REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 52, Número 1, 2011, Páginas 109–120

SOME SLATER TYPE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S. S. DRAGOMIR

Abstract. Some inequalities of the Slater type for convex functions of selfadjoint operators in Hilbert spaces H under suitable assumptions for the involved operators are given. Amongst others, it is shown that if A is a positive definite operator with $Sp(A) \subset [m, M]$ and f is convex and has a continuous derivative on $[m, M]$, then for any $x \in H$ with $\|x\| = 1$ the following inequality holds:

$$0 \le f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \le \frac{1}{4} \cdot \sqrt{\frac{Mf'(M)}{mf'(m)}} (M - m)\left(f'(M) - f'(m)\right).$$

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_{-}(x) \le f'_{+}(x) \le f'_{-}(y) \le f'_{+}(y)$ which shows that both f'_{-} and f'_{+} are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x - a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_{-}, $f'_{+} \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_{-}(x) \le \varphi(x) \le f'_{+}(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following result is well known in the literature as *the Slater inequality:*

2000 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint operators, Positive operators, Slater's inequality, Convex functions, Functions of selfadjoint operators.

Theorem 1 (Slater, 1981, [14]). *If $f : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $x_i \in I, p_i \geq 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$ and $\sum_{i=1}^{n} p_i \varphi(x_i) \neq 0$, where $\varphi \in \partial f$, then*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq f\left(\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \right). \tag{1.1}$$

As pointed out in [2, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I, \tag{1.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

2. An Operator Reverse for the Slater Inequality

Let A be a selfadjoint operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [8, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [8] and the references therein. For other results, see [11], [12], [13], [9] and [10].

The following result holds:

Theorem 2. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a* *selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive definite operator on H then*

$$0 \leq f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \leq B(f', A; x)\left[\langle Af'(A)x, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle\right], \quad (2.1)$$

where

$$B(f', A; x) := \frac{1}{\langle f'(A)x, x\rangle} . f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s) \quad (2.2)$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then we have

$$\langle f'(A) \cdot (t \cdot 1_H - A)x, x\rangle \leq \langle [f(t) \cdot 1_H - f(A)]x, x\rangle \leq \langle f'(t) \cdot (t \cdot 1_H - A)x, x\rangle \quad (2.3)$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (2.3) is equivalent with

$$t\langle f'(A)x, x\rangle - \langle f'(A)Ax, x\rangle \leq f(t) - \langle f(A)x, x\rangle \leq f'(t)t - f'(t)\langle Ax, x\rangle \quad (2.4)$$

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

Now, since A is selfadjoint with $m1_H \leq A \leq M1_H$ and $f'(A)$ is positive definite, then $mf'(A) \leq Af'(A) \leq Mf'(A)$, i.e., $m\langle f'(A)x, x\rangle \leq \langle Af'(A)x, x\rangle \leq M\langle f'(A)x, x\rangle$ for any $x \in H$ with $\|x\| = 1$, which shows that

$$t_0 := \frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} \in [m, M] \text{ for any } x \in H \text{ with } \|x\| = 1.$$

Finally, if we put $t = t_0$ in the equation (2.4), then we get the desired result (2.1). □

Remark 1. *It is important to observe that, the condition that $f'(A)$ is a positive definite operator on H can be replaced with the more general assumption that*

$$\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} \in \mathring{I} \text{ for any } x \in H \text{ with } \|x\| = 1, \quad (2.5)$$

which may be easily verified for particular convex functions f.

We also notice that the first inequality in (2.1) is the operator version of Slater inequality (1.1).

Remark 2. *Now, if the function is concave on* $\mathring{I}$ *and the condition (2.5) holds, then we have the inequality*

$$0 \leq \langle f(A)x, x\rangle - f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) \leq B(f', A; x)\left[\langle Ax, x\rangle \langle f'(A)x, x\rangle - \langle Af'(A)x, x\rangle\right], \quad (2.6)$$

for any $x \in H$ *with* $\|x\| = 1$.

The following examples are of interest:

Example 1. *If* A *is a positive definite operator on* H, *then*

$$(0 \leq) \langle \ln Ax, x\rangle - \ln\left(\left\langle A^{-1}x, x\right\rangle^{-1}\right) \leq \langle Ax, x\rangle \cdot \left\langle A^{-1}x, x\right\rangle - 1, \quad (2.7)$$

for any $x \in H$ *with* $\|x\| = 1$.

Indeed, we observe that if we consider the concave function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then

$$\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} = \frac{1}{\langle A^{-1}x, x\rangle} \in (0, \infty), \text{ for any } x \in H \text{ with } \|x\| = 1$$

and by the inequality (2.6) we deduce the desired result (2.7).

The following example concerning powers of operators is of interest as well:

Example 2. *If* A *is a positive definite operator on* H, *then for any* $x \in H$ *with* $\|x\| = 1$ *we have*

$$0 \leq \langle A^p x, x\rangle^{p-1} - \left\langle A^{p-1}x, x\right\rangle^p \leq p\langle A^p x, x\rangle^{p-2}\left[\langle A^p x, x\rangle - \langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle\right] \quad (2.8)$$

for $p \geq 1$,

$$0 \leq \left\langle A^{p-1}x, x\right\rangle^p - \langle A^p x, x\rangle^{p-1} \leq p\langle A^p x, x\rangle^{p-2}\left[\langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle - \langle A^p x, x\rangle\right] \quad (2.9)$$

for $0 < p < 1$, *and*

$$0 \leq \langle A^p x, x\rangle^{p-1} - \left\langle A^{p-1}x, x\right\rangle^p \leq (-p)\langle A^p x, x\rangle^{p-2}\left[\langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle - \langle A^p x, x\rangle\right] \quad (2.10)$$

for $p < 0$.

The proof follows from the inequalities (2.1) and (2.6) for the convex (concave) function $f(t) = t^p, p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0, 1))$ by performing the required calculation. The details are omitted.

3. Some Lemmas of Interest

In order to provide other reverses for the operator version of Slater's inequality, we need the following lemmas that are of interest in their own right:

Lemma 1. *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A) \subseteq [m,M]$ for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$\begin{aligned}&\Big|\langle f(A)\,g(A)\,y,y\rangle - \langle f(A)\,y,y\rangle \cdot \langle g(A)\,x,x\rangle \\ &\qquad -\frac{\gamma+\Gamma}{2}\left[\langle g(A)\,y,y\rangle - \langle g(A)\,x,x\rangle\right]\Big| \\ &\leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)\,y\|^2 + \langle g(A)\,x,x\rangle^2 - 2\langle g(A)\,x,x\rangle\langle g(A)\,y,y\rangle\right]^{1/2} \end{aligned} \tag{3.1}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Proof. First of all, observe that, for each $\lambda\in\mathbb{R}$ and $x,y\in H$, $\|x\|=\|y\|=1$ we have the identity

$$\begin{aligned}&\langle (f(A)-\lambda\cdot 1_H)(g(A)-\langle g(A)\,x,x\rangle\cdot 1_H)\,y,y\rangle \\ &= \langle f(A)\,g(A)\,y,y\rangle - \lambda\cdot[\langle g(A)\,y,y\rangle - \langle g(A)\,x,x\rangle] - \langle g(A)\,x,x\rangle\langle f(A)\,y,y\rangle .\end{aligned} \tag{3.2}$$

Taking the modulus in (3.2) we have

$$\begin{aligned}&|\langle f(A)\,g(A)\,y,y\rangle - \lambda\cdot[\langle g(A)\,y,y\rangle - \langle g(A)\,x,x\rangle] \\ &- \langle g(A)\,x,x\rangle\langle f(A)\,y,y\rangle| \\ &= |\langle (g(A)-\langle g(A)\,x,x\rangle\cdot 1_H)\,y,(f(A)-\lambda\cdot 1_H)\,y\rangle| \\ &\leq \|g(A)\,y-\langle g(A)\,x,x\rangle\,y\|\,\|f(A)\,y-\lambda y\| \\ &= \left[\|g(A)\,y\|^2+\langle g(A)\,x,x\rangle^2-2\langle g(A)\,x,x\rangle\langle g(A)\,y,y\rangle\right]^{1/2} \\ &\times\|f(A)\,y-\lambda y\| \\ &\leq \left[\|g(A)\,y\|^2+\langle g(A)\,x,x\rangle^2-2\langle g(A)\,x,x\rangle\langle g(A)\,y,y\rangle\right]^{1/2} \\ &\times\|f(A)-\lambda\cdot 1_H\|\end{aligned} \tag{3.3}$$

for any $x,y\in H$, $\|x\|=\|y\|=1$.

Now, since $\gamma=\min_{t\in[m,M]} f(t)$ and $\Gamma=\max_{t\in[m,M]} f(t)$, then by the property (P) we have that $\gamma\leq\langle f(A)\,y,y\rangle\leq\Gamma$ for each $y\in H$ with $\|y\|=1$ which is clearly equivalent with

$$\left|\langle f(A)\,y,y\rangle - \frac{\gamma+\Gamma}{2}\|y\|^2\right| \leq \frac{1}{2}(\Gamma-\gamma)$$

or with

$$\left|\left\langle\left(f(A)-\frac{\gamma+\Gamma}{2}1_H\right)y,y\right\rangle\right| \leq \frac{1}{2}(\Gamma-\gamma)$$

for each $y\in H$ with $\|y\|=1$.

Taking the supremum in this inequality we get

$$\left\|f(A)-\frac{\gamma+\Gamma}{2}\cdot 1_H\right\|\le\frac{1}{2}(\Gamma-\gamma),$$

which together with the inequality (3.3) applied for $\lambda=\frac{\gamma+\Gamma}{2}$ produces the desired result (3.1). □

Corollary 1. *With the assumptions in Lemma 1 we have*

$$\begin{aligned}&|\langle f(A)\,g(A)\,x,x\rangle-\langle f(A)\,x,x\rangle\cdot\langle g(A)\,x,x\rangle|\\&\quad\le\frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)\,x\|^2-\langle g(A)\,x,x\rangle^2\right]^{1/2}\left(\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)\end{aligned}\tag{3.4}$$

for each $x\in H$ *with* $\|x\|=1$, *where* $\delta:=\min_{t\in[m,M]}g(t)$ *and* $\Delta:=\max_{t\in[m,M]}g(t)$.

Proof. The first inequality follows from (3.1) by putting $y=x$.

Now, if we write the first inequality in (3.1) for $f=g$ we get

$$\begin{aligned}0\le\|g(A)\,x\|^2-\langle g(A)\,x,x\rangle^2&=\langle g^2(A)\,x,x\rangle-\langle g(A)\,x,x\rangle^2\\&\le\frac{1}{2}(\Delta-\delta)\left[\|g(A)\,x\|^2-\langle g(A)\,x,x\rangle^2\right]^{1/2}\end{aligned}$$

which implies that

$$\left[\|g(A)\,x\|^2-\langle g(A)\,x,x\rangle^2\right]^{1/2}\le\frac{1}{2}(\Delta-\delta)$$

for each $x\in H$ with $\|x\|=1$.

This together with the first part of (3.1) proves the desired bound (3.4). □

The following lemmas, that are of interest in their own right, collect some Grüss type inequalities for vectors in inner product spaces obtained earlier by the author:

Lemma 2. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, $u,v,e\in H$, $\|e\|=1$, *and* $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ *such that*

$$\operatorname{Re}\langle\beta e-u,u-\alpha e\rangle\ge 0,\operatorname{Re}\langle\delta e-v,v-\gamma e\rangle\ge 0\tag{3.5}$$

or, equivalently,

$$\left\|u-\frac{\alpha+\beta}{2}e\right\|\le\frac{1}{2}|\beta-\alpha|,\left\|v-\frac{\gamma+\delta}{2}e\right\|\le\frac{1}{2}|\delta-\gamma|.\tag{3.6}$$

Then

$$\begin{aligned}&|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle|\\&\quad\le\frac{1}{4}\cdot|\beta-\alpha|\,|\delta-\gamma|-\begin{cases}\left[\operatorname{Re}\langle\beta e-u,u-\alpha e\rangle\operatorname{Re}\langle\delta e-v,v-\gamma e\rangle\right]^{\frac{1}{2}},\\[2mm]\left|\langle u,e\rangle-\frac{\alpha+\beta}{2}\right|\left|\langle v,e\rangle-\frac{\gamma+\delta}{2}\right|.\end{cases}\end{aligned}\tag{3.7}$$

The first inequality has been obtained in [3] (see also [7, p. 44]) while the second result was established in [4] (see also [7, p. 90]). They provide refinements of the earlier result from [1] where only the first part of the bound, i.e., $\frac{1}{4}|\beta-\alpha||\delta-\gamma|$ has been given. Notice that, as pointed out in [4], the upper bounds for the Grüss functional incorporated in (3.7) cannot be compared in general, meaning that one is better than the other depending on appropriate choices of the vectors and scalars involved.

Another result of this type is the following one:

Lemma 3. *With the assumptions in Lemma 2 and if* $\operatorname{Re}(\beta\overline{\alpha})>0, \operatorname{Re}(\delta\overline{\gamma})>0$ *then*

$$|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle| \leq \begin{cases} \frac{1}{4}\cdot\frac{|\beta-\alpha||\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{\frac{1}{2}}}|\langle u,e\rangle\langle e,v\rangle|, \\ \\ \left[\left(|\alpha+\beta|-2[\operatorname{Re}(\beta\overline{\alpha})]^{\frac{1}{2}}\right)\left(|\delta+\gamma|-2[\operatorname{Re}(\delta\overline{\gamma})]^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\ \cdot[|\langle u,e\rangle\langle e,v\rangle|]^{\frac{1}{2}}. \end{cases} \tag{3.8}$$

The first inequality has been established in [5] (see [7, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [6]. The details are omitted.

4. Further Reverses for the Slater's Inequality

The following results that provide perhaps more useful upper bounds for the nonnegative quantity

$$f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle \text{ for } x\in H \text{ with } \|x\|=1,$$

can be stated:

Theorem 3. *Let I be an interval and $f:I\to\mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. Assume that A is a selfadjoint operator on the Hilbert space H with $Sp(A)\subseteq[m,M]\subset\mathring{I}$ and $f'(A)$ is a positive definite operator on H. Then we have the inequalities*

$$\begin{aligned}(0\leq)\, & f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle \\ & \leq B(f',A;x)\times\begin{cases}\frac{1}{2}\cdot(M-m)\left[\|f'(A)x\|^2-\langle f'(A)x,x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}\cdot(f'(M)-f'(m))\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2}\end{cases} \\ & \leq \frac{1}{4}(M-m)(f'(M)-f'(m))B(f',A;x)\end{aligned} \tag{4.1}$$

and

$$
\begin{aligned}
(0\le)\, f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) - \langle f(A)x,x\rangle \\
\le B(f',A;x)\times\Bigg[\frac{1}{4}(M-m)(f'(M)-f'(m)) \\
-\begin{cases} [\langle Mx-Ax,Ax-mx\rangle\langle f'(M)x-f'(A)x,f'(A)x-f'(m)x\rangle]^{\frac{1}{2}}, \\ \left|\langle Ax,x\rangle-\frac{M+m}{2}\right|\left|\langle f'(A)x,x\rangle-\frac{f'(M)+f'(m)}{2}\right| \end{cases}\Bigg] \\
\le \frac{1}{4}(M-m)(f'(M)-f'(m))\,B(f',A;x) \qquad (4.2)
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$, respectively.

Moreover, if A is a positive definite operator, then

$$
\begin{aligned}
(0\le)\, f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) - \langle f(A)x,x\rangle \\
\le B(f',A;x)\times\begin{cases} \frac{1}{4}\cdot\frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}}\langle Ax,x\rangle\langle f'(A)x,x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)[\langle Ax,x\rangle\langle f'(A)x,x\rangle]^{\frac{1}{2}}, \end{cases} \\
\le B(f',A;x)\times\begin{cases} \frac{1}{4}\cdot\sqrt{\frac{Mf'(M)}{mf'(m)}}(M-m)(f'(M)-f'(m)), \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)\sqrt{Mf'(M)}, \end{cases} \qquad (4.3)
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$.

Proof. By Corollary 1 we can state that

$$
\begin{aligned}
\langle Af'(A)x,x\rangle - \langle Ax,x\rangle\cdot\langle f'(A)x,x\rangle \\
\le \frac{1}{2}\cdot(M-m)\left[\|f'(A)x\|^2-\langle f'(A)x,x\rangle^2\right]^{1/2} \\
\le \frac{1}{4}(M-m)(f'(M)-f'(m)) \qquad (4.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle Af'(A)x,x\rangle - \langle Ax,x\rangle\cdot\langle f'(A)x,x\rangle \\
\le \frac{1}{2}\cdot(f'(M)-f'(m))\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2} \\
\le \frac{1}{4}(M-m)(f'(M)-f'(m)), \qquad (4.5)
\end{aligned}
$$

for each $x\in H$ with $\|x\|=1$, which together with (2.1) provide the desired result (4.1).

On making use of the Lemma 2, we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \le \frac{1}{4}(M-m)(f'(M)-f'(m))$$
$$- \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \left|\langle Ax,x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x,x\rangle - \frac{f'(M)+f'(m)}{2}\right|, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (4.2).

Finally, on making use of Lemma 3 we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle$$
$$\le \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \langle Ax,x\rangle \langle f'(A)x,x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)[\langle Ax,x\rangle \langle f'(A)x,x\rangle]^{\frac{1}{2}}, \end{cases} \tag{4.6}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (4.3). □

Remark 3. *If A is a positive definite operator with $Sp(A) \subset [m,M]$, then obviously*

$$B(f',A;x) \le \frac{f'(M)}{f'(m)}$$

and from (4.2) we have

$$(0 \le) f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) - \langle f(A)x,x\rangle$$
$$\le \frac{1}{4} \cdot \frac{f'(M)}{f'(m)}(M-m)(f'(M)-f'(m)) \tag{4.7}$$

while from (4.3) we have

$$(0 \le) f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) - \langle f(A)x,x\rangle$$
$$\le \begin{cases} \frac{1}{4} \cdot \sqrt{\frac{Mf'(M)}{mf'(m)}}(M-m)(f'(M)-f'(m)), \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)\frac{f'(M)}{f'(m)}\sqrt{Mf'(M)}. \end{cases} \tag{4.8}$$

In order to compare the upper bound for the difference $f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle$ *provided by (4.7) with the first bound from (4.8), consider the quantity*

$$K(f',M.m):=\frac{\frac{f'(M)}{f'(m)}}{\sqrt{\frac{Mf'(M)}{mf'(m)}}}=\sqrt{\frac{\frac{f'(M)}{f'(m)}}{\frac{M}{m}}}.$$

For the convex function $f(t)=t^p, p\geq 1$ *we have*

$$K(f',M.m)=\left(\frac{M}{m}\right)^{\frac{p-2}{2}}$$

which shows that for $p\geq 2$ *the bound from (4.7) is not as good as the first bound from (4.8). The conclusion is the other way around if* $1\leq p<2$.

Similar comments can be made for the other bounds. The details are omitted.

Problem 1. *It is an open problem for the author whether or not, for a given convex function* f *and a given positive definite operator* A *with* $Sp(A)\subset[m,M]$ *there exists a vector* $x\in H$ *with* $\|x\|=1$ *realizing the equality case in either of the above inequalities (4.7) and (4.8).*

Remark 4. *We observe, from the first inequality in (4.6), that*

$$(1\leq)\frac{\langle Af'(A)x,x\rangle}{\langle Ax,x\rangle\langle f'(A)x,x\rangle}\leq\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1$$

which implies that

$$f'\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)\leq f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right),$$

for each $x\in H$ *with* $\|x\|=1$, *since* f' *is monotonic nondecreasing and* A *is positive definite.*

Now, the first inequality in (4.3) implies the following result

$$\begin{aligned}(0\leq)f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle&\\ \leq\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}&\\ \times f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right)\langle Ax,x\rangle&\\ \leq\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]M\right)M&\end{aligned}\tag{4.9}$$

for each $x\in H$ *with* $\|x\|=1$.

From the second inequality in (4.3) we also have

$$
\begin{aligned}
(0 \le) f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) &- \langle f(A)x,x\rangle \\
&\le \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right) \\
\times f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right)&\left[\frac{\langle Ax,x\rangle}{\langle f'(A)x,x\rangle}\right]^{\frac{1}{2}} \\
&\le \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right) \\
\times f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]M\right)&\sqrt{\frac{M}{f'(m)}} \qquad (4.10)
\end{aligned}
$$

for each $x \in H$ with $\|x\| = 1$.

Remark 5. *If the condition that $f'(A)$ is a positive definite operator on H from the Theorem 3 is replaced by the condition (2.5), then the inequalities (4.1) and (4.2) will still hold. Similar inequalities can be stated for concave functions. However, the details are not provided here.*

Acknowledgement. The author would like to thank the anonymous referee for valuable comments that have been incorporated in the final version of the paper.

References

[1] S.S. Dragomir, A generalisation of Grüss' inequality in inner product spaces and applications, *J. Mathematical Analysis and Applications,* **237** (1999), 74-82. 115

[2] S.S. Dragomir, *Discrete Inequalities of the Cauchy-Bunyakovsky-Schwarz Type*, Nova Science Publishers, NY, 2004. 110

[3] S.S. Dragomir, Some Grüss type inequalities in inner product spaces, *J. Inequal. Pure & Appl. Math.*, **4**(2) (2003), Article 42. (Online http://jipam.vu.edu.au/article.php?sid=280). 115

[4] S.S. Dragomir, On Bessel and Grüss inequalities for orthornormal families in inner product spaces, *Bull. Austral. Math. Soc.*, **69**(2) (2004), 327-340. 115

[5] S.S. Dragomir, Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces, *J. Inequal. Pure & Appl. Math.,* **5**(3) (2004), Article 76. (Online: http://jipam.vu.edu.au/article.php?sid=432). 115

[6] S.S. Dragomir, Reverses of the Schwarz inequality in inner product spaces generalising a Klamkin-McLenaghan result, *Bull. Austral. Math. Soc.* **73**(1)(2006), 69-78. 115

[7] S.S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces,* Nova Science Publishers Inc, New York, 2005, x+249 p. 115

[8] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005. 110

[9] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), no. 2-3, 551–564. 110

[10] J. Mićić, Y.Seo, S.-E. Takahasi and M. Tominaga, Inequalities of Furuta and Mond-Pečarić, *Math. Ineq. Appl.*, **2**(1999), 83-111. 110

[11] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19**(1993), 405-420. 110

[12] B. Mond and J. Pečarić, On some operator inequalities, *Indian J. Math.*, **35**(1993), 221-232. 110
[13] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166. 110
[14] M.S. Slater, A companion inequality to Jensen's inequality, *J. Approx. Theory*, **32**(1981), 160-166. 110

S. S. Dragomir
Research Group in Mathematical Inequalities & Applications
School of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
sever.dragomir@vu.edu.au
http://www.staff.vu.edu.au/rgmia/dragomir/
School of Computational and Applied Mathematics
University of the Witwatersrand, Private Bag 3
Wits 2050, Johannesburg, South Africa

Recibido: 29 de octubre de 2009
Aceptado: 12 de octubre de 2010