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# TRIALITY AND THE NORMAL SECTIONS OF CARTAN'S ISOPARAMETRIC HYPERSURFACES

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Abstract. The present paper is devoted to study the algebraic sets of normal sections of the so called Cartan's isoparametric hypersurfaces $M_{\mathbb{R}}$, $M_{\mathbb{C}}$, $M_{\mathbb{H}}$ and $M_{\mathbb{O}}$ of complete flags in the projective planes $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$. It presents a connection between "normed trialities" and the polynomial defining the algebraic sets of normal sections of these hypersurfaces. It contains also a geometric and topological description of these algebraic sets by means of the isotropy actions of the isoparametric hypersurfaces.

## 1. Introduction

In the present paper we indicate some properties of the algebraic sets of normal sections of the so called Cartan's isoparametric hypersurfaces on spheres (see [10] for background on the subject). In general, isoparametric hypersurfaces in spheres are interesting as geometric objects due to their many remarkable properties that have been rather profoundly studied by several authors (see the survey article [11] and references therein). Our interest in these objects comes from the possibility of using them as testing ground in the study of normal sections of submanifolds of Euclidean spaces. The study of submanifolds of Euclidean spaces by their normal sections is by no means a new method in Differential Geometry but for homogeneous submanifolds its interest was renewed by the well known result of B. Y. Chen, [4] (see also [10]).

The unit tangent vectors defining planar normal sections at a point $p$ of an arbitrary compact *spherical* (i.e. contained in a sphere) submanifold $M^n$ of a Euclidean space $\mathbb{R}^{n+k}$ form a real algebraic set $\widehat{X}_p[M]$ of the unit sphere in the tangent space of $M^n$ at $p$ and alternatively in the real projective space $\mathbb{R}P(T_p(M^n))$. This set was previously studied, for certain submanifolds (natural embedded manifolds of complete flags of a compact simple Lie group $G$) in several papers. These results have been summarized in the survey article [6].

In [10] the consideration of general isoparametric submanifolds was initiated and new results were obtained in the particular setting of *isoparametric hypersurfaces of spheres* (or equivalently isoparametric submanifolds of rank 2 in Euclidean spaces). A particularly interesting feature of isoparametric submanifolds is the

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fact that they are defined by polynomial functions which simplifies the problem of finding the equations defining the algebraic sets of normal sections. The reason for restricting our attention to homogeneous (in fact *extrinsically homogeneous*) isoparametric submanifolds is that, for them, the algebraic sets of normal sections associated to each point are in fact "independent" of the chosen point and so became objects associated to the manifold globally. Also in [10] the study is extended to the consideration of *all* normal sections of these submanifolds. It is shown there, that the tangent unit sphere at a given point is divided into algebraic sets determined by the "nature" of the normal section that they define. In particular, for the isoparametric hypersurfaces in the spheres known as Cartan's isoparametric hypersurfaces it is proven that the algebraic sets of tangent unit vectors defining *non-planar normal sections* are *smooth submanifolds* of the unit tangent spheres. It is then possible to separate normal sections at $p$ (in fact their unit tangent vectors) into a one parameter family of submanifolds of the unit tangent sphere with three singular members, one of which is the algebraic set of planar normal sections.

In the present paper we continue the study of these one parameter families for Cartan's isoparametric hypersurfaces obtaining a better understanding of their structure and properties. We observe that the algebraic sets of normal sections of these submanifolds, are defined by the rather special *"normed trialities"* described in [1, p. 35 ; Ch. 14-16] (see also [2, p. 159-163]). This seems to be an interesting fact that connects the normal sections of the four Cartan's hypersurfaces with the rather special phenomenon of triality. The transitive *groups* of Cartan's hypersurfaces (which are also those of the projective planes $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$) are the connected components of the groups of automorphisms of the Jordan algebras $H_3(F)$ ($F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) [8, p. 24] (see Section 3.1) and the isotropy subgroups are related to the corresponding *groups of automorphisms* of the trialities associated to the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

The algebraic sets of non-planar normal sections of Cartan's hypersurfaces have natural cohomogeneity 2 actions of the isotropy subgroups and they all have the same quotient spaces which allows us to describe with great detail their structure. The mentioned one-parameter family of these algebraic sets (in this particular case and in general for isoparametric hypersurfaces of spheres) seems to be an object that deserves attention and may be studied either in the sphere or considering its image in the associated real projective space. In some sense, they are a weak (or more general) version of isoparametric families.

The contents of the present paper are the following. In the next section we indicate basic facts from [10] required to describe the new results. In Section 3 we recall Cartan's isoparametric submanifolds and in its subsection 3.1 we present the construction of these submanifolds in a subspace of the Jordan algebras $H_3(F)$ describing their Cartan-Münzner polynomial. We also recall the polynomial $P(X)$ defining the algebraic set of unit vectors in tangent space which generate planar normal sections and Theorem 5 obtained in [10]. In Section 4 we mention basic facts from [1], [2] about trialities indicating the connection between normed trialities and the polynomial $P(X)$ defining our algebraic sets. We recall in Subsection 4.1 an interesting theorem due to B. Wilking (Theorem 6) [12] and describe some of its consequences relevant here (see(16)). It follows from these facts that the problem of describing the structure of the level sets $P^{-1}(d)$ for the manifolds $M_{\mathbb{H}}$ and $M_{\mathbb{O}}$ is reduced to study those in $M_{\mathbb{C}}$ (notation in Section 3). The next three sections, which constitute the core of the present paper, are devoted to the study of the level sets for $M_{\mathbb{C}}$. In Section 5 we indicate some particular notation to be used in the study of $M_{\mathbb{C}}$ and in Section 6 we study the algebraic set $\widehat{X}_E[M_{\mathbb{C}}] = P^{-1}(0)$ by means of the action of the torus, obtaining a description of its nature. In Section 7 we use similar methods to study the submanifolds $P^{-1}(d) \subset S^5$, for $d \in (0, m_o)$.

We present in Section 8 the case of $M_{\mathbb{R}}$ which is rather particular and in Section 9 the situation of $M_{\mathbb{O}}$ and $M_{\mathbb{H}}$. Finally in Section 10 we see that the highest (and lowest) level sets consist of a single orbit of the corresponding isotropy groups.

For any unexplained notation or terminology we refer the reader to [10].

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## 2. Normal Sections

By definition, *normal sections* are the curves obtained by cutting a submanifold $M^n$ of $\mathbb{R}^{n+k}$ with the affine subspace generated by a unit tangent vector and the normal space, at a given point $p$ of $M^n$. This curve can be given a $C^\infty$ parametrization around $p$ which is *regular* and can therefore be locally parametrized by arc-length. These curves are normal sections only at the point $p = \gamma(0)$ because as soon as we leave $p$ and move to $\gamma(s)$, the curve is, in general, no longer a normal section at $\gamma(s)$.

Let us recall that

**Definition 1.** *A curve $\gamma(s)$ parametrized by arc-length in the submanifold $M^n \subset \mathbb{R}^{n+k}$ is said to be planar at $p$ if $p = \gamma(0)$ and its first three derivatives $\gamma'(0)$, $\gamma''(0)$ and $\gamma'''(0)$ are* linearly dependent in $T_p(\mathbb{R}^{n+k})$.

Let $\alpha$ denote the second fundamental form of $M^n$ in $\mathbb{R}^{n+k}$ and let $(\overline{\nabla}_Y \alpha)$ denote its usual covariant derivative. One can prove the following basic fact ([10]):

**Lemma 2.** *Let $M^n$ be a compact submanifold of the sphere $S^{n+k-1} \subset \mathbb{R}^{n+k}$. The normal section $\gamma$ of $M$ at $p$ in the direction of $X \in T_p(M)$ is planar at $p$ if and only if the covariant derivative of the second fundamental form vanishes on the vector $X = \gamma'(0)$. That is, if and only if $X$ satisfies the equation $(\overline{\nabla}_X \alpha)(X, X) = 0$.*■

Given a point $p \in M$ we shall denote as in [10], [6],

$$\widehat{X}_p[M] := \{X \in S(T_p(M)) : \|X\| = 1, \gamma_X \text{ is planar at } p\}$$

or equivalently

$$\widehat{X}_p[M] = \left\{Y \in T_p(M) : \|Y\| = 1, \left(\overline{\nabla}_Y \alpha\right)(Y, Y) = 0\right\}. \tag{1}$$

This is then a real algebraic set. We call this the algebraic set of planar normal sections of $M$. Furthermore if $X \in \widehat{X}_p[M]$ then clearly $(-X)$ defines a planar normal section (the same curve in opposite direction) then one may identify $X$ with $(-X)$ and so obtain an algebraic set in the real projective space $\mathbb{R}P\left(T_p\left(M\right)\right)$ which we denote by $X_p\left[M\right]$. In principle this gives us a way to study sections planar at $p$.

We assume that $M$ is *compact* and *extrinsically homogeneous* [3, p.35] that is, for any two points $p, q \in M$ there is an isometry $g$ of $\mathbb{R}^{n+k}$ such that $g\left(M\right) = M$ and $g(p) = q$. Then, $\widehat{X}_p\left[M\right]$ does not "depend" on the point $p$. In fact if $p$ and $q$ are two points in $M^n$ and $g$ is an isometry of $\mathbb{R}^{n+k}$ such that $g\left(M\right) = M$ and $g(p) = q$. Then

$$\left(\overline{\nabla}_{\left(\left. g_* \right|_p X\right)} \alpha\right) \left(\left. g_* \right|_p X, \left. g_* \right|_p X\right) = \left. g_* \right|_p \left(\overline{\nabla}_X \alpha\right) \left(X, X\right),$$

and we clearly have that

$$\widehat{X}_q\left[M\right] = \widehat{X}_{g(p)}\left[M\right] = \left. g_* \right|_p \left(\widehat{X}_p\left[M\right]\right)$$

and we may free ourselves from the point $p$. This isomorphism obviously goes to the projective space.

In order to study the normal sections of $M^n$ in $\mathbb{R}^{n+k}$, it is convenient to consider the polynomials

$$P_j\left(X\right) = \left\langle \omega_j, \left(\overline{\nabla}_X \alpha\right)\left(X, X\right)\right\rangle, \quad j = 1, 2, \ldots, k. \tag{2}$$

where $\{\omega_1, \ldots, \omega_k\}$ is any basis of the normal space $T_p\left(M\right)^{\perp}$ at $p$. Obviously they define the algebraic set $\widehat{X}_p\left[M\right]$ by the conditions

$$P_j\left(X\right) = 0, \quad j = 1, 2, \ldots, k \quad \left\|X\right\| = 1.$$

These polynomials make sense on $T_p\left(M\right)$ but since, by definition, normal sections are generated by unit vectors we consider them, generally, on the unit sphere $S\left(T_p\left(M\right)\right)$.

We restrict our considerations to *isoparametric hypersurfaces in spheres*. A more general setting can be found in [10]. Let $M^n$ be an isoparametric hypersurface in $S^{n+1} \subset \mathbb{R}^{n+2}$. Form the theory of isoparametric hypersurfaces in spheres it follows that there exists a so called Cartan-Münzner polynomial $f : S^{n+1} \to \mathbb{R}$ such that

$$M = f^{-1}\left(0\right).$$

The reader is referred to the survey by G. Thorbergsson [11] where the development of the theory is described, indicating the sources in the abundant literature on the subject. Alternatively we may consider $M$ as a compact rank 2 isoparametric submanifold of $\mathbb{R}^{n+2}$ then $M$ is spherical [9, 6.3.11 p.123], [3, 5.2.10] and we may think that the sphere has center $0 \in \mathbb{R}^{n+k}$ and radius 1. $M$ is a regular level set of an isoparametric polynomial map $H : \mathbb{R}^{n+2} \to \mathbb{R}^2$ which has components $H = (h_1, h_2)$, where usually one takes $h_1\left(X\right) = \left\|X^2\right\| - 1$ and $h_2$ the Cartan-Münzner polynomial $f : \mathbb{R}^{n+2} \to \mathbb{R}$ (i.e. $h_2 = f$) and then one takes $M^n = H^{-1}\left(0, 0\right)$. The main reason of the importance of isoparametric submanifolds for our study is that the gradients $\{\nabla h_1, \nabla h_2\}$ provide a $\nabla^{\perp}$-parallel frame of the normal bundle of $M$ (see for instance [3, p.142-3]) We shall use this natural basis of the normal space instead of $\{\omega_1, \omega_2\}$.

Then in this case our polynomials become:

$$P_j(X) = \left\langle \nabla h_j(p), \left(\overline{\nabla}_X \alpha\right)(X,X)\right\rangle, \quad j = 1,2$$

and they define $\widehat{X}_p[M]$ by the conditions

$$P_j(X) = 0, \quad j = 1,2, \quad \|X\| = 1.$$

We have another simple fact of great importance ([10]):

**Lemma 3.** *Let $M^n$ be a compact submanifold of the sphere $S^{n+k-1} \subset \mathbb{R}^{n+k}$ and $p$ a point in $M^n$. Then, for a unitary $X \in T_p(M)$ we have*
$\left\langle \left(\overline{\nabla}_X \alpha\right)(X,X), p\right\rangle = 0.$■

Since $\nabla h_1(p) = 2p$ this says that the first polynomial $P_1(X)$ is identically zero on $T_p(M)$ and the second one is

$$\begin{aligned} P_2(X) &= \left\langle \nabla h_2(p), \left(\overline{\nabla}_X \alpha\right)(X,X)\right\rangle = \left\langle \mathrm{grad}_{Sn+1}(h_2), \left(\overline{\nabla}_X \alpha\right)(X,X)\right\rangle \\ &= \left\langle \left(\nabla h_2(p) - \langle \nabla h_2(p), p\rangle p\right), \left(\overline{\nabla}_X \alpha\right)(X,X)\right\rangle \end{aligned}$$

This says that all we need to define the algebraic set $\widehat{X}_p[M]$ is the polynomial

$$P_2(X) = \left\langle \nabla h_2(p), \left(\overline{\nabla}_X \alpha\right)(X,X)\right\rangle. \tag{3}$$

The action of the isotropy subgroup of the point $p \in M^n$ (i.e. the group of extrinsic isometries. $g$ such that $g(M) = M$ and $g(p) = p$) is trivial on the normal space $T_p(M)^{\perp}$, hence the polynomial (3) is invariant by the isotropy group of the orbit $M$ at $p$ and hence so are all their level sets.

Then we need to care only about the polynomial $P_2(X)$ which we shall denote simply by $P(X)$ from now on. The reader is referred to [10] for a description of the method to get $P(X)$. For extrinsic homogeneous isoparametric hypersurfaces it is relatively "easy" to compute explicitly the single polynomial defining $\widehat{X}_p[M]$. Furthermore for these spaces we have a general fact that it is convenient to mention [10].

**Proposition 4.** *Let $M^n \subset \mathbb{R}^{n+2}$ be an extrinsic homogeneous isoparametric hypersurface in a sphere. Let $X \in T_p(M)$ be a unit tangent vector at a point $p \in M^n$ and let $\gamma(s)$ be the normal section of $M^n$ at $p$ generated by $X$ ($\gamma$ is a curve contained in the three dimensional Euclidean space Span $\left\{X, T_p(M)^{\perp}\right\}$). Let $\kappa = \kappa(X)$ and $\tau = \tau(X)$ be respectively the curvature and torsion of $\gamma$ at $p$. Then for the polynomial (3) there is a constant $A$ such that $P(X) = A\kappa^2\tau$.*■

This Proposition indicates that the polynomial $P(X)$ does not just define the algebraic set $\widehat{X}_p[M]$ of normal sections planar at $p$, it can also be used to produce some sort of *"classification"* of the other normal sections at $p$. In fact, the level set

$$P^{-1}(a) = \{X \in S(T_p(M)) : P(X) = a\} \tag{4}$$

(which is also an algebraic set in $S(T_p(M))$) contains all unit tangent vectors $X$ which generate normal sections with the same value of the product $\kappa^2\tau$.

It is important to recall that in order to consider the Frenet frame and the torsion of a curve $\gamma(s)$ in $\mathbb{R}^3$ at a particular point (say $p = \gamma(0)$) the curve is required to be regular in an open interval $(-\varepsilon, \varepsilon)$ containing $s = 0$ and also must have positive Frenet curvature $\kappa$ on $(-\varepsilon, \varepsilon)$. *This condition is satisfied here.* In fact since our submanifold $M^n$ is compact it contains a point $q$ where, $\forall X \in T_q(M)$, $X \neq 0$, $\alpha_q(X, X) \neq 0$ [7, p.27]; but since it is homogeneous this happens at every point. Then

$$\kappa(X) = \|\alpha_p(X, X)\| > 0, \qquad \forall X \in T_q(M), X \neq 0, \quad \forall p \in M$$

Let us notice also that if a unit vector $X \in P^{-1}(a)$ then $(-X) \in P^{-1}(-a)$ which corresponds to the same section in opposite direction. These two sets are interchanged by the antipodal map of the sphere $S(T_p(M))$ and therefore they go to a single algebraic set in the projective space $\mathbb{R}P(T_p(M))$. These sets with $X_p[M]$ form a one-parameter family of algebraic sets in $\mathbb{R}P(T_p(M))$ for the parameter $a \in [0, d]$ where $d$ is the maximum value of $P(X)$ on the sphere $S(T_p(M))$. It is rather natural then to study this one-parameter family naturally associated to our space $M^n$ or, as we rather do, to study the family in (4) defined for $a \in [-d, d]$.

An interesting problem that arises here is what can be said about the smoothness of the algebraic sets $P^{-1}(a) \subset S(T_p(M))$. In this respect it is important to notice that it is clear at this point that the algebraic set $\widehat{X}_p[M] = P^{-1}(0)$ *is never smooth* [10, Prop. 4.1].

We shall concentrate here on the so called Cartan's isoparametric hypersurfaces that are described in the next section.

## 3. Cartan's isoparametric submanifolds

The so called Cartan's isoparametric submanifolds are the manifolds of complete flags in the projective planes $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$ (real, complex, quaternionic and Cayley projective planes). We shall denote them by $M_{\mathbb{R}}$, $M_{\mathbb{C}}$, $M_{\mathbb{H}}$ and $M_{\mathbb{O}}$ respectively. A complete flag in any of the projective planes is a pair $(p, l)$ where $p$ is a point in the plane and $l$ a line (real, complex, quaternionic or octonionic) containing the point $p$. In each projective plane, the group of isometries. acts transitively on flags which yields the homogeneous representations

$$\begin{aligned} M_{\mathbb{R}} &= SO(3) / (Z_2 \times Z_2) \\ M_{\mathbb{C}} &= SU(3) / T^2 \\ M_{\mathbb{H}} &= Sp(3) / (Sp(1))^3 \\ M_{\mathbb{O}} &= F_4 / Spin(8). \end{aligned}$$

The dimensions of these manifolds are, respectively, 3, 6, 12 and 24. They are easily seen to be tubes over the corresponding projective planes $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$ and $M_{\mathbb{R}} \subset M_{\mathbb{C}} \subset M_{\mathbb{H}} \subset M_{\mathbb{O}}$.

By taking any homogeneous (invariant) metric on $M_{\mathbb{O}}$ and the induced one on the other manifolds, each one is totally geodesic in those containing it. These manifolds are isoparametric submanifolds of rank 2 in some Euclidean spaces. We briefly recall their definition in the next subsection.

3.1. **The simple formally real Jordan algebras $H_3(F)$ and the manifolds $M_{\mathbb{R}} \subset M_{\mathbb{C}} \subset M_{\mathbb{H}} \subset M_{\mathbb{O}}$.** To describe the Cartan-Münzner polynomials that define our isoparametric families, we recall some well known facts (see, for instance, the presentation in [8]). Let $F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ and denote by $M_3(F)$ the $3 \times 3$ matrices with entries in $F$. Let $H_3(F) = \left\{u \in M_3(F) : \overline{u}^t = u\right\}$ where $x \mapsto \overline{x}$ denotes conjugation in $F$. An element $u \in H_3(F)$ is called a Hermitian matrix and will be denoted by

$$u = \begin{bmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{bmatrix}, \xi_j \in \mathbb{R}, x_j \in F. \tag{5}$$

Naturally $H_3(\mathbb{R}) \subset H_3(\mathbb{C}) \subset H_3(\mathbb{H}) \subset H_3(\mathbb{O})$.

They are real Jordan Algebras with the product

$$u \circ v = \frac{1}{2}(uv + vu),$$

In the first three cases, each algebra is part of a family of "special" Jordan algebras $H_n(F)$ ($F = \mathbb{R}, \mathbb{C}, \mathbb{H}$) [1, p. 119] and $H_3(\mathbb{O})$ is an *exceptional Jordan algebra* [1, p.120]. We consider the algebra $H_3(\mathbb{O})$, on the others everything works by restriction.

One has in this algebra three natural functions, [1, p.113], namely

$$\begin{aligned} &\ell(A) = tr(A) \\ &b(A,B) = \ell(A \circ B) \\ &T(A,B,C) = b(A \circ B, C) \end{aligned} \tag{6}$$

which are determined by the product [1, p.113]. The *group of automorphisms* of this algebra is isomorphic to $F_4$, [1, 16.7,(i)]. The functions (6) are invariant by this action [1, 16.5].

Let us consider the subspace $U = \{u \in H_3(F) : \ell(u) = tr(u) = 0\}$. The group $F_4$ preserves $U$ and this defines the *26-dimensional irreducible real representation* of $F_4$. The restriction of the above action preserves the restriction of the functions.

Let us consider in $U$ the inner product

$$\langle u, v \rangle = \frac{1}{2} b(u, v). \tag{7}$$

This subspace $U$ with the inner product (7) will be our ambient Euclidean space for the manifolds $M_{\mathbb{R}}$, $M_{\mathbb{C}}$, $M_{\mathbb{H}}$ and $M_{\mathbb{O}}$. Note that $\dim_{\mathbb{R}}(U) = 5, 8, 14, 26$ for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. Consider now on $U$ the function defined by

$$f(u) = \left(\frac{\sqrt{3}}{2}\right) T(u, u, u)$$

and its restriction to the unit sphere $S \subset U$. This is the Cartan-Münzner polynomial. To simplify notation we use also $M$ to indicate $M_F$ when there is no need to specify which is the $F$ under consideration.

The point
$$E = diag\left(-1, 0, 1\right) \tag{8}$$
is a point in $M = f^{-1}(0)$. The point $E$ belongs to $M_F$ for all $F$.

Let us consider in $U$ the subspace:
$$a = \left\{ diag\left(\xi_1, \xi_2, \xi_3\right) : \sum_j \xi_j = 0 \right\}. \tag{9}$$
It is not hard to see that the normal space to all $M_F$ at $E$ is the same for all $F$ namely $T_E\left(M_F\right)^{\perp} = a$ and the tangent spaces at $E$ satisfy $T_E\left(M_{\mathbb{R}}\right) \subset T_E\left(M_{\mathbb{C}}\right) \subset T_E\left(M_{\mathbb{H}}\right) \subset T_E\left(M_{\mathbb{O}}\right)$ and so the tangent space at $E$ is just the affine subspace
$$T_E\left(M_F\right) = E + \left\{ \begin{bmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & x_1 \\ x_2 & \overline{x_1} & 0 \end{bmatrix}, x_j \in F \right\}.$$

These manifolds have the interesting property [10] that the polynomial $P(X)$ associated to the normal sections of $M$ as in Section 2 is just the restriction of the function $f$ to the tangent space i.e.
$$P(X) = f(X), \qquad X \in T_E(M).$$

Then it is easy to see that the polynomial $P(X)$ takes the form
$$P(X) = 3\sqrt{3}t\left(x_1x_2x_3\right), \qquad t\left(x_1x_2x_3\right) = 2\mathrm{Re}\left((x_1x_2)x_3\right). \tag{10}$$
The trilinear function $\mathrm{Re}\left((x_1x_2)x_3\right)$ has the following properties:
$$\begin{array}{l} \mathrm{Re}\left((ab)\,c\right) = \mathrm{Re}\left(a\,(bc)\right), \\ \mathrm{Re}\left((ab)\,c\right) \text{ is invariant by cyclic permutation} \end{array} \tag{11}$$

Then we have our four Cartan Hypersurfaces $M_F$ (for each $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) as:
$$M_F = \left\{ u \in S(U) : f(u) = 0 \right\}.$$
We may consider that all tangent spaces $T_E\left(M_F\right)$ (translated to the origin) coincide with the subspaces $a^{\perp}$ in each case. If we denote an $u \in H_3(F)$ as in (5) by $u = \left(\xi_1, \xi_2, \xi_3, x_1, x_2, x_3\right)$ then we have
$$\begin{array}{l} a^{\perp} = \left\{ (0, 0, 0, x_1, x_2, x_3) : x_j \in F \right\}, \\ T_E\left(M_F\right) = E + a^{\perp}. \end{array}$$
and our polynomial $P(X)$ is given by (10) in $a^{\perp}$. The normal section $\gamma$ at the point $E$ of $M_F$ defined by $X = (0, 0, 0, x_1, x_2, x_3) \in a^{\perp}$ is a planar normal section if and only if the vector $X$ satisfies the equations
$$\|X\| = 1, \quad P(X) = 0.$$

For $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, it was proven in [10] that

**Theorem 5.** *The polynomial $P(X)$ has only three critical values on the unit sphere $S\left(a^{\perp}\right)$ in $a^{\perp}$ namely $0$, its maximum $(m_o)$ and its minimum $(-m_o)$. Hence the level sets $P^{-1}(r)$ for $r \in (-m_o, 0) \cup (0, m_o)$ are smooth submanifolds (hypersurfaces) of the sphere $S\left(a^{\perp}\right)$.*■

If there is need to indicate which $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is under consideration, we may eventually use the notations $P_F(X)$ and $t_F(x_1x_2x_3)$ for the polynomials $P(X)$ and $t(x_1x_2x_3)$ respectively. In all cases the value of $m_o$ is $m_o = 2$.

## 4. Triality

We want to bring "triality" into the picture so we recall a few required facts about it, referring the reader to [1] and [2, p.159-64] for the omitted details.

Let $V_1$, $V_2$ and $V_3$ be three real vector spaces with inner product and corresponding norms $\|*\|_j$, a *normed triality* is a trilinear map $\varphi : V_1 \times V_2 \times V_3 \longrightarrow \mathbb{R}$ which satisfies

$$|\varphi(v_1, v_2, v_3)| \leq \|v_1\|_1 \|v_2\|_2 \|v_3\|_3 \tag{12}$$

and fixing any two variables there exists a *non-zero* value of the third one where the bound (12) is attained. In [2, p. 162] it is observed that normed division algebras go hand in hand with normed trialities and with $H_3(F)$ (for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$). The relevance of these objects here, comes from the following observation:

**Remark 1.** [1, Th. 15.14] *For $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ the trilinear map*

$$\varphi(x_1, x_2, x_3) = \frac{1}{2} t_F(x_1x_2x_3)$$

*is a normed triality.*

The action of the isotropy groups of the four isoparametric hypersurfaces $M_\mathbb{R} \subset M_\mathbb{C} \subset M_\mathbb{H} \subset M_\mathbb{O}$ at the common point $E$ on the tangent spaces $T_E(M_\mathbb{R}) \subset T_E(M_\mathbb{C}) \subset T_E(M_\mathbb{H}) \subset T_E(M_\mathbb{O})$ is intimately related to this phenomenon of triality. The corresponding isotropy groups are respectively $Z_2^2 \subset T^2 \subset (Sp(1))^3 \subset Spin(8)$. As mentioned above, it follows from the definition of the polynomial $P(X)$ that it is invariant by the action of the corresponding isotropy group on the tangent spaces. In this fashion the isotropy groups of our four manifolds are *related* to the corresponding *groups of automorphisms* of the trialities associated to the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

An *automorphism* of a *normed triality* $\varphi$ is a triple of *orthogonal* maps $(f_1, f_2, f_3)$ ($f_i : V_i \to V_i$, $i = 1, 2, 3$) which satisfies:

$$\varphi(f_1(v_1), f_2(v_2), f_3(v_3)) = \varphi(v_1, v_2, v_3), \quad \forall v_i \in V_i. \tag{13}$$

The triples $(f_1, f_2, f_3)$ form a group with the composition which is denoted by $Aut(\varphi)$.

A word should be said about the actions of $(O(1))^3$, $(U(1))^3$, $(Sp(1))^3$ and $Spin(8)$ on the space $F^3 = a^\perp$ where the trialities are defined. By [1, (5.7), (15.14), (15.15)] $Aut(t_\mathbb{O})$ coincides with the group $Spin(8)$ and the actions of the subgroups $(O(1))^3$, $(U(1))^3$, $(Sp(1))^3$ of $Spin(8)$ are the restrictions of that action. The way in which the subgroup $(Sp(1))^3$ is contained in $Spin(8)$ is the following. Since $Sp(1)$ consists of quaternions of norm 1 we may define for $(a_1, a_2, a_3) \in (Sp(1))^3$ a triad of orthogonal maps (each : $\mathbb{H} \to \mathbb{H}$) as

$$(a_1, a_2, a_3)(x_1, x_2, x_3) = \left(a_1x_1a_2^{-1}, a_2x_2a_3^{-1}, a_3x_3a_1^{-1}\right). \tag{14}$$

This triad is in $Spin\,(8)$ since it preserves the triality $\varphi$ restricted to $\mathbb{H}^3 \subset \mathbb{O}^3$. In fact, for $x_j \in \mathbb{H}$, $j = 1, 2, 3$,

$$\mathrm{Re}\,((x_1x_2)x_3) = \mathrm{Re}\left(\left(a_1x_1a_2^{-1}a_2x_2a_3^{-1}\right)a_3x_3a_1^{-1}\right).$$

Note that the triads $(1, 1, 1)$ and $(-1, -1, -1)$ act trivially and one has (as indicated in [2, p. 162]):

$$Aut\,(t_{\mathbb{H}}) \simeq (Sp\,(1))^3 \,/\, \{\pm Id\}\,.$$

Now the groups $O\,(1)$, $U\,(1)$ are the sets of unit elements in the corresponding algebras $\mathbb{R}$, and $\mathbb{C}$. We are interested in the subgroups $(O\,(1))^3$, $(U\,(1))^3$ of $(Sp\,(1))^3$ so note that, as a result of restricting the action (14) to $(U\,(1))^3$, we get

$$\left(e^{is_1}, e^{is_2}, e^{is_3}\right)(x_1, x_2, x_3) = \left(e^{i(s_1-s_2)}x_1, e^{i(s_2-s_3)}x_2, e^{i(s_3-s_1)}x_3\right). \qquad (15)$$

and since $(s_1 - s_2) + (s_2 - s_3) + (s_3 - s_1) = 0$, the resulting action is that of

$$\left\{(g_1, g_2, g_3) \in U\,(1)^3 : g_1g_2g_3 = 1\right\} \simeq T^2,$$

and similarly, for $O\,(1)^3$, we get

$$\left\{(g_1, g_2, g_3) \in O\,(1)^3 : g_1g_2g_3 = 1\right\} \simeq (Z_2)^2\,.$$

4.1. **The nature of these actions.** Thanks to the following result, due to B. Wilking, [12] we have a basic tool to study the above actions of the isotropy groups and also to describe the structure of the level sets $P^{-1}\,(r)$ of our polynomial (10) on the spheres $S\,(T_E\,(M)) = S\left(a^{\perp}\right)$ for $r \in [-m_o, m_o]$. The interval $[-m_o, m_o]$ is the image by $P$ of that sphere.

**Theorem 6.** *(B. Wilking) Given the point $E \in M_{\mathbb{C}}$ and a vector $(v_1, v_2, v_3) \in T_E\,(M_{\mathbb{O}})$ there is an element $(h_1, h_2, h_3)$ of the isotropy subgroup of $M_{\mathbb{O}}$ at $E$, namely $Spin\,(8)$, that moves $(v_1, v_2, v_3)$ to $T_p\,(M_{\mathbb{C}}) \subset T_E\,(M_{\mathbb{O}})$ that is:*

$$(h_1, h_2, h_3)\,(x_1, x_2, x_3) \in T_E\,(M_{\mathbb{C}})\,.$$

*Furthermore if $(v_1, v_2, v_3) \in T_E\,(M_{\mathbb{H}}) \subset T_E\,(M_{\mathbb{O}})$ then the element $(h_1, h_2, h_3)$ can be taken in the isotropy group $(Sp\,(1))^3$ of $M_{\mathbb{H}}$ at $E$.*■

It may be of interest to the reader to see how this theorem may be proven in the case where $(x_1, x_2, x_3) \in \mathbb{H}^3$. By the theorem of Cayley, [5, p. 215], we can take unit quaternions $(a_1, a_2, 1)$ such that $a_1x_1a_2^{-1}$ is real. Acting with this triad on $(x_1, x_2, x_3)$ we get $\left(r_1, a_2x_2, x_3a_1^{-1}\right)$ $(r_1 \in \mathbb{R})$. Now we can take the triad $(b, b, b_3)$ such that $b\,(a_2x_2)\,b_3^{-1} = r_2$ is also real. Then applying the two triads we get

$$(b, b, b_3)\,((a_1, a_2, 1)\,(x_1, x_2, x_3)) = \left(r_1, r_2, b_3\left(x_3a_1^{-1}\right)b^{-1}\right) = (r_1, r_2, y_3)\,.$$

Now we write the third component $y_3$, as $y_3 = r_3 + u_3$ with $r_3$ real and $u_3$ a pure quaternion $(u_3 \in \mathrm{Im}\,(\mathbb{H}))$. By the Theorem of Hamilton [5, p. 216] we may find a unit quaternion $a$ such that $au_3a^{-1} = qi$ ($q \in \mathbb{R}$, $i$ the imaginary unit in $\mathbb{C} \subset \mathbb{H}$) then the triad $(a, a, a)$ takes $(r_1, r_2, y_3)$ to $(r_1, r_2, r_3 + qi)$. The composition $(a, a, a) \circ (b, b, b_3) \circ (a_1, a_2, 1)$ gives the desired result.

**Remark 2.** *It is then clear that we may take any tangent vector* $X = (x_1, x_2, x_3)$ $(x_j \in \mathbb{O}, \mathbb{H}, \mathbb{C}, \forall j)$ *to one of the form* $(r_1, r_2, y_3)$*, with* $r_j \in \mathbb{R}$ *and* $y_3 \in \mathbb{C}$*.*

As a consequence of this theorem we have now a first description of the structure of the level sets $P^{-1}(r)$ for $M_{\mathbb{H}}$ and $M_{\mathbb{O}}$, in terms of those of $M_{\mathbb{C}}$. We may write:

$$\begin{array}{ll} \text{(i)} & P_{\mathbb{O}}^{-1}(d) = Spin\,(8)\, P_{\mathbb{C}}^{-1}(d)\,, \\ \text{(ii)} & P_{\mathbb{O}}^{-1}(0) = Spin\,(8)\, P_{\mathbb{C}}^{-1}(0)\,, \\ \text{(iii)} & P_{\mathbb{H}}^{-1}(d) = (Sp\,(1))^3\, P_{\mathbb{C}}^{-1}(d)\,, \\ \text{(iv)} & P_{\mathbb{H}}^{-1}(0) = (Sp\,(1))^3\, P_{\mathbb{C}}^{-1}(0)\,. \end{array} \tag{16}$$

This fact shows that in order to understand these algebraic sets we may reduce our study to the manifold $M = M_{\mathbb{C}}$. Then our objective is to describe the structure of the sets $P_{\mathbb{C}}^{-1}(0) = \widehat{X}_E\,[M_{\mathbb{C}}]$ and $P_{\mathbb{C}}^{-1}(d)$.

## 5. The case of $M_{\mathbb{C}}$

We concentrate then on the manifold $M = M_{\mathbb{C}} = SU\,(3)\,/T^2$. We need to fix some notation for the tangent space $T_E\,(M_{\mathbb{C}})$. We have here $X = (x_1, x_2, x_3)$ such that $x_1$, $x_2$, and $x_3$ are *complex numbers*. Then we write them as

$$x_1 = (a_1 + ia_2)\,, \quad x_2 = (b_1 + ib_2)\,, \quad x_3 = (c_1 + ic_2) \tag{17}$$

or alternatively as

$$\begin{array}{rcl} X & = & (a\,(\cos\eta + i\sin\eta)\,, b\,(\cos\alpha + i\sin\alpha)\,, c\,(\cos\beta + i\sin\beta)) \\ a & = & |x_1|\,, b = |x_2|\,, c = |x_3|\,. \end{array} \tag{18}$$

The function $t\,(x_1x_2x_3)$, with the notation (17), takes the form:

$$t\,(x_1x_2x_3) = 2\,(a_1b_1c_1 - a_1b_2c_2 - a_2b_1c_2 - a_2b_2c_1)$$

and in terms of (18) it is

$$t\,(x_1x_2x_3) = 2abc\cos(\eta + \alpha + \beta)\,.$$

In turn, the polynomial $P\,(X)$ in notation (18), looks like

$$P\,(X) = 3\sqrt{3}t\,(x_1x_2x_3) = 6\sqrt{3}abc\cos(\eta + \alpha + \beta)\,. \tag{19}$$

Recall that we consider our polynomial $P\,(X)$ defined on the unit sphere $S\left(a^{\perp}\right) \simeq S^5$ so we also have the condition

$$1 = a^2 + b^2 + c^2. \tag{20}$$

The action of the isotropy group $T^2$ at $E$ is, as noted above, given by (15). So, in notation (18), we have

$$\begin{array}{l} g\,(X) = (x_1', x_2', x_3') \\ x_1' = a\,(\cos((s_1 - s_2) + \eta) + i\sin((s_1 - s_2) + \eta)) \\ x_2' = b\,(\cos((s_2 - s_3) + \alpha) + i\sin((s_2 - s_3) + \alpha)) \\ x_3' = c\,(\cos((s_3 - s_1) + \beta) + i\sin((s_3 - s_1) + \beta))\,. \end{array} \tag{21}$$

Notice that, by Remark 2 and (21), we can take any $X$ in (18) to a vector of the form

$$X = (a, b, c\,(\cos(\eta + \alpha + \beta) + i\sin(\eta + \alpha + \beta)))\,. \tag{22}$$

The following two sections are devoted to the study of the level sets $P^{-1}(r)$ ($r \in [-m_o, m_o]$) for $M_{\mathbb{C}}$.

## 6. The algebraic set $\widehat{X}_E[M_{\mathbb{C}}] = P_{\mathbb{C}}^{-1}(0)$

The form (19) of the polynomial of the normal sections, suggests that we may consider in $\widehat{X}_E[M_{\mathbb{C}}] = P_{\mathbb{C}}^{-1}(0)$ the following two subsets such that $\widehat{X}_E[M_{\mathbb{C}}] = \Omega_o \cup \Omega_1$.

$$\Omega_o = \left\{X \in \widehat{X}[M_{\mathbb{C}}] : a \text{ or } b \text{ or } c \text{ is zero}\right\} \tag{23}$$

$$\Omega_1 = \left\{X \in \widehat{X}[M_{\mathbb{C}}] : a \neq 0, b \neq 0, c \neq 0\right\} \tag{24}$$

The set $\Omega_o$ consists of three totally geodesic spheres $S^3$ in $S^5$ and each one of them is invariant by the action of the torus. These three spheres intersect, in pairs, in unit circles given, respectively, by:

$$a = 1, \qquad b = 1, \qquad c = 1 \tag{25}$$

which are themselves orbits of the torus. These three one dimensional spheres form the singular set of $\widehat{X}_E[M_{\mathbb{C}}]$ (the set where the gradient of $P$ vanishes) and these are also the singular orbits of the torus, where the isotropy subgroups are one dimensional tori. The other orbits on the totally geodesic spheres $S^3$ in $S^5$ are 2-dimensional tori.

We study now the orbits of the points in $\Omega_1$. If $(a,b,c) \neq (a_1,b_1,c_1)$ then $X$ and $X_1$ are not in the same orbit of $T^2$, so we may consider inside $\Omega_1$ the subsets determined by the moduli namely

$$\Omega_1(a,b,c) = \{X \in \Omega_1 : |x_1| = a, |x_2| = b, |x_3| = c\}$$

and study the orbits of $T^2$ on these sets. Consider the point $X$ (notation as in (18)) in $\Omega_1(a,b,c)$. In this situation we must have $\cos(\eta+\alpha+\beta) = 0$ or equivalently $\eta+\alpha+\beta = (2k+1)\frac{\pi}{2}$, and then, by of the action of the torus, we can take $X$ to the point with components

$$(a, b, ci\sin(\eta+\alpha+\beta)) = (a, b, ci\sin(\gamma)). \tag{26}$$

Since $\sin\left((2k+1)\frac{\pi}{2}\right) = (-1)^k$, there are two possibilities for this point namely

$$X_2 = (a, b, ic), \qquad X_2' = (a, b, -ic). \tag{27}$$

These two points are not in the same orbit of $T^2$. In fact, if $\gamma = (2k+1)\frac{\pi}{2}$ and there is a $g \in T^2$ such that $g(X_2) = X_2'$ then

$$\sin((s_3 - s_1) + \gamma) = -\sin(\gamma)$$

but since

$$\left.\begin{array}{l}(s_1 - s_2) = 2k_1\pi \\ (s_2 - s_3) = 2k_2\pi\end{array}\right\} \Rightarrow (s_3 - s_1) = 2h\pi$$

we get a contradiction and then in each set $\Omega_1(a,b,c)$ there are two different orbits namely those of $X_2$ and $X_2'$ respectively. Any point $X \in \Omega_1(a,b,c)$ can be taken either to $X_2$ or to $X_2'$.

Since, by definition, in $\Omega_1$ all the moduli $a, b, c$ are greater than zero and each triple of moduli $(a, b, c)$ determines two orbits we see that the *orbit space of the action of the torus on* $\Omega_1$ "essentially" consists of *two* copies of the *open* set

$$A = \left\{(a, b, c) \in S^2 : a, b, c > 0 \right\} \tag{28}$$

in $S^2$ (the third axis is the imaginary one in $\mathbb{R} \times \mathbb{R} \times \mathbb{C}$). The boundary of this open set is, of course, the spherical triangle with sides

$$(a = 0, b > 0, c > 0),\ (a > 0, b = 0, c > 0),\ (a > 0, b > 0, c = 0) \tag{29}$$

and vertices

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (0, 0, 1)\,. \tag{30}$$

The points on this boundary, correspond to the orbits on $\Omega_o$. Note that each triad $(a, b, c)$ with either $a = 0$ or $b = 0$ or $c = 0$ determines a single orbit. Hence the two *closed sets* $\overline{A}$ are glued by their boundaries with the identity mapping. The singular set corresponds to the vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Since this set is connected, it is also clear that $P^{-1}(0) = \widehat{X}_E [M_{\mathbb{C}}]$ *is connected.*

The set formed by the two copies of $\overline{A}$ identifying their boundaries, is clearly homeomorphic to the sphere $S^2$ and we may, *roughly*, describe the structure of the algebraic set $\widehat{X}_E [M_{\mathbb{C}}]$. We see that $\widehat{X}_E [M_{\mathbb{C}}]$ "looks" like $T^2 \times S^2$ with three fibers where the torus $T^2$ is *replaced* by $T^1$.

## 7. The submanifolds $P_{\mathbb{C}}^{-1}(d) \subset S^5$, $d \in (0, m_o)$

Now we study the submanifold $P^{-1}(d) \subset S^5$. In this section we keep *fixed* $d \in (0, m_o)$.

A point $X \in P^{-1}(d)$ may be written as (18) and we have in (19) the form of the polynomial $P(X)$.

Recall that we consider our polynomial $P(X)$ defined on the unit sphere $S^5$ so

$$P^{-1}(d) = \left\{X \in S^5 : P(X) = d\right\} \subset S^5.$$

We fix the notation

$$k = \frac{d}{6\sqrt{3}} < \frac{m_o}{6\sqrt{3}} = k_o. \tag{31}$$

Notice that, as was indicated above, $m_o = 2$ and so $k_o = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$.

It is important to consider the "real" points in $P^{-1}(d)$ that is those that, in the notation (18), have $\eta = \alpha = \beta = 0$. We denote this set by *Real* $\left(P^{-1}(d)\right)$. We have the following Proposition whose proof is left to the reader.

**Proposition 7.** *Real* $\left(P^{-1}(d)\right)$ *is a smooth 1-dimensional submanifold of* $S^2$ *whose four connected components are smooth simple closed curves in* $S^2$ *One of these components is contained in the open set* $A \subset S^2$ *(28).*■

For $k$ as in (31), we denote by $C_k \subset$ *Real* $\left(P^{-1}(d)\right)$ the component contained in the open set $A \subset S^2$ (28).

It is important to indicate here, that the other three components of *Real* $\left(P^{-1}(d)\right)$ are images of of $C_k$ by elements of the torus.

Let us consider , for each $r$ such that $k \leq r \leq k_o$, the smooth curves $C_r$ (note that $C_{k_o}$ reduces to the point $q_o = \frac{1}{\sqrt{3}}(1,1,1)$) and define, *associated to each* $C_r$, two particular angles by

$$\theta_r^+ = \cos^{-1}\left(\frac{k}{r}\right), \quad \theta_r^+ \in \left[0, \frac{\pi}{2}\right), \quad \theta_r^- = -\theta_r^+.$$

If $r = k$ then $\theta_r^+ = 0$ while for $r = k_o$

$$\theta_{k_o}^+ = \cos^{-1}\left(3\sqrt{3}k\right) = \cos^{-1}\left(\frac{d}{2}\right), \quad 0 < \frac{d}{2} < 1.$$

If $(a,b,c) \in C_r$ the two points

$$X\left(a,b,c,\theta_r^\pm\right) = \left(a,b,c\left(\cos\theta_r^\pm + i\sin\theta_r^\pm\right)\right)$$

belong to the manifold $P^{-1}(d)$ because

$$\begin{aligned} P\left(X\left(a,b,c,\theta_r^\pm\right)\right) &= 6\sqrt{3}abc\cos\left(\theta_r^\pm\right) = 6\sqrt{3}r\cos\left(\pm\cos^{-1}\left(\frac{k}{r}\right)\right) \\ &= 6\sqrt{3}r\left(\frac{k}{r}\right) = 6\sqrt{3}\frac{d}{6\sqrt{3}} = d. \end{aligned}$$

Then associated to each curve $C_r$ (for $k < r \leq k_o$) we have two opposite angles, namely $\theta_r^\pm$, while for $r = k$ the curve $C_k$ has only one $\theta_k^\pm = 0$. We may define then the following two subsets of $S^5$

$$\begin{aligned} \Theta_d^+ &= \left\{X\left(a,b,c,\theta_r^+\right) \in S^5 : (a,b,c) \in C_r,\ k \leq r \leq k_o\right\} \\ \Theta_d^- &= \left\{X\left(a,b,c,\theta_r^-\right) \in S^5 : (a,b,c) \in C_r,\ k \leq r \leq k_o\right\} \end{aligned}$$

Setting now these two pieces together we define

$$\Theta_d = \Theta_d^+ \cup \Theta_d^-, \tag{32}$$

and it is clear that we have:

$$C_k = \Theta_d^+ \cap \Theta_d^-.$$

It is also clear that both $\Theta_d^+$ and $\Theta_d^-$ are homeomorphic to the closed unit disk in $\mathbb{R}^2$ (the boundary is the *simple closed* curve $C_k$) and then in turn, the union (32) is homeomorphic to the sphere $S^2$. Note that on the curve $C_k \subset \Theta_d$ the points *are all real* ($x_j \in \mathbb{R}$, $j = 1,2,3$) but the other points on $\Theta_d$ have one complex component. In particular the "central" point in $\Theta_d^+$ is $X\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \theta_{k_o}^+\right)$ and similarly in $\Theta_d^-$ it is $X\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\theta_{k_o}^+\right)$.

Furthermore, any point $X_1 \in P^{-1}(d) \subset S^5$ can be taken, by the action of the torus $T^2$, to one and only one point in $\Theta_d$ from which it follows that $P^{-1}(d)$ is connected.

We have then:

**Theorem 8.** *For $d \in (-m_o, 0) \cup (0, m_o)$, the smooth hypersurface $P^{-1}(d) \subset S^5$ is connected and homeomorphic to $T^2 \times S^2$.*■

## 8. The case $M = M_{\mathbb{R}}$

It follows from the above comments that, for $M_{\mathbb{R}} = SO(3)/(Z_2 \times Z_2)$, the algebraic set $\widehat{X}_E[M_{\mathbb{R}}] = P^{-1}(0)$ consists of the three circles in $S^2 \subset \mathbb{R}^3$ given by $a = 0$, $b = 0$ and $c = 0$ respectively. It is connected since it is the union of three unit spheres of dimension one joined by the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. On the other hand the smooth submanifold $P^{-1}(d) \subset S^3$, for $d \in (-m_o, 0) \cup (0, m_o)$, consists of four copies of the curve $C_k$ ($k$ in (31)) and so it is *not connected*. Indeed for $d \in (0, m_o)$ the smooth curve $C_k$ is in $P_2^{-1}(d)$ but there are also the images of $C_k$ by $(Z_2 \times Z_2) \subset T^2$ which are:

$$\begin{aligned} C_k^{(3)} &= \left\{(a,b,c) \in S^2 : a, b < 0, c > 0 \text{ and } abc = k\right\}, \\ C_k^{(2)} &= \left\{(a,b,c) \in S^2 : a, c < 0, b > 0 \text{ and } abc = k\right\}, \\ C_k^{(1)} &= \left\{(a,b,c) \in S^2 : b, c < 0, a > 0 \text{ and } abc = k\right\}. \end{aligned}$$

## 9. The case $M = M_{\mathbb{O}}, M_{\mathbb{H}}$

It follows now from the facts described in Sections 6, 7 and also from (16) that:

**Corollary 9.** *$\widehat{X}[M_{\mathbb{O}}]$, $P_{\mathbb{O}}^{-1}(d)$, $\widehat{X}[M_{\mathbb{H}}]$ and $P_{\mathbb{H}}^{-1}(d)$ are connected.*■

## 10. The top level sets $P^{-1}(\pm m_o)$

Here the set $\Theta_{m_o}$ consists only of the point $X\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$ and so $P^{-1}(m_o)$ is, in each case, the orbit of this point by each one of the groups $(Z_2 \times Z_2)$, $T^2$, $(Sp(1))^3$ and $Spin(8)$. Notice that, as was indicated above, $m_o = 2$. For instance, in the first case (that of $M_{\mathbb{R}}$) $P^{-1}(m_o)$ consists of the four points $\frac{1}{\sqrt{3}}(1,1,1)$, $\frac{1}{\sqrt{3}}(-1,-1,1)$, $\frac{1}{\sqrt{3}}(-1,1,-1)$, $\frac{1}{\sqrt{3}}(1,-1,-1)$). They are the points of the orbit of $\frac{1}{\sqrt{3}}(1,1,1)$ by $(Z_2 \times Z_2)$. The same is true for $\Theta_{-m_o}$.

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