathbb{G} = (\mathbb{R}^N, \circ)$ (not necessarily homogeneous nor nilpotent) such that the X_j's are left invariant on $\mathbb{G}$ (so that $\mathcal{L} = \sum_{j=1}^m X_j + X_0$ is left invariant on $\mathbb{G}$)?

Problem (P2) Does there exist a function $\gamma \in C^\infty(\mathbb{R}^N \setminus \{0\})$, $\gamma \geq 0$ such that

$$\Gamma(x, y) = \gamma(y^{-1} \circ x), \;\; x, y \in \mathbb{R}^N, \;\; x \neq y$$

is a fundamental solution for $\mathcal{L}$?

We have solved problems **(P1)** and **(P2)** for a class of Hörmander operators containing, e.g.,

(1) The Kolmogorov-Fokker-Planck operator $\mathcal{L}$, already studied with S. Polidoro, evolution counterpart of the previous degenerate Ornstein-Uhlenbeck operators:

$$\mathcal{L} := div\,(A\nabla) + \langle x, B\nabla\rangle - \partial_t;$$

(2) The Fokker-Planck operators introduced by Mumford in computer vision:

$$\mathcal{M}_f = \partial^2_{x_1} + \cos x_1 \partial_{x_2} + \sin x_1 \partial_{x_3} - \partial_t$$

$$\mathcal{M}_b = (x_2\partial_{x_1} - x_1\partial_{x_2} + \partial_{x_3})^2 + \partial_{x_1} + \partial_t;$$

(3) Non-autonomous Kolmogorov operators with periodic coefficients like

$$\mathcal{L} = \partial^2_{x_1} + (\cos t\, \partial_{x_2} + \sin t\, \partial_{x_3})^2 - (\partial_t + x_1\partial_{x_4})$$

contained in the class recently studied by Da Prato an Lunardi.

In order to show our answer to problem **(P1)**, we need to recall some notation and basic results from Lie groups theory in $\mathbb{R}^N$. The set

$$T(\mathbb{R}^N) := \{X = \sum_{k=1}^{N} a^k_j\, \partial_{x_k}\} \equiv \{(a^1, \dots, a^N)^T\} = C^\infty(\mathbb{R}^N, \mathbb{R}^N)$$

is a Lie algebra when endowed with its natural linear structure and the Lie bracket operation

$$[X, Y] := XY - YX.$$

The set $\mathfrak{a} \subset T(\mathbb{R}^N)$ is called a Lie algebra of vector fields in $\mathbb{R}^N$ if $\mathfrak{a}$ is *linear* and closed with respect to $[\cdot, \cdot]$.

Let $\mathfrak{a}$ be a Lie algebra of vector fields in $\mathbb{R}^N$ and let $x \in \mathbb{R}^N$. Then

$$\mathfrak{a}(x) = \{X(x) : X \in \mathfrak{a}\}$$

is a linear subspace of $\mathbb{R}^N$. Its dimension is called the *rank* of $\mathfrak{a}$ at x:

$$\text{rank } \mathfrak{a}(x) := \dim\, \mathfrak{a}(x).$$

We stress that, in general, $\dim\, \mathfrak{a}(x) \le \dim \mathfrak{a}$ for every $x \in \mathbb{R}^N$. The Lie algebra generated by a family $\mathbf{Z} \subset T(\mathbb{R}^N)$ is defined as follows:

$$\text{Lie } \mathbf{Z} := \text{intersection of the Lie algebras containing } \mathbf{Z}.$$

Given $X \in T(\mathbb{R}^N)$ we denote with $\exp(t\, X)(x)$ the solution to the Cauchy problem

$$\begin{cases} \dot\gamma = X(\gamma) \\ \gamma(0) = x. \end{cases}$$

A vector field $X \in T(\mathbb{R}^N)$ is said to be *complete*, if $\exp(t\, X)(x)$ is well defined for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

Now, we are able to state a crucial *necessary condition for* (**(P1)**).

Let $\mathfrak{a} := \mathcal{L}\text{ie}\{X_0, X_1, \dots, X_m\}$ and assume (P1) is solvable. Then

(H1) Every $X \in \mathfrak{a}$ is complete and

$$\dim\, \mathfrak{a} \;=\; N$$

(see e.g. [BLU], Corollary 1.2.23 and Proposition 1.2.7).

Condition (H1) has the following important consequence: the map

$$\text{Exp} : \mathfrak{a} \to \mathbb{R}^N, \quad \text{Exp}(X) = \exp(tX)(0)|_{t=1}$$

is well defined, and smooth. Moreover, since $\text{Exp}(X) = X(0) + o(X)$ as $X \to 0$, Exp is a *diffeomorphism* from W, a neighborhood of 0 in $\mathfrak{a}$ to V, a neighborhood of 0 in $\mathbb{R}^N$. Then, if we let

$$\text{Log} := (\text{Exp}/W)^{-1}$$

the following maps are well defined for every $x, y \in V$:

$$x \circ y := \exp(\text{Log}(y))(x), \qquad x^{-1} := \text{Exp}(-\text{Log}(x)). \tag{8}$$

We explicitly remark that, if it would exist a composition law $\circ$ making the X_j' left invariant, then $\circ$ would satisfy (8).

Our main result regarding problem (**P1**) is the following theorem, obtained in collaboration with A. Bonfiglioli in [BoL].

Theorem 3.1. *Let $\mathbf{X} = \{X_0, X_1, \ldots, X_m\} \subset T(\mathbb{R}^N)$ be a family of real analytic vector fields satisfying the Hörmander rank condition and suppose that $\mathfrak{a} := Lie(\mathbf{X})$ satisfies (H1). Assume the maps*

(i) $(x, y) \mapsto x \circ y$ *and*
(ii) $x \mapsto x^{-1}$

defined in (8) admit a real analytic extension to all $\mathbb{R}^N \times \mathbb{R}^N$ and to all $\mathbb{R}^N$, respectively.

Then $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a Lie group whose Lie algebra is $\mathfrak{a}$.

We proved the existence of a *global* fundamental solution for a Hörmander operator under the following assumptions (L1) and (L2). We say that the operator

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 \tag{9}$$

satisfies (L1) if one of the following conditions holds:

(i) $\mathcal{L}$ is *asymptotically elliptic* along $\nu \in \mathbb{R}^N$, $\nu \neq 0$, i.e.

$$\sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 > 0, \;\; \forall\, x, \;\xi \;\in \mathbb{R}^N, \;|x| > M, \;|\xi - \nu| < \delta$$

for suitable positive constants M and δ;

(ii) $\mathcal{L}$ is homogeneous with respect to to a group of dilations $(\delta_r)_{r>0}$, and not totally degenerate at the origin, i.e.

$$\sum_{j=1}^{m} \langle X_j(0), \xi\rangle^2 > 0, \;\text{ for a suitable }\; \xi \neq 0.$$

We say that the operator (9) satisfies (L2) if it is of *parabolic type*, i.e., if it can be written as

$$\mathcal{L} = \sum_{i,j=1}^{N-1} a_{i,j}(x)\partial^2_{x_i,x_j} + \sum_{j=1}^{N-1} b_j(x)\partial_{x_j} \pm \partial_{x_N}.$$

With the previous definitions at hand, we can state our main result regarding problem (**P2**). Its proof is contained in [BoL].

Theorem 3.2. *Let $\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0$ be a Hörmander operator in $\mathbb{R}^N$ satisfying conditions (L1) and (L2). Then there exists a function Γ s.t.*

(i) *The map $(x, y) \mapsto \Gamma(x, y)$ is defined, non-negative and smooth away from the set $\{(x, y) \in \mathbb{R}^N : x = y\}$.*

(ii) *For every fixed $y, x \in \mathbb{R}^N$, $\Gamma(\cdot, y)$ and $\Gamma(x, \cdot)$ are locally integrable and, for every test function φ:*

$$\mathcal{L} \int_{\mathbb{R}^N} \Gamma(\cdot, y)\, \varphi(y)\, dy = -\varphi = \int_{\mathbb{R}^N} \Gamma(\cdot, y)\, \mathcal{L}\varphi(y)\, dy.$$

(iii) *If $\mathcal{L}$ is left invariant on a Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$, and the Lebesgue measure is left invariant on $\mathbb{G}$, then*

$$\Gamma(x, y) = \Gamma(y^{-1} \circ x, 0) \text{ for every } x, y \in \mathbb{R}^N.$$

Let us now show examples to which our main theorems apply.

EXAMPLE 1. The Kolmogorov-Fokker-Planck operator

$$\mathcal{A} = div\,(A\nabla) + \langle x, B\nabla\rangle - \partial_t$$

satisfies all the hypotheses of Theorem 3.1 and of Theorem 3.2.

It is left translation invariant on the Lie group $\mathbb{K} = (\mathbb{R}^N \times \mathbb{R}, \circ)$ with the composition law

$$(x, t) \circ (y, \tau) = (y + E(\tau)x, t + \tau),$$

where $E(\tau) = \exp(-\tau B)$.

Then we rediscovered a result first appeared in [LaPo].

EXAMPLE 2. The forward Mumford operator

$$\mathcal{M}_f = \partial_{x_1}^2 + \cos x_1 \partial_{x_2} + \sin x_1 \partial_{x_3} - \partial_t$$

satisfies all the hypotheses of Theorem 3.1. It is left translation invariant on the Lie group $\mathbb{M}_f = (\mathbb{R}^3 \times \mathbb{R}, \circ)$ with composition law

$$(x, t) \circ (y, \tau) = (x_1 + y_1,\ x_2 + y_2 \ \cos x_1 - y_3 \ \sin x_1,\ x_3 + y_2 \ \sin x_1 + y_3 \ \cos x_1,\ t + \tau).$$

REMARK. We want to emphasize that, in this case, the relevant Exp map is *not* a global diffeomorphism.

Since the operator $\mathcal{M}_f$ is of parabolic type, and elliptic along $(1, 0, \ldots, 0)$, by Theorem 3.2 it has a global fundamental solution, which is left invariant on $\mathbb{M}_f$ because, as it can be easily verified, the Lebesgue measure is left invariant on this group.

EXAMPLE 3. The backward Mumford operator

$$\mathcal{M}_b = (x_2\partial_{x_1} - x_1\partial_{x_2} + \partial_{x_3})^2 + \partial_{x_1} + \partial_t$$

is left translation invariant on the Lie group $\mathbb{M}_b = (\mathbb{R}^3 \times \mathbb{R}, \circ)$ with composition law

$$(x, t) \circ (y, \tau) = (x_1 \ \cos y_3 + x_2 \ \sin y_3 + y_1,\ x_2 \ \cos y_3 - x_1 \ \sin y_3 + y_2,\ x_3 + y_3,\ t + \tau).$$

This Lie group is isomorphic to $\mathbb{M}_f$. The operator $\mathcal{M}_b$ has a global fundamental solution which is left invariant on $\mathbb{M}_f$.

EXAMPLE 4. The Da Prato-Lunardi-type operator, the non autonomous Kolmogorov operator with periodic coefficients

$$\mathcal{L} = \partial_{x_1}^2 + (\cos t\ \partial_{x_2} + \sin t\ \partial_{x_3})^2 - (\partial_t + x_1 \partial_{x_4}),$$

satisfies all the hypotheses of our main theorems. It is left translation invariant on the Lie group $\mathbb{G} = (\mathbb{R}^4 \times \mathbb{R}, \circ)$ with composition law

$$(x,t)\circ(y,\tau) = (x_1+y_1, x_2+y_2\cos t - y_3 \sin t, x_3+y_2\cos t + y_3 \sin t, x_4+y_4+\tau x_1, t+\tau).$$

$\mathcal{L}$ is of parabolic type, and elliptic along $(1,0,\dots,0)$. Then it has a global fundamental solution, which is left invariant on $\mathbb{G}$.

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