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APPROXIMATION AND SHAPE PRESERVING PROPERTIES OF THE TRUNCATED BASKAKOV OPERATOR OF MAX-PRODUCT KIND

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Abstract. Starting from the study of the *Shepard nonlinear operator of max-prod type* in [2], [3], in the recent monograph [4], Open Problem 5.5.4, pp. 324-326, the *Baskakov max-prod type operator* is introduced and the question of the approximation order by this operator is raised. The aim of this note is to obtain the order of uniform approximation $C\omega_1(f;\frac{1}{\sqrt{n}})$ (with the explicit constant $C = 24$) of another operator called the *truncated max-prod Baskakov operator* and to prove by a counterexample that in some sense, for arbitrary f this type of order of approximation with respect to $\omega_1(f;\frac{1}{\sqrt{n}})$ cannot be improved. However, for some subclasses of functions including for example the nondecreasing concave functions, the essentially better order of approximation $\omega_1(f;\frac{1}{n})$ is obtained. Finally, some shape preserving properties are proved.

1. Introduction

Starting from the study of the *Shepard nonlinear operator of max-prod type* in [2], [3], by the Open Problem 5.5.4, pp. 324-326 in the recent monograph [4], the following *nonlinear Baskakov operator of max-prod type* is introduced (here $\bigvee$ means maximum)

$$V_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} b_{n,k}(x)},$$

where $b_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$, $n \geq 1$, $x \in [0,1]$. Note that $b_{n,0}(x) = \frac{1}{(1+x)^n}$ for $0 < x \leq 1$ and $b_{n,0}(0) = 1$ by convention.

The aim of this note is to obtain the order of uniform approximation of $f : [0,1] \to \mathbb{R}$ by the so-called *truncated Baskakov operator of max-product kind*, defined as follows :

$$U_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} b_{n,k}(x)}, x \in [0,1], n \in \mathbb{N}, n \geq 1.$$

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We will prove that the order of uniform approximation is $\omega_1(f; 1/\sqrt{n})$ with explicit constant in front of it and that in some sense, in general this type of order of approximation with respect to $\omega_1(f; \cdot)$ cannot be improved. However, for some subclasses of functions, including for example the nondecreasing concave functions, the essentially better order $\omega_1(f; 1/n)$ is obtained. This allows us to put in evidence large classes of functions (e.g. nondecreasing concave polygonal lines) for which the order of approximation is essentially better than the order given by the most sequences of linear Bernstein-type operators. Finally, some shape preserving properties are presented.

Section 2 presents some general results on nonlinear operators, in Section 3 we prove several auxiliary lemmas, Section 4 contains the approximation results while in Section 5 we present some shape preserving properties.

2. **Preliminaries**

For the proof of the main result we need some general considerations on the so-called nonlinear operators of max-prod kind. Over the set of positive reals, $\mathbb{R}_+$, we consider the operations $\vee$ (maximum) and $\cdot$, product. Then $(\mathbb{R}_+, \vee, \cdot)$ has a semiring structure and we call it as Max-Product algebra.

Let $I \subset \mathbb{R}$ be a closed bounded or unbounded interval, and

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\}.$$

The general form of $L_n : CB_+(I) \to CB_+(I)$, (called here a discrete max-product type approximation operator) studied in the paper will be

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x, x_i) \cdot f(x_i),$$

or

$$L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i) \cdot f(x_i),$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(\cdot, x_i) \in CB_+(I)$ and $x_i \in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy a pseudo-linearity condition of the form

$$L_n(\alpha \cdot f \vee \beta \cdot g)(x) = \alpha \cdot L_n(f)(x) \vee \beta \cdot L_n(g)(x), \forall \alpha, \beta \in \mathbb{R}_+, f, g : I \to \mathbb{R}_+.$$

In this section we present some general results on these kinds of operators which will be useful later in the study of the truncated Baskakov max-product kind operator considered in Introduction.

Lemma 2.1. ([1]) *Let $I \subset \mathbb{R}$ be a bounded or unbounded interval,*

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties :

(i) if $f, g \in CB_+(I)$ satisfy $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in N$;

(ii) $L_n(f + g) \leq L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.

Then for all $f, g \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f-g|)(x).$$

Proof. Since it is very simple, we reproduce here the proof in [1]. Let $f, g \in CB_+(I)$. We have $f = f - g + g \leq |f-g| + g$, which by the conditions $(i)-(ii)$ successively implies $L_n(f)(x) \leq L_n(|f-g|)(x) + L_n(g)(x)$, that is $L_n(f)(x) - L_n(g)(x) \leq L_n(|f-g|)(x)$.

Writing now $g = g - f + f \leq |f-g| + f$ and applying the above reasonings, it follows $L_n(g)(x) - L_n(f)(x) \leq L_n(|f-g|)(x)$, which combined with the above inequality gives $|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f-g|)(x)$. □

Remarks. 1) It is easy to see that the truncated Baskakov max-product operator satisfy the conditions in Lemma 2.1, (i), (ii). In fact, instead of (i) it satisfies the stronger condition

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x),\ f, g \in CB_+(I).$$

Indeed, taking in the above equality $f \leq g$, $f, g \in CB_+(I)$, it easily follows $L_n(f)(x) \leq L_n(g)(x)$.

2) In addition, it is immediate that the truncated Baskakov max-product operator is positive homogenous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \geq 0$.

3) Since in the main results of the present paper in fact we take $I = [0,1]$, the following two corollaries are stated just for the case when I is bounded.

Corollary 2.2. ([1]) *Let* I *be a bounded interval,* $L_n : CB_+(I) \to CB_+(I)$, $n \in N$ *be a sequence of operators satisfying the conditions (i)-(ii) in Lemma 1 and in addition being positively homogenous. Then for all* $f \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|f(x) - L_n(f)(x)| \leq \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega_1(f;\delta)_I + |f(x)| \cdot |L_n(e_0)(x) - 1|,$$

where $\delta > 0$, $e_0(t) = 1$ *for all* $t \in I$, $\varphi_x(t) = |t-x|$ *for all* $t \in I$, $x \in I$, $\omega_1(f;\delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x-y| \leq \delta\}$.

Proof. The proof is identical with that for positive linear operators and because of its simplicity we reproduce it below. Indeed, from the identity

$$L_n(f)(x) - f(x) = [L_n(f)(x) - f(x) \cdot L_n(e_0)(x)] + f(x)[L_n(e_0)(x) - 1],$$

it follows (by the positive homogeneity and by Lemma 2.1)

$$|f(x) - L_n(f)(x)| \leq |L_n(f(x))(x) - L_n(f(t))(x)| + |f(x)| \cdot |L_n(e_0)(x) - 1| \leq$$

$$L_n(|f(t) - f(x)|)(x) + |f(x)| \cdot |L_n(e_0)(x) - 1|.$$

Now, since for all $t, x \in I$ we have

$$|f(t) - f(x)| \leq \omega_1(f; |t-x|)_I \leq \left[\frac{1}{\delta}|t-x| + 1\right] \omega_1(f;\delta)_I,$$

replacing above we immediately obtain the estimate in the statement. □

An immediate consequence of Corollary 2.2 is the following.

Corollary 2.3. ([1]) *Suppose that in addition to the conditions in Corollary 2.2, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in N$. Then for all $f \in CB_+(I)$, $n \in N$ and $x \in I$ we have*

$$|f(x) - L_n(f)(x)| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega_1(f;\delta)_I.$$

The truncated max-product operator $U_n^{(M)}(f)(x)$ satisfies the following useful result.

Lemma 2.4. *For any arbitrary function $f : [0,1] \to \mathbb{R}_+$, $U_n^{(M)}(f)(x)$ is positive, continuous on $[0,1]$ and satisfies $U_n^{(M)}(f)(0) = f(0)$, for all $n \in \mathbb{N}$, $n \geq 2$.*

Proof. Since $b_{n,k}(x) > 0$ for all $x \in (0,1]$, $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0,\ldots,n\}$, it follows that the denominator $\bigvee_{k=0}^{n} b_{n,k}(x) > 0$ for all $x \in (0,1]$ and $n \in \mathbb{N}$, $n \geq 2$. But the numerator is a maximum of finite number of continuous functions on $[0,1]$, so it is a continuous function on $[0,1]$ and this implies that $U_n^{(M)}(f)(x)$ is continuous on $(0,1]$. To prove now the continuity of $U_n^{(M)}(f)(x)$ at $x = 0$, we observe that $b_{n,k}(0) = 0$ for all $k \in \{1,2,\ldots,n\}$ and $b_{n,k}(0) = 1$ for $k = 0$, which implies that $\bigvee_{k=0}^{n} b_{n,k}(x) = 1$ in the case of $x = 0$. The fact that $U_n^{(M)}(f)(x)$ coincides with $f(x)$ at $x = 0$ immediately follows from the above considerations, which proves the lemma. □

Remark. From the above considerations, it is clear that $U_n^{(M)}(f)(x)$ satisfies all the conditions in Lemma 2.1, Corollary 2.2 and Corollary 2.3 for $I = [0,1]$

3. **Auxiliary Results**

For the proofs of the main results we need some notations and auxiliary results, as follows.

Remark. Note that since by Lemma 2.4 we have $U_n^{(M)}(f)(0) - f(0) = 0$ for all n, it follows that in the notations, proofs and statements of the all approximation results, that is in Lemmas 3.1-3.3, Theorem 4.1, Lemma 4.2, Corollaries 4.4, 4.5, in fact we always may suppose that $x > 0$.

For each $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0,1,2,\ldots,n\}$, $j \in \{0,1,2,\ldots,n-2\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, let us denote

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left|\frac{k}{n} - x\right|,$$

where $m_{k,n,j}(x) = \frac{b_{n,k}(x)}{b_{n,j}(x)}$ for $x \in (0,1]$, $m_{0,n.0}(0) = 1$ and $m_{k,n,0}(0) = 0$ for all $k \in \{1,2,\ldots,n\}$.

It is clear that if $k \geq j+2$ then

$$M_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n} - x)$$

and if $k \leq j$ then

$$M_{k,n,j}(x) = m_{k,n,j}(x - \frac{k}{n}).$$

Also, for each $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, n-2\}$, $k \geq j+3$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, let us denote

$$\overline{M}_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n-1} - x)$$

and for each $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, n-2\}$, $k \leq j-1$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ let us denote

$$\underline{M}_{k,n,j}(x) = m_{k,n,j}(x)(x - \frac{k}{n-1}).$$

Lemma 3.1. *Let $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ and $n \in \mathbb{N}$, $n \geq 2$.*

(i) For all $k \in \{0, 1, 2, \ldots, n\}$, $j \in \{0, 1, 2, \ldots, n-2\}$, $k \geq j+3$ we have

$$M_{k,n,j}(x) \leq \overline{M}_{k,n,j}(x) \leq 2M_{k,n,j}(x).$$

(ii) For all $k \in \{0, 1, 2, \ldots, n\}$, $j \in \{0, 1, 2, \ldots, n-2\}$, $k \leq j-1$ we have

$$\underline{M}_{k,n,j}(x) \leq M_{k,n,j}(x) \leq 2\underline{M}_{k,n,j}(x).$$

Proof. (i) The inequality $M_{k,n,j}(x) \leq \overline{M}_{k,n,j}(x)$ is immediate.

On the other hand, taking account of the fact that the function $h(x) = \frac{\frac{k}{n-1} - x}{\frac{k}{n} - x}$ is nondecreasing on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ we get

$$\begin{aligned} \overline{M}_{k,n,j}(x) &= M_{k,n,j}(x) \cdot \frac{\frac{k}{n-1} - x}{\frac{k}{n} - x} \leq M_{k,n,j}(x) \cdot \frac{\frac{k}{n-1} - \frac{j+1}{n-1}}{\frac{k}{n} - \frac{j+1}{n-1}} \\ &= M_{k,n,j}(x) \cdot \frac{n(k-j-1)}{kn-k-nj-n}. \end{aligned}$$

We have $\frac{n(k-j-1)}{kn-k-nj-n} \leq \frac{n(k-j-1)}{kn-n-nj-n} = \frac{k-j-1}{k-j-2} = 1 + \frac{1}{k-j-2} \leq 2$ which proves (i).

(ii) The inequality $\underline{M}_{k,n,j}(x) \leq M_{k,n,j}(x)$ is immediate.

On the other hand, taking account of the fact that the function $h(x) = \frac{x - \frac{k}{n}}{x - \frac{k}{n-1}}$ is nonincreasing on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ we get

$$\begin{aligned} M_{k,n,j}(x) &= \underline{M}_{k,n,j}(x) \cdot \frac{x - \frac{k}{n}}{x - \frac{k}{n-1}} \leq \underline{M}_{k,n,j}(x) \cdot \frac{\frac{j}{n-1} - \frac{k}{n}}{\frac{j}{n-1} - \frac{k}{n-1}} \\ &= \underline{M}_{k,n,j}(x) \cdot \frac{nj - nk + k}{n(j-k)}. \end{aligned}$$

We have $\frac{nj-nk+k}{n(j-k)} \leq \frac{nj-nk+n}{n(j-k)} = \frac{j-k+1}{j-k} = 1 + \frac{1}{j-k} \leq 2$ which proves (ii). □

Lemma 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$. For all $k \in \{0, 1, 2, \ldots, n\}$, $j \in \{0, 1, 2, \ldots, n-2\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have*

$$m_{k,n,j}(x) \leq 1.$$

Proof. First let us notice that for $x = 0$ we necessarily have $j = 0$ which implies $m_{0,n,0}(x) = 1$ and $m_{k,n,0}(x) = 0$ for all $k \in \{1, 2, \ldots, n\}$.

Suppose now that $x > 0$ when clearly $m_{k,n,j}(x) > 0$. We have two cases: 1) $k \geq j$ and 2) $k \leq j$.

Case 1). Since clearly the function $h(x) = \frac{1+x}{x}$ is nonincreasing on $[j/n-1, (j+1)/n-1]$ (or $(0, \frac{1}{n-1}]$ for $j = 0$) it follows

$$\begin{aligned}\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} &= \frac{k+1}{n+k} \cdot \frac{1+x}{x} \geq \frac{k+1}{n+k} \cdot \frac{n+j}{j+1} \\ &= \frac{(n+k)(j+1) + (n-1)(k-j)}{(n+k)(j+1)} \geq 1\end{aligned}$$

which implies $m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq m_{j+2,n,j}(x) \geq \ldots \geq m_{n,n,j}(x)$.

Case 2). We get

$$\begin{aligned}\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} &= \frac{n+k-1}{k} \cdot \frac{x}{x+1} \geq \frac{n+k-1}{k} \cdot \frac{j}{n+j-1} \\ &= \frac{k(n+j-1) + (n-1)(j-k)}{k(n+j-1)} \geq 1,\end{aligned}$$

which immediately implies

$$m_{j,n,j}(x) \geq m_{j-1,n,j}(x) \geq m_{j-2,n,j}(x) \geq \ldots \geq m_{0,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$, the conclusion of the lemma is immediate. □

Lemma 3.3. *Let* $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ *and* $n \in \mathbb{N}$, $n \geq 2$.

(i) If $k \in \{j+3, j+4, \ldots, n-1\}$ *is such that* $k - \sqrt{2(k+1)} \geq j$, *then* $\overline{M}_{k,n,j}(x) \geq \overline{M}_{k+1,n,j}(x)$.

(ii) If $k \in \{1, 2, \ldots, j-1\}$ *is such that* $j - \sqrt{2j} \geq k$, *then* $\underline{M}_{k,n,j}(x) \geq \underline{M}_{k-1,n,j}(x)$.

Proof. (i) We observe that

$$\overline{M}_{k+1,n,j}(x) = \overline{M}_{k,n,j}(x) \cdot \frac{n+k}{k+1} \cdot \frac{x}{x+1} \cdot \frac{\frac{k+1}{n-1} - x}{\frac{k}{n-1} - x}.$$

Since the function $g(x) = \frac{x}{x+1} \cdot \frac{\frac{k+1}{n-1} - x}{\frac{k}{n-1} - x}$ clearly is nondecreasing, it follows that $g(x) \leq g(\frac{j+1}{n-1}) = \frac{j+1}{n+j} \cdot \frac{k-j}{k-j-1}$ for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Then

$$\overline{M}_{k+1,n,j}(x) \leq \overline{M}_{k,n,j}(x) \cdot \frac{n+k}{k+1} \cdot \frac{j+1}{n+j} \cdot \frac{k-j}{k-j-1}.$$

By simple calculations and taking into account the fact that $k - \sqrt{2(k+1)} \geq j$ we obtain

$$\begin{aligned}& (k+1)(n+j)(k-j-1) - (n+k)(j+1)(k-j) \\ =\ & n[(k-j)^2 - (k+1)] + kj - j^2 - k^2 - j \\ \geq\ & n[2(k+1) - k - 1] + kj - j^2 - k^2 - n \\ =\ & kn + kj - j^2 - k^2 > 0,\end{aligned}$$

which proves (i).

(ii) We observe that

$$\underline{M}_{k,n,j}(x) = \underline{M}_{k-1,n,j}(x) \cdot \frac{n+k-1}{k} \cdot \frac{x}{x+1} \cdot \frac{x-\frac{k}{n-1}}{x-\frac{k-1}{n}}.$$

Since the function $h(x) = \frac{x}{x+1} \cdot \frac{x-\frac{k}{n-1}}{x-\frac{k-1}{n-1}}$ is nondecreasing, it follows that $h(x) \geq h(\frac{j}{n-1}) = \frac{j}{n+j-1} \cdot \frac{j-k}{j-k+1}$ for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Then

$$\underline{M}_{k,n,j}(x) \geq \underline{M}_{k-1,n,j}(x) \cdot \frac{n+k-1}{k} \cdot \frac{j}{n+j-1} \cdot \frac{j-k}{j-k+1}.$$

By simple calculations and taking into account the fact that $j - \sqrt{2j} \geq k$ we obtain

$$\begin{aligned} & j(n+k-1)(j-k) - k(n+j-1)(j-k+1) \\ = \; & n[(j-k)^2 - k] + kj - j^2 - k^2 + k] \geq n(2j-k) + kj - j^2 - k^2 + k \\ \geq \; & nj + kj - j^2 - k^2 + k > 0 \end{aligned}$$

which proves (ii) and the lemma. □

Also, a key result in the proofs of the main results is the following.

Lemma 3.4. *Let $n \in \mathbb{N}$, $n \geq 2$. We have*

$$\bigvee_{k=0}^{n} b_{n,k}(x) = b_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right], \, j = 0, 1, \ldots, n-2.$$

Proof. First we show that for fixed $n \in \mathbb{N}, n \geq 2$ and $0 \leq k < k+1 \leq n$ we have

$$0 \leq b_{n,k+1}(x) \leq b_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n-1)].$$

Indeed, the inequality one reduces to

$$0 \leq \binom{n+k}{k+1} \frac{x^{k+1}}{(1+x)^{n+k+1}} \leq \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

which after simple calculus is obviously equivalent to

$$0 \leq x \leq \frac{k+1}{n-1}.$$

By taking $k = 0, 1, .., n-1$ in the inequality just proved above, we get

$$b_{n,1}(x) \leq b_{n,0}(x), \text{ if and only if } x \in [0, 1/(n-1)],$$

$$b_{n,2}(x) \leq b_{n,1}(x), \text{ if and only if } x \in [0, 2/(n-1)],$$

$$b_{n,3}(x) \leq b_{n,2}(x), \text{ if and only if } x \in [0, 3/(n-1)],$$

so on,

$$b_{n,k+1}(x) \leq b_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n-1)],$$

and so on until finally

$$b_{n,n-1}(x) \leq b_{n,n-2}(x), \text{ if and only if } x \in [0, 1]$$

and

$$b_{n,n}(x) \leq b_{n,n-1}(x), \text{ if and only if } x \in [0, 1].$$

From all these inequalities, reasoning by recurrence we easily obtain:

$$\text{if } x \in [0, 1/(n-1)] \text{ then } b_{n,k}(x) \le b_{n,0}(x), \text{ for all } k = 0, 1, \dots, n,$$

$$\text{if } x \in [1/(n-1), 2/(n-1)] \text{ then } b_{n,k}(x) \le b_{n,1}(x), \text{ for all } k = 0, 1, \dots, n,$$

$$\text{if } x \in [2/(n-1), 3/(n-1)] \text{ then } b_{n,k}(x) \le b_{n,2}(x), \text{ for all } k = 0, 1, \dots, n,$$

and so on, in general

$$\text{if } x \in [j/(n-1), (j+1)/(n-1)] \text{ then } b_{n,k}(x) \le b_{n,j}(x), \text{ for all } k = 0, 1, \dots, n.$$

Combining these last implications with the above "if and only if" equivalences and writing

$$\bigvee_{k=0}^{n} b_{n,k}(x) = \max\left\{\bigvee_{k=0}^{j-1} b_{n,k}(x), \bigvee_{k=j}^{n} b_{n,k}(x)\right\},$$

the lemma is immediate. □

4. **Approximation Results**

If $U_n^{(M)}(f)(x)$ represents the *truncated Baskakov operator of max-product kind* defined in the Introduction, then the first main result of this section is the following.

Theorem 4.1. *Let $f : [0,1] \to \mathbb{R}_+$ be continuous. Then we have the estimate*

$$|U_n^{(M)}(f)(x) - f(x)| \le 24\omega_1\left(f, \frac{1}{\sqrt{n+1}}\right), n \in \mathbb{N}, n \ge 2, x \in [0,1],$$

where

$$\omega_1(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x - y| \le \delta\}.$$

Proof. It is easy to check that the truncated max-product Baskakov operator fulfils the conditions in Corollary 2.3 and we have

$$|U_n^{(M)}(f)(x) - f(x)| \le \left(1 + \frac{1}{\delta_n} U_n^{(M)}(\varphi_x)(x)\right)\omega_1(f, \delta_n), \tag{1}$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := U_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} b_{n,k}(x)\left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{n} b_{n,k}(x)}, x \in [0,1].$$

Let $x \in [j/(n-1), (j+1)/(n-1)]$, where $j \in \{0, 1, \dots, n-2\}$ is fixed. By Lemma 3.4 we easily obtain

$$E_n(x) = \max_{k=0,1,\dots,n}\{M_{k,n,j}(x)\}, x \in [j/(n-1), (j+1)/(n-1)].$$

So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j \in \{1, \dots, n-2\}$ is fixed, $x \in [j/(n-1), (j+1)/(n-1)]$ and $k \in \{0, 1, \dots, n\}$. In fact we will prove that

$$M_{k,n,j}(x) \le \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}, \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right], k = 0, 1, \dots, n, \tag{2}$$

which immediately will imply that

$$E_n(x) \leq \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}, \text{ for all } x \in [0,1], n \in \mathbb{N},$$

and taking $\delta_n = \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}$ in (1), since $[2\sqrt{3}(\sqrt{2}+2)] = 11$, from the property $\omega_1(f;\lambda\delta) \leq ([\lambda]+1)\omega_1(f;\delta)$, we immediately obtain the estimate in the statement.

In order to prove (2) we distinguish the following cases:

1) $k \in \{j, j+1, j+2\}$; 2) $k \geq j+3$ and 3) $k \leq j-1$.

Case 1). If $k = j$ then $M_{j,n,j}(x) = \left|\frac{j}{n} - x\right| = x - \frac{j}{n}$. Since $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, it easily follows that $M_{j,n,j}(x) \leq \frac{2}{n}$.

If $k = j+1$ then $M_{j+1,n,j}(x) = m_{j+1,n,j}(x)\left|\frac{j+1}{n} - x\right|$. Since by Lemma 3.2 we have $m_{j+1,n,j}(x) \leq 1$, we obtain $M_{j+1,n,j}(x) \leq \left|\frac{j+1}{n} - x\right|$. Because $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, it easily follows that $\left|\frac{j+1}{n} - x\right| \leq \frac{1}{n}$ which implies $M_{j+1,n,j}(x) \leq \frac{1}{n}$.

If $k = j+2$ then $M_{j+2,n,j}(x) = m_{j+2,n,j}(x)(\frac{j+2}{n} - x) \leq \frac{j+2}{n} - \frac{j}{n-1} = \frac{2n-j-2}{n(n-1)} \leq \frac{2}{n}$.

Case 2). Subcase a). Suppose first that $k - \sqrt{2(k+1)} < j$. We get

$$\overline{M}_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n-1} - x) \leq \frac{k}{n-1} - x \leq \frac{k}{n-1} - \frac{j}{n-1} \leq$$

$$\frac{k}{n-1} - \frac{k - \sqrt{2(k+1)}}{n-1} = \frac{\sqrt{2(k+1)}}{n-1} \leq \frac{3\sqrt{2}}{\sqrt{n+1}}.$$

Subcase b). Suppose now that $k - \sqrt{2(k+1)} \geq j$. Since the function $g(x) = x - \sqrt{2(x+1)}$ is nondecreasing on the interval $[0, \infty)$ it follows that there exists $\overline{k} \in \{0, 1, 2, \ldots, n\}$, of maximum value, such that $\overline{k} - \sqrt{2(\overline{k}+1)} < j$. Then for $k_1 = \overline{k}+1$ we get $k_1 - \sqrt{2(k_1+1)} \geq j$. Then

$$\begin{aligned} \overline{M}_{\overline{k}+1,n,j}(x) &= m_{\overline{k}+1,n,j}(x)(\frac{\overline{k}+1}{n-1} - x) \leq \frac{\overline{k}+1}{n-1} - x \\ &\leq \frac{\overline{k}+1}{n-1} - \frac{j}{n-1} \leq \frac{\overline{k}+1}{n-1} - \frac{\overline{k} - \sqrt{2(\overline{k}+1)}}{n-1} \\ &= \frac{\sqrt{2(\overline{k}+1)}+1}{n-1} \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{n+1}}. \end{aligned}$$

Also we have $k_1 \geq j+3$. Indeed, this is a consequence of the fact that g is nondecreasing on the interval $[0, \infty)$ and because through simple calculus we get $g(j+2) < j$. By Lemma 3.3, (i) it follows that $\overline{M}_{\overline{k}+1,n,j}(x) \geq \overline{M}_{\overline{k}+2,n,j}(x) \geq \ldots \geq \overline{M}_{n,n,j}(x)$. We thus obtain $\overline{M}_{k,n,j}(x) \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{n+1}}$ for any $k \in \{\overline{k}+1, \overline{k}+2, \ldots, n\}$.

Therefore, in both subcases, by Lemma 3.1, (i), we get $M_{k,n,j}(x) \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{n+1}}$.

Case 3). Subcase a). Suppose first that $j-\sqrt{2j}<k$. Then we obtain

$$\begin{aligned}\underline{M}_{k,n,j}(x) &= m_{k,n,j}(x)(x-\frac{k}{n-1})\leq\frac{j+1}{n-1}-\frac{k}{n-1}\\ &\leq \frac{j+1}{n-1}-\frac{j-\sqrt{2j}}{n-1}=\frac{\sqrt{2j}+1}{n-1}\leq\frac{\sqrt{2}+1}{\sqrt{n-1}}.\end{aligned}$$

Subcase b). Suppose now that $j-\sqrt{2j}\geq k$. Let $\widetilde{k}\in\{0,1,2,\dots,n\}$ be the minimum value such that $j-\sqrt{2j}<\widetilde{k}$. Then $k_2=\widetilde{k}-1$ satisfies $j-\sqrt{2j}\geq k_2$. Then

$$\begin{aligned}\underline{M}_{\widetilde{k}-1,n,j}(x) &= m_{\widetilde{k}-1,n,j}(x)(x-\frac{\widetilde{k}-1}{n-1})\leq\frac{j+1}{n-1}-\frac{\widetilde{k}-1}{n-1}\\ &\leq \frac{j+1}{n-1}-\frac{j-\sqrt{2j}-1}{n-1}=\frac{\sqrt{2j}+2}{n-1}\leq\frac{\sqrt{2}+2}{\sqrt{n-1}}.\end{aligned}$$

Also, because in this case $j\geq 1$ it is immediate that $k_2\leq j-1$. By Lemma 3.3, (ii) it follows that $\underline{M}_{\widetilde{k}-1,n,j}(x)\geq\underline{M}_{\widetilde{k}-2,n,j}(x)\geq\ldots\geq\underline{M}_{0,n,j}(x)$. We obtain $\underline{M}_{k,n,j}(x)\leq\frac{\sqrt{2}+2}{\sqrt{n-1}}$ for any $k\leq j-1$ and $x\in[\frac{j}{n-1},\frac{j+1}{n-1}]$.

In both subcases, by Lemma 3.1, (ii), we get $M_{k,n,j}(x)\leq\frac{2(\sqrt{2}+2)}{\sqrt{n-1}}\leq\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}$.

In conclusion, collecting all the estimates in the above cases and subcases we easily get the relationship (2), which completes the proof. □

Remark. The order of approximation in terms of ω_1 in Theorem 4.1 cannot be improved, in the sense that the order of the expression $\max_{x\in[0,1]}\{E_n(x)\}$ is exactly $\frac{1}{\sqrt{n+1}}$ (here $E_n(x)$ is defined in the proof of Theorem 4.1). Indeed, for $n\in\mathbb{N}$ let us take $j_n=[\frac{n}{2}]$, $k_n=j_n-[\sqrt{n}]$ and $x_n=\frac{j_n+1}{n-1}$. Then by simple calculation, for all $n\geq 2$ we get

$$\underline{M}_{k_n,n,j_n}(x_n)=$$

$$\begin{aligned}&\frac{\binom{n+k_n-1}{k_n}x_n^{k_n}/(1+x_n)^{n+k_n}}{\binom{n+j_n-1}{j_n}x_n^{j_n}/(1+x_n)^{n+j_n}}\left(x_n-\frac{k_n}{n-1}\right)\\ &= \frac{(n+k_n-1)!}{(n+j_n-1)!}\cdot\frac{j_n!}{k_n!}\left(\frac{x_n}{1+x_n}\right)^{k_n-j_n}\left(x_n-\frac{k_n}{n-1}\right)\\ &= \frac{(k_n+1)(k_n+2)\ldots j_n}{(n+k_n)(n+k_n+1)\ldots(n+j_n-1)}\left(\frac{1+[n/2]}{n+[n/2]}\right)^{k_n-j_n}\frac{[\sqrt{n}]+1}{n-1}\\ &\geq \left(\frac{k_n+1}{n+j_n-1}\right)^{j_n-k_n}\left(\frac{1+[n/2]}{n+[n/2]}\right)^{k_n-j_n}\cdot\frac{1}{\sqrt{n}}\\ &= \left(\frac{[n/2]-[\sqrt{n}]+1}{n+[n/2]-1}\right)^{[\sqrt{n}]}\left(\frac{n+[n/2]}{1+[n/2]}\right)^{[\sqrt{n}]}\cdot\frac{1}{\sqrt{n}}\\ &\geq \left(\frac{[n/2]-[\sqrt{n}]+1}{1+[n/2]}\right)^{[\sqrt{n}]}\cdot\frac{1}{\sqrt{n}}\geq\left(\frac{n/2-\sqrt{n}}{1+n/2}\right)^{\sqrt{n}}\cdot\frac{1}{\sqrt{n}}.\end{aligned}$$

Because $\lim_{n\to\infty}\left(\frac{n/2-\sqrt{n}}{1+n/2}\right)^{\sqrt{n}}=e^{-2}$, there exists $n_0\in\mathbb{N}$ such that

$$\left(\frac{1+[n/2]}{[n/2]+[\sqrt{n}]}\right)^{[\sqrt{n}]}\geq e^{-3},\ \text{for all } n\geq\max\{n_0,2\}.$$

It follows

$$\underline{M}_{k_n,n,j_n}(x_n)\geq\frac{e^{-3}}{\sqrt{n}}\geq\frac{e^{-3}}{\sqrt{n+1}},$$

for all $n\geq\max\{n_0,2\}$. Taking into account Lemma 3.1, (ii) too, it follows that for all $n\geq\max\{n_0,2\}$ we have $M_{k_n,n,j_n}(x_n)\geq\frac{e^{-3}}{\sqrt{n+1}}$, which implies the desired conclusion.

In what follows we will prove that for large subclasses of functions f, the order of approximation $\omega_1(f;1/\sqrt{n+1})$ in Theorem 4.1 can essentially be improved to $\omega_1(f;1/n)$.

For this purpose, for any $n\in\mathbb{N}$, $n\geq 2$, $k\in\{0,1,\dots,n\}$ and $j\in\{0,1,\dots,n-2\}$, let us define the functions $f_{k,n,j}:[\frac{j}{n-1},\frac{j+1}{n-1}]\to\mathbb{R}$,

$$f_{k,n,j}(x)=m_{k,n,j}(x)f\left(\frac{k}{n}\right)=\frac{b_{n,k}(x)}{b_{n,j}(x)}f\left(\frac{k}{n}\right)=\frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}}\left(\frac{x}{1+x}\right)^{k-j}f\left(\frac{k}{n}\right).$$

For any $j\in\{0,1,\dots,n-2\}$ and $x\in[\frac{j}{n-1},\frac{j+1}{n-1}]$ we can write

$$U_n^{(M)}(f)(x)=\bigvee_{k=0}^{n}f_{k,n,j}(x).$$

Also, we need the following auxiliary lemmas.

Lemma 4.2. *Let* $f:[0,1]\to[0,\infty)$ *be such that*

$$U_n^{(M)}(f)(x)=\max\{f_{j,n,j}(x),f_{j+1,n,j}(x),f_{j+2,n,j}(x)\},$$

for all $x\in[j/(n-1),(j+1)/(n-1)]$ *and* $n\in\mathbb{N}$, $n\geq 2$. *Then*

$$\left|U_n^{(M)}(f)(x)-f(x)\right|\leq 2\omega_1\left(f;\frac{1}{n}\right),\ \textit{for all } x\in[j/(n-1),(j+1)/(n-1)].$$

Proof. We distinguish three cases:

Case (i). Let $x\in[j/(n-1),(j+1)/(n-1)]$ be fixed such that $U_n^{(M)}(f)(x)=f_{j,n,j}(x)$. By simple calculation we have

$$0\leq x-\frac{j}{n}\leq\frac{j+1}{n-1}-\frac{j}{n}=\frac{n+j}{n(n-1)}\leq\frac{2}{n}$$

and $f_{j,n,j}(x)=f(\frac{j}{n})$, it follows that

$$\left|U_n^{(M)}(f)(x)-f(x)\right|\leq 2\omega_1\left(f;\frac{1}{n}\right).$$

Case (ii). Let $x\in[j/(n-1),(j+1)/(n-1)]$ be such that $U_n^{(M)}(f)(x)=f_{j+1,n,j}(x)$. We have two subcases:

(ii_a) $U_n^{(M)}(f)(x) \leq f(x)$, when evidently $f_{j,n,j}(x) \leq f_{j+1,n,j}(x) \leq f(x)$ and we immediately get

$$\left|U_n^{(M)}(f)(x) - f(x)\right| = |f_{j+1,n,j}(x) - f(x)|$$
$$= f(x) - f_{j+1,n,j}(x) \leq f(x) - f(j/n) \leq 2\omega_1\left(f; \frac{1}{n}\right).$$

(ii_b) $U_n^{(M)}(f)(x) > f(x)$, when

$$\left|U_n^{(M)}(f)(x) - f(x)\right| = f_{j+1,n,j}(x) - f(x) = m_{j+1,n,j}(x) f(\frac{j+1}{n}) - f(x)$$
$$\leq f(\frac{j+1}{n}) - f(x).$$

Since $0 \leq \frac{j+1}{n} - x \leq \frac{j+1}{n} - \frac{j}{n-1} = \frac{n-(j+1)}{n(n-1)} \leq \frac{1}{n}$ it follows $f(\frac{j+1}{n}) - f(x) \leq \omega_1\left(f; \frac{1}{n}\right)$.

Case (iii). Let $x \in [j/(n-1), (j+1)/(n-1)]$ be such that $U_n^{(M)}(f)(x) = f_{j+2,n,j}(x)$. We have two subcases:

(iii_a) $U_n^{(M)}(f)(x) \leq f(x)$, when evidently $f_{j,n,j}(x) \leq f_{j+2,n,j}(x) \leq f(x)$ and we immediately get

$$\left|U_n^{(M)}(f)(x) - f(x)\right| = |f_{j+2,n,j}(x) - f(x)|$$
$$= f(x) - f_{j+2,n,j}(x) \leq f(x) - f(j/n) \leq 2\omega_1\left(f; \frac{1}{n}\right).$$

(iii_b) $U_n^{(M)}(f)(x) > f(x)$, when

$$\left|U_n^{(M)}(f)(x) - f(x)\right| = f_{j+2,n,j}(x) - f(x) = m_{j+2,n,j}(x) f(\frac{j+2}{n}) - f(x)$$
$$\leq f(\frac{j+2}{n}) - f(x).$$

Since $0 \leq \frac{j+2}{n} - x \leq \frac{j+2}{n} - \frac{j}{n-1} = \frac{2n-(j+2)}{n(n-1)} \leq \frac{2}{n}$ it follows $f(\frac{j+2}{n}) - f(x) \leq 2\omega_1\left(f; \frac{1}{n}\right)$, which proves the lemma. □

Lemma 4.3. *Let $f : [0,1] \to [0,\infty)$ be concave. Then the function $g : (0,1] \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing.*

Proof. Let $x, y \in (0,1]$ be with $x \leq y$. Then

$$f(x) = f\left(\frac{x}{y}y + \frac{y-x}{y}0\right) \geq \frac{x}{y}f(y) + \frac{y-x}{y}f(0) \geq \frac{x}{y}f(y),$$

which implies $\frac{f(x)}{x} \geq \frac{f(y)}{y}$. □

Corollary 4.4. *Let $f : [0,1] \to [0,\infty)$ be a nondecreasing function such that the function $g : (0,1] \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing, then*

$$\left|U_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \text{ for all } x \in [0,1], \text{ and } n \in \mathbb{N}, n \geq 2.$$

Proof. Since f is nondecreasing it follows (see the proof of Theorem 5.3 in the next section)

$$U_n^{(M)}(f)(x) = \bigvee_{k\geq j}^{n} f_{k,n,j}(x), \text{ for all } x \in [j/(n-1), (j+1)/(n-1)].$$

Let $x \in [0,1]$ and $j \in \{0,1,\ldots,n-2\}$ such that $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Let $k \in \{0,1,\ldots,n\}$ be with $k \geq j$. Then

$$\begin{aligned} f_{k+1,n,j}(x) &= \frac{\binom{n+k}{k+1}}{\binom{n+j-1}{j}}(\frac{x}{1+x})^{k+1-j} f(\frac{k+1}{n}) \\ &= \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \cdot \frac{n+k}{k+1}(\frac{x}{1+x})^{k-j}\frac{x}{1+x} f(\frac{k+1}{n}). \end{aligned}$$

Since $g(x)$ is nonincreasing we get $\frac{f(\frac{k+1}{n})}{\frac{k+1}{n}} \leq \frac{f(\frac{k}{n})}{\frac{k}{n}}$ that is $f(\frac{k+1}{n}) \leq \frac{k+1}{k} f(\frac{k}{n})$. From $x \leq \frac{j+1}{n-1}$ it follows

$$\begin{aligned} f_{k+1,n,j}(x) &\leq \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}}(\frac{x}{1+x})^{k-j}\frac{j+1}{n+j} \cdot \frac{n+k}{k+1} \cdot \frac{k+1}{k} f(\frac{k}{n}) \\ &= f_{k,n,j}(x)\frac{j+1}{n+j} \cdot \frac{n+k}{k} = \frac{(n+j)k + n(j+1-k) + k}{(n+j)k} \cdot f_{k,n,j}(x). \end{aligned}$$

It is immediate that for $k \geq j+2$ we have $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$. Thus we obtain

$$f_{j+2,n,j}(x) \geq f_{j+3,n,j}(x) \geq \ldots \geq f_{n,j,n}(x),$$

that is

$$U_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x), f_{j+2,n,j}(x)\},$$

for all $x \in [j/(n-1), (j+1)/(n-1)]$, and from Lemma 4.2 we obtain

$$\left|U_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right).$$

□

Corollary 4.5. *Let $f : [0,1] \to [0,\infty)$ be a nondecreasing concave function. Then*

$$\left|U_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \text{ for all } x \in [0,1].$$

Proof. The proof is immediate by Lemma 4.3 and by Corollary 4.4. □

Remarks. 1) It is easy to see that if $f : [0,1] \to [0,\infty)$ is a convex, nondecreasing function satisfying $\frac{f(x)}{x} \geq f(1)$ for all $x \in [0,1]$, then the function $g : (0,1] \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing and as a consequence for f is valid the conclusion of Corollary 4.4. Indeed, for simplicity let us suppose that $f \in C^1[0,1]$ and denote $F(x) = xf'(x) - f(x)$, $x \in [0,1]$. Then $g'(x) = \frac{F(x)}{x^2}$, for all $x \in (0,1]$. Since the inequality $\frac{f(x)}{x} \geq f(1)$ can be written as $\frac{f(1)-f(x)}{1-x} \leq f(1)$, for all $x \in [0,1)$, passing to limit with $x \to 1$ it follows $f'(1) \leq f(1)$, which implies (since f' is nondecreasing)

$$F(x) \leq xf'(1) - f(x) \leq xf'(1) - xf(1) = x[f'(1) - f(1)] \leq 0, \text{ for all } x \in (0,1],$$

which means that $g(x)$ is nonincreasing.

An example of function satisfying the above conditions is $f(x) = e^x$, $x \in [0,1]$.

2) It is known that for the most sequences of linear Bernstein-type operators $L_n(f)(x)$, $n \in \mathbb{N}$, $x \in [0,1]$, the uniform estimate cannot be better than $\omega_2^{\varphi}(f;1/\sqrt{n})$, where $\omega_2^{\varphi}(f;\delta)$ is the Ditzian-Totik second order modulus of smoothness given by

$$\omega_2^{\varphi}(f;\delta) = \sup\{\sup\{|f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|; x \in I_h\}, h \in [0,\delta]\},$$

with $\varphi(x) = \sqrt{x(1-x)}$, $\delta \leq 1$ and $I_h = \left[\frac{h^2}{1+h^2}, \frac{1}{1+h^2}\right]$.

Now, if f is, for example, a nondecreasing concave polygonal line on $[0,1]$, then we get that $\omega_2^{\varphi}(f;\delta) \sim \delta$ for $\delta \leq 1$, which shows that the order of approximation obtained in this case for the most sequences of linear Bernstein-type operators is exactly $\frac{1}{\sqrt{n}}$. On the other hand, since such of function f obviously is a Lipschitz function on $[0,1]$ (as having bounded all the derivative numbers) by Corollary 4.5 we get that the order of approximation by the truncated max-product Baskakov operator is less than $\frac{1}{n}$, which is essentially better than $\frac{1}{\sqrt{n}}$. In a similar manner, by Corollary 4.4 and by Remark 1 after Corollary 4.5, we can produce many subclasses of functions for which the order of uniform approximation given by the truncated max-product Baskakov operator, is essentially better than the order of approximation given by the most sequences of linear Bernstein-type operators.

In fact, the Corollaries 4.4 and 4.5 have no corespondent in the case of these linear and positive operators. All these prove the advantages we may have in some cases, by using the truncated max-product Baskakov operator. Intuitively, the truncated max-product Baskakov operator has better approximation properties than any linear Benstein-type operator, for non-differentiable functions in a finite number of points (with the graphs having some "corners"), as for example for functions defined as a maximum of a finite number of continuous functions on $[0,1]$.

5. **Shape Preserving Properties**

Remark. By the continuity of $U_n^{(M)}(f)(x)$ on $[0,1]$ in Lemma 2.4, it will suffice to prove the shape properties of $U_n^{(M)}(f)(x)$ only on $(0,1]$. As a consequence, in the notations and proofs below we always may suppose that $x > 0$.

In this section we will present some shape preserving properties. In what follows, as in Section 4 for any $k \in \{0,1,\ldots,n\}$ and $j \in \{0,1,\ldots,n-2\}$, let us consider the functions $f_{k,n,j} : [\frac{j}{n-1}, \frac{j+1}{n-1}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{b_{n,k}(x)}{b_{n,j}(x)} f\left(\frac{k}{n}\right) = \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \cdot \left(\frac{x}{1+x}\right)^{k-j} f\left(\frac{k}{n}\right).$$

For any $j \in \{0, 1, \ldots, n-2\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we can write

$$U_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

Lemma 5.1. *Let $n \in \mathbb{N}$, $n \geq 2$. If $f : [0,1] \to \mathbb{R}_+$ is a nondecreasing function then for any $k \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, n-2\}$ with $k \leq j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$.*

Proof. Since $k \leq j$, by the proof of Lemma 3.1, case 2), it follows that $m_{k,n,j}(x) \geq m_{k-1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k-1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \geq m_{k-1,n,j}(x) f\left(\frac{k-1}{n}\right),$$

which proves the lemma. □

Corollary 5.2. *Let $n \in \mathbb{N}$, $n \geq 2$. If $f : [0,1] \to \mathbb{R}_+$ is nonincreasing then $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$ for any $k \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, n-2\}$ with $k \geq j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.*

Proof. Since $k \geq j$, by the proof of Lemma 3.1, case 1), it follows that $m_{k,n,j}(x) \geq m_{k+1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k+1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \geq m_{k+1,n,j}(x) f\left(\frac{k+1}{n}\right),$$

which proves the corollary. □

Theorem 5.3. *If $f : [0,1] \to \mathbb{R}_+$ is nondecreasing then $U_n^{(M)}(f)$ is nondecreasing, for any $n \in \mathbb{N}$ with $n \geq 2$.*

Proof. Since $U_n^{(M)}(f)$ is continuous on $[0,1]$, it suffices to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, \ldots, n-2\}$, $U_n^{(M)}(f)$ is nondecreasing.

So let $j \in \{0, 1, \ldots, n-2\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. As f is nondecreasing, from Lemma 5.1 it follows that

$$f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq \ldots \geq f_{0,n,j}(x).$$

Then it is immediate that

$$U_n^{(M)}(f)(x) = \bigvee_{k \geq j}^{n} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Clearly that for $k \geq j$ the function $f_{k,n,j}$ is nondecreasing and since $U_n^{(M)}(f)$ is defined as the maximum of nondecreasing functions, it follows that it is nondecreasing. □

Corollary 5.4. *If $f : [0,1] \to \mathbb{R}_+$ is nonincreasing then $U_n^{(M)}(f)$ is nonincreasing, for any $n \in \mathbb{N}$ with $n \geq 2$.*

Proof. Since $U_n^{(M)}(f)$ is continuous on $[0,1]$, it suffices to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, \ldots, n-2\}$, $U_n^{(M)}(f)$ is nonincreasing.

So let $j \in \{0, 1, \ldots, n-2\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. As f is nonincreasing, from Corollary 5.2 it follows that

$$f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \ldots \geq f_{n,n,j}(x).$$

Then it is immediate that

$$U_n^{(M)}(f)(x) = \bigvee_{k \geq 0}^{j} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Clearly that for $k \leq j$ the function $f_{k,n,j}$ is nonincreasing and since $U_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. □

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

Definition 5.5. Let $f : [0,1] \to \mathbb{R}$ be continuous on $[0,1]$. One says that f is quasi-convex on $[0,1]$ if it satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y, \lambda \in [0,1].$$

(see e.g. the book [4], p. 4, (iv)).

Remark. By [5], the continuous function f is quasi-convex on $[0,1]$ equivalently means that there exists a point $c \in [0,1]$ such that f is nonincreasing on $[0,c]$ and nondecreasing on $[c,1]$. The class of quasi-convex functions includes the both classes of nondecreasing functions and of nonincreasing functions (obtained from the class of quasi-convex functions by taking $c = 0$ and $c = 1$, respectively). Also, it obviously includes the class of continuous convex functions on $[0,1]$.

Corollary 5.6. *If $f : [0,1] \to \mathbb{R}_+$ is continuous and quasi-convex on $[0,1]$ then for all $n \in \mathbb{N}, n \geq 2$, $U_n^{(M)}(f)$ is quasi-convex on $[0,1]$.*

Proof. If f is nonincreasing (or nondecreasing) on $[0,1]$ (that is the point $c = 1$ (or $c = 0$) in the above Remark) then by the Corollary 5.4 (or Theorem 5.3, respectively) it follows that for all $n \in \mathbb{N}$, $n \geq 2$, $U_n^{(M)}(f)$ is nonincreasing (or nondecreasing) on $[0,1]$.

Suppose now that there exists $c \in (0,1)$, such that f is nonincreasing on $[0,c]$ and nondecreasing on $[c,1]$. Define the functions $F, G : [0,1] \to \mathbb{R}_+$ by $F(x) = f(x)$ for all $x \in [0,c]$, $F(x) = f(c)$ for all $x \in [c,1]$ and $G(x) = f(c)$ for all $x \in [0,c]$, $G(x) = f(x)$ for all $x \in [c,1]$.

It is clear that F is nonincreasing and continuous on $[0,1]$, G is nondecreasing and continuous on $[0,1]$ and that $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0,1]$.

But it is easy to show (see also Remark 1 after the proof of Lemma 2.1) that

$$U_n^{(M)}(f)(x) = \max\{U_n^{(M)}(F)(x), U_n^{(M)}(G)(x)\}, \text{ for all } x \in [0,1],$$

where by the Corollary 5.4 and Theorem 5.3 , $U_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0,1]$ and $U_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0,1]$. We have two cases : 1) $U_n^{(M)}(F)(x)$ and $U_n^{(M)}(G)(x)$ do not intersect each other ; 2) $U_n^{(M)}(F)(x)$ and $U_n^{(M)}(G)(x)$ intersect each other.

Case 1). We have $\max\{U_n^{(M)}(F)(x), U_n^{(M)}(G)(x)\} = U_n^{(M)}(F)(x)$ for all $x \in [0,1]$ or $\max\{U_n^{(M)}(F)(x), U_n^{(M)}(G)(x)\} = U_n^{(M)}(G)(x)$ for all $x \in [0,1]$, which obviously proves that $U_n^{(M)}(f)(x)$ is quasi-convex on $[0,1]$.

Case 2). In this case it is clear that there exists a point $c' \in [0,1]$ such that $U_n^{(M)}(f)(x)$ is nonincreasing on $[0,c']$ and nondecreasing on $[c',1]$, which by the result in [5] implies that $U_n^{(M)}(f)(x)$ is quasiconvex on $[0,1]$ and proves the corollary. □

It is of interest to exactly calculate $U_n^{(M)}(f)$ for $f(x) = e_0(x) = 1$ and for $f(x) = e_1(x) = x$. In this sense we can state the following.

Lemma 5.7. *For all $x \in [0,1]$ and $n \in \mathbb{N}, n \geq 2$ we have $U_n^{(M)}(e_0)(x) = 1$ and*

$$U_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,0}(x)}{b_{n,0}(x)} = \frac{x}{1+x}, \text{ if } x \in [0, 1/n],$$

$$U_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,1}(x)}{b_{n,0}(x)} = \frac{(n+1)x^2}{(1+x)^2}, \text{ if } x \in [1/n, 1/(n-1)],$$

$$U_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,1}(x)}{b_{n,1}(x)} = \frac{x}{1+x} \cdot \frac{n+1}{n}, \text{ if } x \in [1/(n-1), 2/n],$$

$$U_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,2}(x)}{b_{n,1}(x)} = \frac{x^2}{(1+x)^2} \cdot \frac{(n+1)(n+2)}{2n}, \text{ if } x \in [2/n, 2/(n-1)],$$

$$U_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,2}(x)}{b_{n,2}(x)} = \frac{x}{1+x} \cdot \frac{n+2}{n}, \text{ if } x \in [2/(n-1), 3/n],$$

$$U_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,3}(x)}{b_{n,2}(x)} = \frac{x^2}{(1+x)^2} \cdot \frac{(n+2)(n+3)}{3n}, \text{ if } x \in [3/n, 3/(n-1)],$$

and so on, in general we have

$$U_n^{(M)}(e_1)(x) = \frac{x}{1+x} \cdot \frac{n+j}{n}, \text{ if } x \in [j/(n-1), (j+1)/n],$$

$$U_n^{(M)}(e_1)(x) = \frac{x^2}{(1+x)^2} \cdot \frac{(n+j)(n+j+1)}{n(j+1)}, \text{ if } x \in [(j+1)/n, (j+1)/(n-1)],$$

for $j \in \{0, 1, \ldots, n-2\}$.

Proof. The formula $U_n^{(M)}(e_0)(x) = 1$ is immediate by the definition of $U_n^{(M)}(f)(x)$.

To find the formula for $U_n^{(M)}(e_1)(x)$ we will use the explicit formula in Lemma 3.4 which says that

$$\bigvee_{k=0}^{n} b_{n,k}(x) = b_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right], j = 0, 1, \ldots, n-2,$$

where $b_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$.

Since

$$\max_{k=0,\dots,n}\left\{b_{n,k}(x)\frac{k}{n}\right\}=\max_{k=1,\dots,n}\left\{b_{n,k}(x)\frac{k}{n}\right\}=x\cdot\max_{k=0,\dots,n-1}\{b_{n+1,k}(x)\},$$

and because from the proof of Lemma 3.4 we have $b_{n,n}(x)\le b_{n,n-1}(x)\le b_{n,n-2}(x)$ for all $x\in[0,1]$ we obtain

$$U_n^{(M)}(e_1)(x)=x\cdot\frac{\bigvee_{k=0}^{n+1}b_{n+1,k}(x)}{\bigvee_{k=0}^{n}b_{n,k}(x)}$$

Now the conclusion of the lemma is immediate by applying Lemma 3.4 to both expressions $\bigvee_{k=0}^{n+1}b_{n+1,k}(x)$, $\bigvee_{k=0}^{n}b_{n,k}(x)$, taking into account that we get the following division of the interval $[0,1]$

$$0<\frac{1}{n}\le\frac{1}{n-1}\le\frac{2}{n}\le\frac{2}{n-1}\le\frac{3}{n}\le\frac{3}{n-1}\le\frac{4}{n}\le\frac{4}{n-1}\le\dots$$

□

Remarks. 1) The convexity of f on $[0,1]$ is not preserved by $U_n^{(M)}(f)$ as can be seen from Lemma 5.7. Indeed, while $f(x)=e_1(x)=x$ is obviously convex on $[0,1]$, it is easy to see that $U_n^{(M)}(e_1)$ is not convex on $[0,1]$.

2) Also, if f is supposed to be starshaped on $[0,1]$ (that is $f(\lambda x)\le\lambda f(x)$ for all $x,\lambda\in[0,1]$), then again by Lemma 5.7 it follows that $U_n^{(M)}(f)$ for $f(x)=e_1(x)$ is not starshaped on $[0,1]$, although $e_1(x)$ obviously is starshaped on $[0,1]$.

Despite of the absence of the preservation of the convexity, we can prove the interesting property that for any arbitrary nonincreasing function f, the max-product Baskakov operator $U_n^{(M)}(f)$ is piecewise convex on $[0,1]$. We present the following.

Theorem 5.8. *Let $n\in\mathbb{N}$ be with $n\ge 2$. For any nonincreasing function $f:[0,1]\to[0,\infty)$, $U_n^{(M)}(f)$ is convex on any interval of the form $[\frac{j}{n-1},\frac{j+1}{n-1}]$, $j=0,1,\dots,n-2$.*

Proof. For any $k\in\{0,1,\dots,n\}$ and $j\in\{0,1,\dots,n-2\}$ let us consider the functions $f_{k,n,j}:[\frac{j}{n-1},\frac{j+1}{n-1}]\to\mathbb{R}$,

$$f_{k,n,j}(x)=m_{k,n,j}(x)f(\frac{k}{n})=\frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}}\cdot\left(\frac{x}{1+x}\right)^{k-j}f(\frac{k}{n}).$$

From the proof of Corollary 5.4 we have

$$U_n^{(M)}(f)(x)=\bigvee_{k\ge 0}^{j}f_{k,n,j}(x),$$

for any $j\in\{0,1,\dots,n-2\}$ and $x\in[\frac{j}{n-1},\frac{j+1}{n-1}]$.

We will prove that for any fixed j and $k \leq j$, each function $f_{k,n,j}(x)$ is convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, which will imply that $U_n^{(M)}(f)$ can be written as a maximum of some convex functions on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Since $f \geq 0$ it suffices to prove that the functions $g_{k,j} : [0,1] \to \mathbb{R}_+$, $g_{k,j}(x) = \left(\frac{x}{1+x}\right)^{k-j}$ are convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$.

For $k = j$, $g_{j,j}$ is constant so is convex.

For $k = j-1$ it follows $g_{j-1,j}(x) = \frac{x+1}{x}$ for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Then $g''_{j-1,j}(x) = \frac{2}{x^3} > 0$ for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

If $k \leq j-2$ then $g''_{k,j}(x) = (k-j)\left(\frac{x}{1+x}\right)^{k-j-2} \cdot \frac{1}{(x+1)^4} \cdot (k-j-1-2x) > 0$, for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Since all the functions $g_{k,j}$ are convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, we get that $U_n^{(M)}(f)$ is convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ as maximum of these functions, which proves the theorem. □

References

[1] Bede, B., Gal, S. G., Approximation by nonlinear Bernstein and Favard-Szász-Mirakjan operators of max-product kind, *Journal of Concrete and Applicable Mathematics*, **8**(2010), No. 2, 193–207. 90, 91, 92

[2] Bede, B., Nobuhara, H., Fodor, J., Hirota, K., Max-product Shepard approximation operators, *Journal of Advanced Computational Intelligence and Intelligent Informatics*, **10**(2006), 494–497. 89

[3] Bede, B., Nobuhara, H., Daňková, M., Di Nola, A., Approximation by pseudo-linear operators, *Fuzzy Sets and Systems* **159**(2008), 804–820. 89

[4] Gal, S.G., Shape-Preserving Approximation by Real and Complex Polynomials, Birkhäuser, Boston-Basel-Berlin, 2008. 89, 104

[5] Popoviciu, T., Deux remarques sur les fonctions convexes, *Bull. Soc. Sci. Acad. Roumaine*, **220**(1938), 45-49. 104, 105

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