REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 51, Número 2, 2010, Páginas 121–132

NECESSARY AND SUFFICIENT CONDITIONS FOR THE SCHUR HARMONIC CONVEXITY OR CONCAVITY OF THE EXTENDED MEAN VALUES*

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Abstract. In this paper, we prove that the extended values $E(r,s;x,y)$ are Schur harmonic convex (or concave, respectively) with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ if and only if $(r,s) \in \{(r,s) : s \geq -1, s \geq r, s+r+3 \geq 0\} \cup \{(r,s) : r \geq -1, r \geq s, s+r+3 \geq 0\}$ (or $\{(r,s) : s \leq -1, r \leq -1, s+r+3 \leq 0\}$, respectively).

1. Introduction

For $x, y > 0$ and $r, s \in \mathbb{R}$, the extended mean values $E(r,s;x,y)$ were defined by Stolarsky [27] as follows.

$$E(r,s;x,y) = \left(\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right)^{\frac{1}{s-r}}, \quad rs(r-s)(x-y)\neq 0; \tag{1.1}$$

$$E(r,0;x,y) = \left(\frac{1}{r}\cdot\frac{y^r-x^r}{\log y-\log x}\right)^{\frac{1}{r}}, \quad r(x-y)\neq 0; \tag{1.2}$$

$$E(r,r;x,y) = \frac{1}{e^{\frac{1}{r}}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}}, \quad r(x-y)\neq 0; \tag{1.3}$$

$$E(0,0;x,y) = \sqrt{xy}, \qquad x\neq y; \tag{1.4}$$

$$E(r,s;x,y) = x, \qquad x=y. \tag{1.5}$$

It is not difficult to verify that the extended values $E(r,s;x,y)$ are continuous on the domain $\{(r,s;x,y) : r,s\in\mathbb{R}; x,y>0\}$ and differentiable with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for fixed $r,s\in\mathbb{R}$. They are of symmetry between r and s and between x and y, many basic properties have been obtained by Leach and Sholander in [15]. Many mean values are special cases of the extended mean

2010 *Mathematics Subject Classification.* Primary 26B25; Secondary 26E60.

Key words and phrases. extended mean value, Schur-convex, Schur harmonic convex.

*This work was supported by NSF of China under grant No. 11071069, NSF of Zhejiang Province under grant Nos.Y7080185 and Y7080106, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant No.T200924.

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$E(r,s;x,y)$, for example,

$$\begin{array}{rcll} M_r(x,y) &=& E(r,2r;x,y) & \text{is the power mean or Hölder mean,}\\ S_p(x,y) &=& E(1,p;x,y) & \text{is the extended logarithmic mean,}\\ I(x,y) &=& E(1,1;x,y) & \text{is the identric or exponential mean,}\\ I_p(x,y) &=& E(p,p;x,y) & \text{is the extended identric or exponential mean,}\\ A(x,y) &=& E(1,2;x,y) & \text{is the arithmetic mean,}\\ G(x,y) &=& E(0,0;x,y) & \text{is the geometric mean,}\\ H(x,y) &=& E(-2,-1;x,y) & \text{is the harmonic mean,}\\ L(x,y) &=& E(0,1;x,y) & \text{is the logarithmic mean,}\\ F_r(x,y) &=& E(r,r+1;x,y) & \text{is the one-parameter mean.} \end{array}$$

Study of $E(r,s;x,y)$ is not only interesting but also important, because most of the two-variables mean values are special cases of $E(r,s;x,y)$ and it is challenging to study a function whose formulation is so indeterminate [20].

For convenience of readers, we recall the notations and definitions as follows.

For $x=(x_1,x_2)\in(0,\infty)\times(0,\infty)$ and $\alpha\geq 0$, we denote by

$$x+y=(x_1+y_1,x_2+y_2),$$

$$xy=(x_1y_1,x_2y_2),$$

$$\alpha x=(\alpha x_1,\alpha x_2)$$

and

$$\frac{1}{x}=(\frac{1}{x_1},\frac{1}{x_2}).$$

Definition 1.1. A set $E_1\subseteq\mathbb{R}^2$ is called a convex set if $\frac{x+y}{2}\in E_1$ whenever $x,y\in E_1$. A set $E_2\subseteq(0,\infty)\times(0,\infty)$ is called a harmonic convex set if $\frac{2xy}{x+y}\in E_2$ whenever $x,y\in E_2$.

It is easy to see that $E\subseteq(0,\infty)\times(0,\infty)$ is a harmonic convex set if and only if $\frac{1}{E}=\{\frac{1}{x}:x\in E\}$ is a convex set.

Definition 1.2. Let $E\subseteq\mathbb{R}^2$ be a convex set. A function $f:E\to\mathbb{R}$ is said to be a convex function on E if $f(\frac{x+y}{2})\leq\frac{f(x)+f(y)}{2}$ for all $x,y\in E$. Moreover, f is called a concave function if $-f$ is a convex function.

Definition 1.3. Let $E\subseteq(0,\infty)\times(0,\infty)$ be a harmonic convex set. A function $f:E\to(0,\infty)$ is called a harmonic convex (or concave, respectively) function on E if $f(\frac{2xy}{x+y})\leq$ (or $\geq$, respectively) $\frac{2f(x)f(y)}{f(x)+f(y)}$ for all $x,y\in E$.

Definitions 1.2 and 1.3 have the following consequence.

Fact A. *If $E_1\subseteq(0,\infty)\times(0,\infty)$ is a harmonic convex set and $f:E_1\to(0,\infty)$ is a harmonic convex function, then*

$$F(x)=\frac{1}{f(\frac{1}{x})}:\frac{1}{E_1}\to(0,\infty)$$

is a concave function. Conversely, if $E_2 \subseteq (0,\infty) \times (0,\infty)$ *is a convex set and* $F : E_2 \to (0,\infty)$ *is a convex function, then*

$$f(x) = \frac{1}{F(\frac{1}{x})} : \frac{1}{E_2} \to (0,\infty)$$

is a harmonic concave function.

Definition 1.4. Let $E \subseteq \mathbb{R}^2$ be a set. A function $F : E \to \mathbb{R}$ is called a Schur convex function on E if

$$F(x_1, x_2) \leq F(y_1, y_2)$$

for each pair of two-tuples $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $x \prec y$, i.e.

$$x_{[1]} \leq y_{[1]}$$

and

$$x_{[1]} + x_{[2]} = y_{[1]} + y_{[2]},$$

where $x_{[i]}$ denotes the ith largest component in x. A function F is called a Schur concave function if $-F$ is a Schur convex function.

Definition 1.5. Let $E \subseteq (0,\infty) \times (0,\infty)$ be a set. A function $F : E \to \mathbb{R}$ is called a Schur harmonic convex (or concave, respectively) function on E if

$$F(x_1, x_2) \leq (\text{or } \geq, \text{respectively}) F(y_1, y_2)$$

for each pair of $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$.

Definitions 1.4 and 1.5 have the following consequence.

Fact B. *Let* $E \subseteq (0,\infty) \times (0,\infty)$ *be a set, and* $H = \frac{1}{E} = \{\frac{1}{x} : x \in E\}$*. Then* $f : E \to (0,\infty)$ *is a Schur harmonic convex (or concave, respectively) function on* E *if and only if* $\frac{1}{f(\frac{1}{x})}$ *is a Schur concave (or convex, respectively) function on* H.

The following well-known result was proved by Marshall and Olkin in [17].

Theorem C. *Let* $E \subseteq \mathbb{R}^2$ *be a symmetric convex set with nonempty interior* $intE$ *and* $\varphi : E \to \mathbb{R}$ *be a continuous symmetry function on* E*. If* φ *is differentiable on* $intE$*, then* φ *is Schur convex (or concave, respectively) on* E *if and only if*

$$(y-x)\left(\frac{\partial \varphi}{\partial y} - \frac{\partial \varphi}{\partial x}\right) \geq 0 \ (or \leq 0, respectively)$$

for all $(x,y) \in intE$.

The following Theorem D can easily be derived from Fact B and Theorem C.

Theorem D. *Let* $E \subseteq (0,\infty) \times (0,\infty)$ *be a symmetric harmonic convex set with nonempty interior* $intE$ *and* $\varphi : E \to (0,\infty)$ *be a continuous symmetry function on* E*. If* φ *is differentiable on* $intE$*, then* φ *is Schur harmonic convex (or concave, respectively) on* E *if and only if*

$$(y-x)\left(y^2\frac{\partial \varphi}{\partial y} - x^2\frac{\partial \varphi}{\partial x}\right) \geq 0 \ (or \leq 0, respectively)$$

for all $(x, y) \in intE$.

The theory of convex functions and Schur convex functions is one of the most important theory in the fields of modern analysis and geometry. It can be used in global Riemannian geometry [11, 12], operator inequalities [3], nonlinear PDE of elliptic type [16], combinatorial optimization [13], isoperimetric problem for polytopes [30], linear regression [26], graphs and matrices [6], improperly posed problems [28], inequalities and extremal problems [9], nilpotent groups [10], global surface theory [24] and other related fields.

The notion of generalized convex function was first introduced by Aczél [1]. Later, many authors established inequalities by using harmonic convex functions theory (see [4, 7, 8, 14, 18, 19, 23, 29]). Recently, Anderson, Vamanamurthy and Vuorinen [2] discussed an attractive class of inequalities, which arise from the notion of harmonic convex functions.

The Schur convexity of the extended mean values $E(r, s; x, y)$ with respect to (r, s) and (x, y) are investigated in [21, 22, 25]. Qi [21] first obtained the following result.

Theorem E. *For fixed* $(x, y) \in (0, \infty) \times (0, \infty)$ *with* $x \neq y$, *the extended mean values* $E(r, s; x, y)$ *are Schur convex on* $(-\infty, 0] \times (-\infty, 0]$ *and Schur concave on* $[0, \infty) \times [0, \infty)$ *with respect to* (r, s).

In [22], Qi, Sándor, Dragomir and Sofo tried to obtain the Schur convexity of the extended mean values $E(r, s; x, y)$ with respect to (x, y) for fixed (r, s) and declared an incorrect conclusion as follows. For given (r, s) with $r, s \notin (0, \frac{3}{2})$ (or $r, s \in (0, 1]$, respectively), the extended mean values $E(r, s; x, y)$ are Schur concave (or Schur convex, respectively) with respect to (x, y) on $(0, \infty) \times (0, \infty)$. Shi, Wu and Qi [25] observed that the above conclusion is wrong and obtained the following Theorem F.

Theorem F. *For fixed* $(r, s) \in \mathbb{R}^2$,

(1) if $2 < 2r < s$ *or* $2 \leq 2s \leq r$, *then the extended mean values* $E(r, s; x, y)$ *are Schur convex with respect to* $(x, y) \in (0, \infty) \times (0, \infty)$*;*

(2) if $(r, s) \in \{r < s \leq 2r, 0 < r \leq 1\} \cup \{s < r \leq 2s, 0 < s \leq 1\} \cup \{0 < s < r \leq 1\} \cup \{0 < r < s \leq 1\} \cup \{s \leq 2r < 0\} \cup \{r \leq 2s < 0\}$, *then the extended mean values* $E(r, s; x, y)$ *are Schur concave with respect to* $(x, y) \in (0, \infty) \times (0, \infty)$.

Recently, Chu and Zhang [5] established the necessary and sufficient conditions such that the extended mean mean values $E(r, s; x, y)$ are Schur convex or Schur concave. But none has ever researched the Schur harmonic convexity of the extended mean values $E(r, s; x, y)$. The main purpose of this article is to present the Schur harmonic convexity of the extended mean values $E(r, s; x, y)$ with respect to (x, y) for fixed (r, s). Our main result is the following Theorem 1.1.

Theorem 1.1. *For fixed* $r, s \in \mathbb{R}^2$, *(1) the extended mean values $E(r,s;x,y)$ are Shur harmonic convex with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $(r,s)\in\{(r,s): s\geq -1, s\geq r, s+r+3\geq 0\}\cup\{(r,s): r\geq -1, r\geq s, s+r+3\geq 0\}$;*

(2) the extended mean values $E(r,s;x,y)$ are Shur harmonic concave with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $(r,s)\in\{(r,s): s\leq -1, r\leq -1, s+r+3\leq 0\}$.

From Theorem 1.1 we have five corollaries as follows.

Corollary 1.1. *For all $x,y\in(0,\infty)$,*

(1) $E(r,s;x,y)\geq H(x,y)$ if and only if $(r,s)\in\{(r,s): s\geq -1, s\geq r, s+r+3\geq 0\}\cup\{(r,s): r\geq -1, r\geq s, s+r+3\geq 0\}$;

(2) $E(r,s;x,y)\leq H(x,y)$ if and only if $(r,s)\in\{(r,s): s\leq -1, r\leq -1, s+r+3\leq 0\}$.

Proof. For any $x,y\in(0,\infty)$ we clearly see that

$$\left(\frac{1}{H(x,y)},\frac{1}{H(x,y)}\right)=\left(\frac{x+y}{2xy},\frac{x+y}{2xy}\right)\prec\left(\frac{1}{x},\frac{1}{y}\right). \tag{1.6}$$

Therefore, Corollary 1.1 follows from (1.5) and Definition 1.5 together with (1.6).

Corollary 1.2. *The extended logarithmic mean values $E(1,p;x,y)=S_p(x,y)$ are Schur harmonic convex with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $p\geq -4$.*

Corollary 1.3. *The extended identric or exponential mean values $E(p,p;\ x,y)=I_p(x,y)$ are Schur harmonic convex with $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $p\geq -1$, and Schur harmonic concave if and only if $p\leq -\frac{3}{2}$.*

Corollary 1.4. *The Hölder or power mean values $E(r,2r;x,y)=M_r(x,y)$ are Schur harmonic convex with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $r\geq -1$, and Schur harmonic concave if and only if $r\leq -1$.*

Corollary 1.5. *The one-parameter mean values $E(r,r+1;x,y)=F_r(x,y)$ are Schur harmonic convex with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $r\geq -2$, and Schur harmonic concave with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $r\leq -2$.*

2. Lemmas

In this section we introduce and establish several lemmas, which are used in the proof of Theorem 1.1.

Lemma 2.1. *Let $s,r\in\mathbb{R}, s\neq 0$ and $f(t)=\frac{r}{s}[(s-r)(t^{s+r+1}-1)-s(t^{s+1}-t^r)+r(t^{r+1}-t^s)]$. Then the following statements hold.*

(1) If $s\geq -1, s>r$ and $s+r+3\geq 0$, then $f(t)\geq 0$ for $t\in[1,\infty)$;

(2) If $s>-1$ and $s+r+3<0$, then there exist $t_1,t_2\in(1,\infty)$ such that $f(t_1)>0$ and $f(t_2)<0$;

(3) If $s\leq -1, s>r$ and $s+r+3\leq 0$, then $f(t)\leq 0$ for $t\in[1,\infty)$;

(4) If $s < -1, s > r$ and $s + r + 3 > 0$, then there exist $t_3, t_4 \in (1, \infty)$ such that $f(t_3) > 0$ and $f(t_4) < 0$.

Proof. (1) Let $f_1(t) = t^{-s-r}f'(t), f_2(t) = t^{2+s}f_1'(t)$ and $f_3(t) = t^{2-s+r}f_2''(t)$. Then simple computations yield

$$f(1) = 0, \tag{2.1}$$

$$f'(t) = \frac{r}{s}(s-r)(s+r+1)t^{s+r} - r(s+1)t^s + r^2t^{r-1} + \frac{r^2}{s}(r+1)t^r - r^2t^{s-1},$$

$$f_1(1) = f'(1) = 0, \tag{2.2}$$

$$f_1'(t) = r^2(s+1)(t^{-r-1} - t^{-s-2}) - r^2(r+1)(t^{-s-1} - t^{-r-2}),$$

$$f_2(1) = f_1'(1) = 0, \tag{2.3}$$

$$f_2'(t) = r^2(s+1)(s-r+1)t^{s-r} + r^2(r+1)(s-r)t^{s-r-1} - r^2(r+1),$$

$$f_2'(1) = r^2(s-r)(s+r+3), \tag{2.4}$$

$$f_2''(t) = r^2(s+1)(s-r+1)(s-r)t^{s-r-1} + r^2(r+1)(s-r)(s-r-1)t^{s-r-2},$$

$$f_3(1) = f_2''(1) = r^2(s-r)^2(s+r+3) \tag{2.5}$$

and

$$f_3'(t) = r^2(s+1)(s-r+1)(s-r). \tag{2.6}$$

If $s \geq -1, s > r$ and $s + r + 3 \geq 0$, then from (2.4), (2.5) and (2.6) we see that $f_2'(1) \geq 0, f_3(1) \geq 0$ and $f_3'(t) \geq 0$. Therefore, Lemma 2.2(1) follows from (2.1)-(2.3).

(2) If $s > -1$ and $s + r + 3 < 0$, then $r < -2$ and $s > r$. From (2.4) and (2.5) we clearly see that

$$f_2'(1) < 0 \tag{2.7}$$

and

$$f_3(1) < 0. \tag{2.8}$$

Inequality (2.8) and the continuity of $f_3(t)$ imply that there exists $\delta_1 > 0$ such that $f_3(t) < 0$ for $t \in [1, 1 + \delta_1)$. By (2.7) we know that $f_2'(t) \leq f_2'(1) < 0$ for $t \in [1, 1 + \delta_1)$, then from (2.1)-(2.3) we clearly see that $f(t) < 0$ for $t \in (1, 1 + \delta_1)$.

On the other hand, it is easy to see that $\lim\limits_{t\to\infty} f(t) = +\infty$. Hence Lemma 2.1(2) is true.

(3) If $s \leq -1, s > r$ and $s + r + 3 \leq 0$, then from (2.4),(2.5) and (2.6) we see that $f_2'(1) \leq 0, f_3(1) \leq 0$ and $f_3'(t) \leq 0$. Therefore, Lemma 2.1(3) follows from (2.1)-(2.3).

(4) If $s < -1, s > r$ and $s + r + 3 > 0$, then from (2.4) and (2.5) we see that

$$f_2'(1) > 0 \tag{2.9}$$

and

$$f_3(1) > 0. \tag{2.10}$$

Inequality (2.10) and the continuity of $f_3(t)$ imply that there exists $\delta_2 > 0$ such that $f_3(t) > 0$ for $t \in [1, 1 + \delta_2)$. By (2.9) we know that $f_2'(t) \geq f_2'(1) > 0$ for $t \in [1, 1 + \delta_2)$, then from (2.1)-(2.3) we clearly see that $f(t) > 0$ for $t \in (1, 1 + \delta_2)$.

On the other hand, it is easy to see that $\lim_{t\to\infty} f(t) = -\frac{r}{s}(s-r) < 0$. Hence Lemma 2.1(4) is true.

Lemma 2.2. *Let $t \in [1,\infty), r \in \mathbb{R}$, and $f(t) = r(t^{r+1}+1)\log t - (t^r-1)(t+1)$. If $r < -3$, then there exist $t_1, t_2 \in (1,\infty)$ such that $f(t_1) < 0$ and $f(t_2) > 0$.*

Proof. Let $f_1(t) = t^{1-r}f'(t)$ and $f_2(t) = t^{2+r}f_1''(t)$. Then simple computations yield

$$f(1) = 0, \tag{2.11}$$

$$f'(t) = r(r+1)t^r \log t - t^r - rt^{r-1} + \frac{r}{t} + 1,$$

$$f_1(1) = f'(1) = 0, \tag{2.12}$$

$$f_1'(t) = r(r+1)\log t + (1-r)t^{-r} - r^2t^{-r-1} + r(r+1) - 1,$$

$$f_1'(1) = 0, \tag{2.13}$$

$$f_1''(t) = \frac{r(r+1)}{t} + r^2(r+1)t^{-r-2} + r(r-1)t^{-r-1},$$

$$f_2(1) = f_1''(1) = r^2(r+3), \tag{2.14}$$

$$f_2'(t) = r(r+1)^2t^r + r(r-1)$$

and

$$f_2'(1) = r^2(r+3). \tag{2.15}$$

If $r < -3$, then (2.14) and (2.15) imply that

$$f_2(1) < 0 \tag{2.16}$$

and

$$f_2'(1) < 0. \tag{2.17}$$

From (2.17) and the continuity of $f_2'(t)$ we know that there exists $\delta > 0$ such that $f_2'(t) < 0$ for $t \in [1, 1+\delta)$, then (2.16) leads to that $f_2(t) \le f_2(1) < 0$. Therefore, $f(t) < 0$ for $t \in (1, 1+\delta)$ follows from (2.11)-(2.13)

On the other hand, we clearly see that $\lim_{t\to\infty} = +\infty$.

Lemma 2.3. *Let $t \in [1,\infty)$, $r \in \mathbb{R}$, and $f(t) = -r(t^{r+1}+t^r)\log t + (t^{r+1}+1)(t^r-1)$. If $-\frac{3}{2} < r < -1$, then there exist $t_1, t_2 \in (1,\infty)$ such that $f(t_1) > 0$ and $f(t_2) < 0$.*

Proof. Let $g_1(t) = t^{-2r}f'(t), g_2(t) = t^{r+2}g_1'(t)$. Then simple computations yield

$$f(1) = 0, \tag{2.18}$$

$$f'(t) = -r[(r+1)t^r + rt^{r-1}]\log t + (2r+1)(t^{2r} - t^r),$$

$$g_1(1) = f'(1) = 0, \tag{2.19}$$

$$g_1'(t) = [r^2(r+1)t^{-r-1} + r^2(r+1)t^{-r-2}]\log t + r^2t^{-r-1} - r^2t^{-r-2},$$

$$g_2(1) = g_1'(1) = 0, \tag{2.20}$$

$$g_2'(t) = [r^2(r+1)]\log t + \frac{r^2(r+1)}{t} + r^3 + 2r^2$$

and

$$g_2'(1) = r^2(2r+3). \tag{2.21}$$

If $-\frac{3}{2} < r < -1$, then (2.21) implies that $g_2'(1) > 0$. From the continuity of $g_2'(t)$ we know that there exists $\eta > 0$ such that $g_2'(t) > 0$ for $t \in [1, 1+\eta)$. Therefore, $f(t) > 0$ for $t \in (1, 1+\eta)$ follows from (2.18)-(2.20).

On the other hand, we clearly see that $\lim\limits_{t\to\infty} f(t) = -1 < 0$.

For a set $E \subseteq \mathbb{R}^2$, let $\overline{E}$ be the closure of E. From the continuity of the extended mean values $E(r,s;x,y)$ and the definition of Schur harmonic convex (or concave, respectively), the following Lemma 2.4 is obvious.

Lemma 2.4. *Let E be a set in rs-plane with nonempty interior. If the extended mean values $E(r,s;x,y)$ are Schur harmonic convex (or concave, respectively) with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for $(r,s) \in E$, then $E(r,s;x,y)$ are Schur harmonic convex (or concave, respectively) with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for $(r,s) \in \overline{E}$.*

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We use Theorem D to discuss the nonpositivity and nonnegativity of $(y-x)(y^2\frac{\partial E(r,s;x,y)}{\partial y} - x^2\frac{\partial E(r,s;x,y)}{\partial x})$ for all $(x,y) \in (0,\infty)\times(0,\infty)$ and for fixed $(r,s) \in \mathbb{R}^2$. Since $(y-x)(y^2\frac{\partial E(r,s;x,y)}{\partial y} - x^2\frac{\partial E(r,s;x,y)}{\partial x}) = 0$ for $x = y$ and $(y-x)(y^2\frac{\partial E(r,s;x,y)}{\partial y} - x^2\frac{\partial E(r,s;x,y)}{\partial x})$ is symmetric with respect to x and y, without loss of generality, we assume $y > x$ in the following discussion.

Let

$$\begin{aligned} E_1 &= \{(r,s) : s \geq -1, s \geq r, s+r+3 \geq 0\} \\ &\quad \cup\{(r,s) : r \geq -1, r \geq s, s+r+3 \geq 0\}, \end{aligned}$$

$$E_2 = \{(r,s) : r \leq -1, s \leq -1, s+r+3 \leq 0\}$$

and

$$\begin{aligned} E_3 &= \{(r,s) : s > -1, s+r+3 < 0\} \\ &\quad \cup\{(r,s) : r > -1, s+r+3 < 0\} \\ &\quad \cup\{(r,s) : r < -1, s < -1, s+r+3 > 0\}. \end{aligned}$$

Then $E_1 \cup E_2 \cup E_3 = \mathbb{R}^2$, $E_1 \cap E_3 = \emptyset$, $E_2 \cap E_3 = \emptyset$, and $\mathrm{int}E_1 \cap \mathrm{int}E_2 = \emptyset$. It is obvious that Theorem 1.1 is true if once we prove that $E(r,s;x,y)$ is Schur harmonic convex, Schur harmonic concave, and neither Schur harmonic convex nor Schur harmonic concave with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for $(r,s) \in E_1, E_2,$ and, E_3, respectively. We divide our proof into three cases.

Case 1. $(r,s) \in E_1$. Let $E_{11} = \{(r,s) : s > -1, s > r, s \neq 0, r \neq 0, s+r+3 > 0\}$, $E_{12} = \{(r,s) : r > -1, r > s, r \neq 0, s \neq 0, s+r+3 > 0\}$ and $F(r,s;x,y) = \frac{r}{s}\frac{y^s - x^s}{y^r - x^r}$. Then

$$E_1 = \overline{E_{11}} \cup \overline{E_{12}} \tag{3.1}$$

and (1.1) leads to the following identity

$$\begin{aligned}&(y-x)(y^2\frac{\partial E(r,s;x,y)}{\partial y}-x^2\frac{\partial E(r,s;x,y)}{\partial x}) \\ =\ & \frac{1}{s-r}\frac{y-x}{(y^r-x^r)^2}x^{s+r+1}F^{\frac{1}{s-r}-1}\times\frac{r}{s}[(s-r)((\frac{y}{x})^{s+r+1}-1) \\ &-s((\frac{y}{x})^{s+1}-(\frac{y}{x})^r)+r((\frac{y}{x})^{r+1}-(\frac{y}{x})^s)]\end{aligned} \tag{3.2}$$

for $(r,s)\in E_{11}$ and $y>x$. From Theorem D, Lemma 2.1(1), (3.2) and the assumption $y>x$ we know that $E(r,s;x,y)$ is Schur harmonic convex with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for $(r,s)\in E_{11}$. Then Lemma 2.4 and (3.1) together with the symmetry of $E(r,s;x,y)$ with respect to (r,s) imply that $E(r,s;x,y)$ is Schur harmonic convex with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for $(r,s)\in E_1$.

Case 2. $(r,s)\in E_2$. Let $E_{21}=\{(r,s):s<-1,s>r,s+r+3<0\}$ and $E_{22}=\{(r,s):r<-1,r>s,s+r+3<0\}$. Then

$$E_2=\overline{E_{21}}\cup\overline{E_{22}}, \tag{3.3}$$

and (1.1), Theorem D, Lemma 2.1(3), (3.2) and the assumption $y>x$ imply that $E(r,s;x,y)$ is Schur harmonic concave with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for $(r,s)\in E_{21}$. From Lemma 2.4 and (3.3) together with the symmetry of $E(r,s;x,y)$ with respect to r and s we know that $E(r,s;x,y)$ is Schur harmonic concave with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for $(r,s)\in E_2$

Case 3. $(r,s)\in E_3$. Let $E_{31}=\{(r,s):s>-1,s+r+3<0\}$, $E_{32}=\{(r,s):r>-1,s+r+3<0\}$ and $E_{33}=\{(r,s):r<-1,s<-1,s+r+3>0\}$. Then $E_3=E_{31}\cup E_{32}\cup E_{33}$. We divide the discussion of this case into three sub-cases.

Sub-case 3.1. $(r,s)\in E_{31}$. Let $E_{311}=\{(r,s):s>-1,s\neq 0,s+r+3<0\}$, and $E_{312}=\{(r,s):s=0,r<-3\}$, then $E_{31}=E_{311}\cup E_{312}$. We divide the discussion of this sub-case into two sub-sub-cases.

Sub-sub-case 3.1.1. $(r,s)\in E_{311}$. Then from (1.1), Theorem D, Lemma 2.1(2), (3.2) and the assumption $y>x$ we clearly see that $E(r,s;x,y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x,y)\in(0,\infty)\times(0,\infty)$.

Sub-case 3.1.2. $(r,s)\in E_{312}$. We note that (1.2) leads to the following identity

$$\begin{aligned}&(y-x)(y^2\frac{\partial E(r,0;x,y)}{\partial y}-x^2\frac{\partial E(r,0;x,y)}{\partial x}) \\ =\ & \frac{y-x}{(r(\log y-\log x))^2}x^{r+1}E(r,0;x,y)^{1-r} \\ &\times[r((\frac{y}{x})^{r+1}+1)\log\frac{y}{x}-((\frac{y}{x})^r-1)((\frac{y}{x})+1)].\end{aligned} \tag{3.4}$$

From Theorem D, Lemma 2.2, (3.4) and the assumption $y > x$ we see that $E(r, s; x, y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for $(r, s) \in E_{312}$.

Therefore, $E(r, s; x, y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for $(r, s) \in E_{31}$.

Sub-case 3.2. $(r, s) \in E_{32}$. The symmetry of $E(r, s; x, y)$ with respect to (r, s) and sub-case 3.1 imply that $E(r, s; x, y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for $(r, s) \in E_{32}$.

Sub-case 3.3. $(r, s) \in E_{33}$. Let $E_{331} = \{(r, s) : s < -1, s > r, s + r + 3 > 0\}$, $E_{332} = \{(r, s) : r < -1, r > s, s+r+3 > 0\}$ and $E_{333} = \{(r, s) : -\frac{3}{2} < s = r < -1\}$. Then

$$E_{33} = E_{331} \cup E_{332} \cup E_{333}. \tag{3.5}$$

We divide the discussion of this sub-case into three sub-sub-cases.

Sub-sub-case 3.3.1. $(r, s) \in E_{331}$. Then from (1.1), Theorem D, Lemma 2.1(4), (3.2) and the assumption $y > x$ we see that $E(r, s; x, y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

Sub-sub-case 3.3.2. $(r, s) \in E_{332}$. The symmetry of $E(r, s; x, y)$ with respect to (r, s) and sub-sub-case 3.3.1 show that $E(r, s; x, y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

Sub-sub-case 3.3.3. $(r, s) \in E_{333}$. We note that (1.3) leads to the following identity.

$$\begin{aligned} & (y-x)(y^2\frac{\partial E(r,s;x,y)}{\partial y} - x^2\frac{\partial E(r,s;x,y)}{\partial x}) \qquad (3.6) \\ = \quad & \frac{y-x}{(y^r - x^r)^2} E(r, r; x, y) x^{2r+1} \\ & \times[-r((\frac{y}{x})^{r+1} + (\frac{y}{x})^r)\log\frac{y}{x} + ((\frac{y}{x})^{r+1} + 1)((\frac{y}{x})^r - 1)]. \end{aligned}$$

From Theorem D, Lemma 2.3, (3.6) and the assumption $y > x$ we see that $E(r, s; x, y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for $(r, s) \in E_{333}$.

Sub-sub-cases 3.3.1-3.3.3 and (3.5) show that $E(r, s; x, y)$ is neither Schur harmonic convex nor Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for $(r, s) \in E_{33}$.

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Recibido: 14 de diciembre de 2009
Aceptado: 23 de julio de 2010