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PROPERTY (ω) AND QUASI-CLASS (A,k) OPERATORS

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Abstract. In this paper, we prove the following assertions: (i) If T is of quasi-class (A,k), then T is polaroid and reguloid; (ii) If T or T^* is an algebraically of quasi-class (A,k) operator, then Weyls theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$; (iii) If T^* is an algebraically of quasi-class (A,k) operator, then a-Weyls theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$; (iv) If T^* is algebraically of quasi-class (A,k) then property (ω) holds for T.

1. Introduction

Throughout this paper let $\mathbf{B}(\mathcal{H})$, $\mathbf{F}(\mathcal{H})$, $\mathbf{K}(\mathcal{H})$, denote, respectively, the algebra of bounded linear operators, the ideal of finite rank operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in \mathbf{B}(\mathcal{H})$ we shall write $\ker(T)$ and $\mathcal{R}(T)$ (or $ran(T)$) for the null space and range of T, respectively. Also, let $\alpha(T) := \dim\ker(T)$, $\beta(T) := co\dim\mathcal{R}(T)$, and let $\sigma(T), \sigma_a(T), \sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

T is called *Weyl* if it is Fredholm of index 0, and *Browder* if it is Fredholm "of finite ascent and descent".

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$ and if such integer does not exist we put $d(T) = \infty$. The essential spectrum $\sigma_F(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$$

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respectively. Evidently

$$\sigma_F(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_F(T) \cup acc\sigma(T)$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.

Following [6], we say that *Weyl's theorem* holds for T if $\sigma(T) \setminus \sigma_w(T) = E_0(T)$, where $E_0(T)$ is the set of all eigenvalues λ of finite multiplicity isolated in $\sigma(T)$. And *Browder's theorem* holds for T if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$, where $\pi_0(T)$ is the set of all poles of T of finite rank.

Let $SF_+(\mathcal{H})$ be the class of all upper semi-Fredholm operators, $SF_+^-(\mathcal{H})$ be the class of all $T \in SF_+(\mathcal{H})$ with $i(T) \leq 0$, and for any $T \in \mathbf{B}(\mathcal{H})$ let

$$\sigma_{SF_+^-}(T) = \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(\mathcal{H})\right\}.$$

Let E_0^a be the set of all eigenvalues of T of finite multiplicity which are isolated in $\sigma_a(T)$. According to [19], we say that T satisfies *a-Weyl's theorem* if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$. It follows from [19, Corollary 2.5] a-Weyl's theorem implies Weyl's theorem.

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [9] we say that $T \in \mathbf{B}(\mathcal{H})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathcal{H}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions, it easily follows that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [17, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

2. Properties of quasi-class (A, k) operators

Definition 2.1. [21] An operator T is said to be a *quasi-class* (A, k) (and we write $T \in \mathcal{Q}(A, k)$) if

$$T^{k^*}(|T^2| - |T|^2)T^k \geq 0, \quad \text{for } k \in \mathbb{N}.$$

If $k = 0$, T is said to be class A (in symbols, $T \in \mathcal{A}$), where T^0 is the identity operator and if $k = 1$, T is said to be quasi-class A (and we write $T \in \mathcal{QA}$).

T. Furuta and T. Yamazaki [10], I.H. Jeon and I. H. Kim [14] and K. Tanahashi et al. [21] introduced class A, quasi-class A and quasi-class (A, k) operators, respectively. It was known that these operators share many interesting properties with hyponormal operators (see [7, 10, 13]).

In this section we prove some properties of quasi-class (A, k) operators. We need the following lemmas.

Lemma 2.2. [21, Theorem 1.] *Let $T \in \mathcal{A}$. Then the following assertions hold:*

(1) $\left\||T^2| - |T|^2\right\| \le \left\|\tilde{T}_{1,1} - \tilde{T^*}_{1,1}\right\| \le \frac{1}{\pi} meas\, \sigma(T)$, *where* $T = U|T|$ *is the polar decomposition of* T *and* $T_{1,1} = |T|U|T|$. *Moreover, if* $meas\, \sigma(T) = 0$, *then* T *is normal.*
(2) *The operator* T *has Bishop's property* (β).
(3) *The restriction* $T|_M$ *to an invariant subspace* M *of* T *is also of class* A.

Lemma 2.3. [21] *Let* $T \in \mathcal{QA}$. *Assume that* $\mathcal{R}(T^k)$ *is not dense, and decompose*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \quad on \quad \mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \ker(T^{k^*}).$$

Then $T_1 = T|_{\overline{\mathcal{R}(T^k)}}$, *the restriction of* T *to* $\overline{\mathcal{R}(T^k)}$ *is class* A, $T_3^k = 0$. *Furthermore,* $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Lemma 2.4. (**Hansen inequality**, [12]) *If* $T, S \in \mathbf{B}(\mathcal{H})$ *satisfy* $T \ge 0$ *and* $\|S\| \le 1$, *then*

$$(S^*TS^*)^\lambda \ge S^*T^\lambda S$$

for all $\lambda \in [0, 1]$.

Hölder-McCarthy Inequality. Let T be a positive operator. Then the following inequalities hold for all $x \in \mathcal{H}$:
(i) $\langle T^r x, x\rangle \le \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $0 < r \le 1$.
(ii) $\langle T^r x, x\rangle \ge \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r \ge 1$.

Theorem 2.5. *Let* $T \in \mathcal{Q}(A, k)$ *for positive integer* k. *Then the following assertions hold.*

(i) $\left\|T^{n+1}x\right\|^2 \le \left\|T^{n+2}x\right\| \|T^n x\|$ *for all unit vector* $x \in \mathcal{H}$ *and all positive integer* $n \ge k$.
(ii) $\left\|T^{n+1}\right\|^n \le \|T^n\|^n r(T^n)$ *for all positive integer* $n \ge k$, *where* $r(T^n)$ *denotes the spectral radius of* T^n.

Proof. (i) It is obvious that if $T \in \mathcal{Q}(A, k)$ then its $\mathcal{Q}(A, k+1)$. We may assume that $k = n$. Since

$$\begin{aligned} \left\langle T^{k^*}|T|^2 T^k x, x\right\rangle &= \langle T^{k+1}x, T^{k+1}x\rangle \\ &= \left\|T^{k+1}x\right\|^2, \end{aligned}$$

and

$$\begin{aligned} \left\langle T^{k^*}(|T^2|)T^k x, x\right\rangle &= \left\langle (T^{2^*}T^2)^{\frac{1}{2}} T^k x, T^k x\right\rangle \\ &\le \langle T^{k+2}x, T^{k+2}x\rangle^{\frac{1}{2}} \left\|T^k x\right\| \quad \text{(by Hölder-McCarthy Inequality)} \\ &= \left\|T^{k+2}\right\| \left\|T^k x\right\|. \end{aligned}$$

But T is quasi-class (A, k). Then

$$\left\|T^{k+1}x\right\|^2 \le \left\|T^{k+2}\right\| \left\|T^k x\right\|.$$

(ii) We may assume that $k = n$, hence we prove

$$\left\|T^{k+1}\right\|^k \leq \left\|T^k\right\|^k r(T^k).$$

If $T^n = 0$ for some $n > k$, then $T^k = 0$ and in this case $r(T^k) = 0$. Hence (2) is obvious. Hence we may assume $T^n \neq 0$ for all $n \geq k$. Then

$$\frac{\|T^{k+1}\|}{\|T^k\|} \leq \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \leq \frac{\|T^{k+3}\|}{\|T^{k+2}\|} \leq \cdots \leq \frac{\|T^{km}\|}{\|T^{km-1}\|}$$

hold by part (1). Hence, we have

$$(\frac{\|T^{k+1}\|}{\|T^k\|})^{mk-k-2} \leq \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \times \leq \frac{\|T^{k+3}\|}{\|T^{k+2}\|} \times \cdots \times \frac{\|T^{km}\|}{\|T^{km-1}\|} = \frac{\|T^{km}\|}{\|T^k\|}.$$

Thus

$$(\frac{\|T^{k+1}\|}{\|T^k\|})^{k-\frac{k}{m}-\frac{2}{m}} \leq \frac{\|T^{km}\|^{\frac{1}{m}}}{\|T^k\|^{\frac{1}{m}}}.$$

Now, letting $m \longrightarrow \infty$ we have

$$\left\|T^{k+1}\right\|^k \leq \left\|T^k\right\|^k r(T^k).$$

□

Lemma 2.6. [21] *Let $T \in \mathcal{Q}(A,k)$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$, and $(T-\lambda)^{k+1} = 0$ if $\lambda = 0$.*

Lemma 2.7. *Let $T \in \mathcal{Q}(A,k)$. Then the restriction $T|_M$ of quasi-class (A,k) T on $\mathcal{H}$ to an invariant subspace M of T is also $\mathcal{Q}(A,k)$.*

Proof. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $\mathcal{H}$ onto M. Put $T_1 = T|_M$. Then $TP = PTP$ and $T_1 = (PTP)|_M$. Since T is a $\mathcal{Q}(A,k)$, we have

$$PT^{k^*}|T^2|T^kP \geq PT^{k^*}|T|^2T^kP.$$

Since $PT^kP = T^kP$ and $PT^{k^*} = PT^{k^*}P$, we have

$$\begin{aligned} PT^{k^*}|T^2|T^kP &= PT^{k^*}P|T^2|PT^kP \\ &= PT^{k^*}P(T^*T^*TT)^{\frac{1}{2}}PT^kP \\ &\leq PT^{k^*}(PT^*T^*TTP)^{\frac{1}{2}}T^kP \qquad \text{(By Lemma 2.4)} \\ &= \begin{pmatrix} T_1^{k^*}|T_1^2|T_1^k & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} PT^{k^*}|T|^2T^kP &= PT^{k^*}P|T|^2PT^kP \\ &= \begin{pmatrix} T_1^{k^*}|T_1|^2T_1^k & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

we have

$$\begin{pmatrix} T_1^{k^*}|T_1^2|T_1^k & 0 \\ 0 & 0 \end{pmatrix} \geq PT^{k^*}|T^2|T^kP \geq PT^{k^*}|T|^2T^kP$$
$$= \begin{pmatrix} T_1^{k^*}|T_1|^2T_1^k & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that T_1 is $\mathcal{Q}(A,k)$ operator. □

Definition 2.8. [5] An operator T is said to have *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if for every open neighborhood G of λ, the function $f_n \in Hol(G)$ with $(T-\lambda)f_n(\mu) \to 0$ uniformly on every compact subset of G implies that $f_n(\mu) \to 0$ uniformly on every compact subset of G,where $Hol(G)$ means the space of all analytic functions on G. When T has Bishop's property (β) at each $\lambda \in \mathbb{C}$, simply say that T has property (β).

Lemma 2.9. [15] *Let G be open subset of complex plane $\mathbb{C}$ and let $f_n \in Hol(G)$ be functions such that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, then $f_n(\mu) \to 0$ uniformly on every compact subset of G.*

Lemma 2.10. *Let $T \in \mathcal{Q}(A,k)$. Then T has Bishop's property (β).*

Proof. Let $f_n(z)$ be analytic on G. Let $(T-z)f_n(z) \longrightarrow 0$ uniformly on each compact subset of G. Then, using the representation of Lemma 2.3 we have

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix}\begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (T_1-z)f_{n1}(z) + T_2 f_{n2}(z) \\ (T_3-z)f_{n2}(z) \end{pmatrix} \longrightarrow 0.$$

Since T_3 is nilpotent, T_3 has Bishop's property (β). Hence $f_{n2}(z) \longrightarrow 0$ uniformly on every compact subset of G. Then $(T_1-z)f_{n1}(z) \longrightarrow 0$. Since T_1 is of class A, T_1 has Bishop's property (β) by Lemma 2.2. hence $f_{n1}(z) \longrightarrow 0$ uniformly on every compact subset of G. Thus T has Bishop's property (β). □

Lemma 2.11. [21, Lemma 13.] *Let $T \in \mathcal{Q}(A,k)$. If $(T-\lambda)x = 0$ and $\lambda \neq 0$, then $(T-\lambda)^*x = 0$.*

Lemma 2.12. *Let $T \in \mathcal{Q}(A,k)$. Then* $\ker(T-\lambda)^{k+1} = \ker(T-\lambda)^{k+2}$ *for all $\lambda \in \mathbb{C}$. Hence $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$.*

Proof. It follows from Theorem 2.5 that

$$\left\|T^{k+1}x\right\|^2 \leq \left\|T^{k+2}x\right\|\left\|T^k x\right\|$$

for all $x \in \mathcal{H}$, we have $\ker T^{k+1} = \ker T^{k+2}$. Let $(T-\lambda)^{k+2}x = 0$ for $\lambda \neq 0$. Then it follows from Lemma 2.11 that $(T-\lambda)^*(T-\lambda)^{k+1}x = 0$. Hence

$$\left\|(T-\lambda)^{k+1}x\right\|^2 = \left\langle (T-\lambda)^*(T-\lambda)^{k+1}x, (T-\lambda)^k x\right\rangle = 0.$$

So the proof is achieved. □

Definition 2.13. ([4]) An operator $T \in \mathbf{B}(\mathcal{H})$ is called algebraically $\mathcal{Q}(A,k)$ if there exists a nonconstant complex polynomial p such that $p(T)$ is a $\mathcal{Q}(A,k)$.

Lemma 2.14. *Let $T \in \mathbf{B}(\mathcal{H})$ be an algebraically $\mathcal{Q}(A,k)$ operator and $\sigma(T) = \{\mu_0\}$, then $T-\mu_0$ is nilpotent.*

Proof. Assume $p(T)$ is $\mathcal{Q}(A,k)$ for some nonconstant polynomial $p(z)$. Since $\sigma(p(T)) = p(\sigma(T)) = \{p(\mu_0)\}$, the operator $p(T)-p(\mu_0)$ is nilpotent by Lemma 2.6. Let

$$p(z)-p(\mu_0) = a(z-\mu_0)^{k_0}(z-\mu_1)^{k_1}\cdots(z-\mu_t)^{k_t},$$

where $\mu_j \neq \mu_s$ for $j \neq s$. Then

$$0 = \{p(T)-p(\mu_0)\}^m = a^m(T-\mu_0)^{mk_0}(T-\mu_1)^{mk_1}\cdots(T-\mu_t)^{mk_t}$$

and hence $(T-\mu_0)^{mk_0} = 0$. □

An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be polaroid if $iso\sigma(T) \subseteq \pi(T)$, where $\pi(T)$ is the set of all poles of T. In general, if T is polaroid then it is isoloid. However, the converse is not true. Consider the following example. Let $T \in \ell^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, and so T is isoloid. However, since T does not have finite ascent, T is not polaroid.

In [7] they showed that every $\mathcal{QA}$ operator is isoloid. We can prove more:

Proposition 2.15. *Let T be an algebraically $\mathcal{Q}(A,k)$ operator. Then T is polaroid.*

Proof. Suppose T is an algebraically $\mathcal{Q}(A,k)$ operator. Then $p(T) \in \mathcal{Q}(A,k)$ for some nonconstant polynomial p. Let $\lambda \in iso(\sigma(T))$. Using the spectral projection $P := \frac{1}{2i\pi}\int_{\partial D}(\mu - T)^{-1}\,d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{and} \quad \sigma(T_1) = \{\lambda\} \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is algebraically $\mathcal{Q}(A,k)$ operator and $\sigma(T_1) = \{\lambda\}$. But $\sigma(T_1 - \lambda I) = \{0\}$ it follows from Lemma 2.14 that $T_1 - \lambda I$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda I$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda I$ has finite ascent and descent. Therefore λ is a pole of the resolvent of T. Thus if $\lambda \in iso(\sigma(T))$ implies $\lambda \in \pi(T)$, and so $iso(\sigma(T)) \subset \pi(T)$. Hence T is polaroid. □

An operator $T \in \mathbf{B}(\mathcal{H})$ is called a-isoloid if $iso\sigma_a(T) \subseteq \sigma_p(T)$. Clearly, if T is a-isoloid then it is isoloid. However, the converse is not true . Consider the following example: Let $T = U \oplus Q$, where U is the unilateral forward shift on ℓ^2 and Q is an injective quasinilpotent on ℓ^2, respectively. Then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$. Therefore T is isoloid but not a-isoloid.

Corollary 2.16. *Let T be an algebraically $\mathcal{Q}(A,k)$ operator. Then T is a-isoloid.*

For $T \in \mathbf{B}(\mathcal{H})$,$\lambda \in \sigma(T)$ is said to be a regular point if there exists $S \in \mathbf{B}(\mathcal{H})$ such that $T - \lambda I = (T-\lambda I)S(T-\lambda I)$. T is is called reguloid if every isolated point of $\sigma(T)$ is a regular point. It is well known [11, Theorems 4.6.4 and 8.4.4] that $T-\lambda I=(T-\lambda I)S(T-\lambda I)$ for some $S\in \mathbf{B}(\mathcal{H}) \Longleftrightarrow T-\lambda I$ has a closed range.

Since polaroid implies reguloid, we have the following corollary as a consequence of Proposition 2.15

Corollary 2.17. *Let T be an algebraically $\mathcal{Q}(A,k)$ operator. Then T is reguloid.*

Proposition 2.18. ([16]) *Let $T\in \mathbf{B}(\mathcal{H})$. If T^* has the SVEP, then $\sigma_{SF_+^-}(T)=\sigma_w(T)$.*

3. Weyl's theorem for algebraically of quasi-class (A,k) operator

Theorem 3.1. *Suppose T or T^* is an algebraically $\mathcal{Q}(A,k)$ operator. Then Weyl's theorem holds for $f(T)$ for every $f\in Hol(\sigma(T))$.*

Proof. Let p be a non-trivial polynomial such that $p(T)$ (resp., $p(T^*)$) is $\mathcal{Q}(A,k)$. Then, see Lemma 2.7 and Proposition 2.15, T (resp., T^*) is hereditarily polaroid (i.e., the restriction of the operator to every of its invariant subspaces is again polaroid) [8, Example 2.5, Page 368]. Hence $f(T)$ (resp., $f(T^*)$) satisfies Weyl's theorem for every $f\in Hol(\sigma(T))$ [8, Theorem 3.6]. □

Proposition 3.2. *Suppose T or T^* is an algebraically of $\mathcal{Q}(A,k)$ operator. Then $\sigma_w(f(T))=f(\sigma_w(T))$ for every $f\in Hol(\sigma(T))$.*

Proof. Let $f\in Hol(\sigma(T))$. To show that $\sigma_w(f(T))=f(\sigma_w(T))$ it is sufficient to show that $f(\sigma_w(T))\subseteq \sigma_w(f(T))$. Suppose that $\lambda\notin\sigma_w(f(T))$. Then $f(T)-\lambda I$ is Weyl. Since T^* is algebraically of $\mathcal{Q}(A,k)$, it has SVEP. It follows from Proposition 2.18 that $i(T-\alpha_j)\geq 0$ for each $j=1,2,\cdots,n$. Since

$$0\leq\sum_{j=1}^{n} i(T-\alpha_j)=i(f(T)-\lambda I)=0,$$

$T-\alpha_j$ is Weyl for each $j=1,\cdots,n$. Hence $\lambda\notin f(\sigma_w(T))$, and so $f(\sigma_w(T))\subseteq\sigma_w(f(T))$. Thus $f(\sigma_w(T))=\sigma_w(f(T))$ for each $f\in Hol(\sigma(T))$. Since Weyls theorem holds for T and T is isoloid, Weyls theorem holds for $f(T)$ for every $f\in Hol(\sigma(T))$. This completes the proof. □

4. a-Weyl's theorem for algebraically of quasi-class (A,k) operator

Let $T\in\mathbf{B}(\mathcal{H})$. It is well known that the inclusion $\sigma_{SF_+^-}(f(T))\subseteq f(\sigma_{SF_+^-}(T))$ holds for every $f\in Hol(\sigma(T))$ with no restriction on T [20]. The next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically of quasi-class (A,k) operator.

Theorem 4.1. *Suppose T^* or T is an algebraically $\mathcal{Q}(A,k)$ operator. Then*

$$\sigma_{SF_+^-}(f(T))=f(\sigma_{SF_+^-}(T))$$

Proof. Since $p(T)$ (resp., $p(T^*)$) has SVEP (by Lemma 2.10 or 2.12) $\Longrightarrow T$ (resp., T^*) has SVEP $\Longrightarrow f(T)$ (resp., $f(T^*)$) has SVEP [18, Theorem 3.3.6]. Since the *upper Browder* and the *upper Weyl* equal $\sigma_{SF_+^-}$ spectra of an operator with SVEP coincide, and since the upper Browder spectrum satisfies the spectral mapping theorem [1, Theorem 3.69], the proof follows. □

It is easily seen that quasi-nilpotent operators do not satisfy a-Weyl's theorem, in general. For instance, if

$$T(x_1, x_2, \cdots) = (0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots), \qquad (x_n) \in \ell^2(\mathbb{N})$$

then T is quasi-nilpotent but a-Weyl's theorem fails for T, since $\sigma(T) = \sigma_a(T) = \sigma_{SF_+^-}(T) = \{0\} = E_0^a(T)$.

Theorem 4.2. *Suppose T^* is an algebraically of quasi-class (A, k) operator. Then a-Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. $P(T^*) \in \mathcal{Q}(A, k)$ implies T^* has SVEP implies $f(T^*)$ has SVEP; hence $\sigma(f(T^*)) = \overline{\sigma_a(f(T))}$, $\sigma_w(f(T^*)) = \overline{\sigma_{SF_+^-}(f(T)}$ and $E_0(f(T^*)) = \overline{E_0^a(f(T)}$. Now apply Theorem 3.1. □

5. Property (ω)

Definition 5.1. [2] A bounded operator $T \in \mathbf{B}(\mathcal{H})$ is said to satisfy *property* (ω) if

$$E_0(T) = \Delta^a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T).$$

As observed in [2], we have either of a-Weyls theorem or property (ω) for $T \Rightarrow$ Weyl's theorem holds for T.

Examples of operators satisfying Weyl's theorem but not property (ω) may be found in [2]. Property (ω) is independent from a-Weyl's theorem: in [2] there are examples of operators $T \in \mathbf{B}(\mathcal{H})$ satisfying property (ω) but not a-Weyl's theorem and vice versa. Generally, property (ω), as well as Weyl's theorems, does not survive under perturbations. More can be said: Weyls theorems and property (ω) for a bounded operator T are liable to fail also under small perturbations K, if "small" is interpreted in the sense of compact or quasi-nilpotent operators. In [3] some sufficient conditions are given for which we have the stability of property (ω), under perturbations by finite rank operators, compact operators, or quasi-nilpotent operator commuting with T.

The following example shows that a-Weyls theorem and Weyls theorem does not imply property (ω).

Example 5.2. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and let U defined by

$$U(x_1, x_2, \cdots) = (0, x_2, x_3, ...), (x_n) \in \ell^2(\mathbb{N}).$$

If $T = R \oplus U$, then $\sigma(T) = D(0, 1)$ the closed unit disc in $\mathbb{C}$, $iso\sigma(T) = \emptyset$ and $\sigma_a(T) = C(0, 1) \cup \{0\}$, where $C(0, 1)$ is unit circle of $\mathbb{C}$. It easily to see that $\sigma_{SF_+^-}(T) = C(0, 1)$. Moreover, we have $E_0(T) = \emptyset$ and $E_0^a(T) = \{0\}$. Hence T

satisfies a- Weyls theorem and so T satisfies Weyls theorem. But T does not satisfy property (ω).

Lemma 5.3. [2] *Suppose that* $T \in \mathbf{B}(\mathcal{H})$.
(i) *If* T^* *has the SVEP then* $\sigma_{SF_+^-}(T) = \sigma_b(T)$.
(ii) *If* T *has the SVEP then* $\sigma_{SF_+^-}(T^*) = \sigma_b(T)$.

Theorem 5.4. *Let* $T \in \mathbf{B}(\mathcal{H})$.

(i) *If* T^* *is algebraically* $\mathcal{Q}(A,k)$ *then property* (ω) *holds for* T.
(ii) *If* T *is algebraically* $\mathcal{Q}(A,k)$ *then property* (ω) *holds for* T^*.

Proof. (i) Since T^* is algebraically of quasi-class (A,k), then T^* is the SVEP and T is polaroid by Proposition 2.15 because T is polaroid if and only if T^* is polaroid. Consequently $\sigma(T) = \sigma_a(T)$. If $iso\sigma(T) = \emptyset$, then $E_0(T) = \emptyset$. We show that $\sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ is empty. By Lemma 5.3 we have $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_b(T)$ and the last set is empty, since $\sigma(T)$ has no isolated points. Therefore, T satisfies property (ω).

Consider the other case, $iso\sigma(T) \neq \emptyset$. Suppose that $\lambda \in E_0(T)$. Then λ is isolated in $\sigma(T)$ and hence, by the polaroid condition, λ is a pole of the resolvent of T , i.e. $a(T-\lambda) = d(T-\lambda) < \infty$. By assumption $\alpha(T-\lambda) < \infty$, so by [1, Theorem 3.1] $\beta(T-\lambda) < \infty$, and hence $T-\lambda$ is a Fredholm operator. Therefore, by Lemma 5.3, $\lambda \in \sigma(T) \setminus \sigma_b(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Conversely, if $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_b(T)$ then λ is an isolated point of $\sigma(T)$. Clearly, $0 < \alpha(T-\lambda) < \infty$, so $\lambda \in E_0(T)$ and hence T satisfies property (ω).
(ii) First note that since T has SVEP then $\sigma_a(T^*) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not onto}\} = \sigma(T) = \sigma(T^*)$. Suppose first that $iso\sigma(T) = iso\sigma(T^*) = \emptyset$. Then $E_0(T^*) = \emptyset$. By Lemma 5.3 we have $\sigma_a(T^*) \setminus \sigma_{SF_+^-}(T^*) = \sigma(T) \setminus \sigma_b(T) = \emptyset$, so T^* satisfies property ω.

Suppose that $iso\sigma(T) \neq \emptyset$ and let $\lambda \in E_0(T^*)$. Then λ is isolated in $\sigma(T) = \sigma(T^*)$, hence a pole of the resolvent of T^*, since T^* is polaroid by Proposition 2.15. By assumption $\alpha(T^* - \overline{\lambda})^p < \infty$ and since the ascent and the descent of $T^* - \overline{\lambda}$ are both finite it then follows by [1, Theorem 3.1] that $\alpha(T-\lambda) = \beta(T-\lambda) < \infty$, so $T^* - \overline{\lambda}$ is Browder and hence also $T-\lambda$ Browder. Therefore, $\lambda \in \sigma(T) \setminus \sigma_b(T)$ and by Lemma 5.3 it then follows that $\lambda \in \sigma_a(T^*) \setminus \sigma_{SF_+^-}(T^*)$.

Conversely, if $\lambda \in \sigma_a(T^*) \setminus \sigma_{SF_+^-}(T^*) = \sigma(T) \setminus \sigma_b(T)$, then λ is an isolated point of the spectrum of $\sigma(T) = \sigma(T^*)$. Hence $T-\lambda$ is Browder, or equivalently $T^* - \overline{\lambda}$ is Browder. Since $\alpha(T^* - \overline{\lambda}) = \beta(T^* - \overline{\lambda})$ we then have $\alpha(T^* - \overline{\lambda}) > 0$ (otherwise $\lambda \notin \sigma(T^*)$). Clearly, $\alpha(T^* - \overline{\lambda}) < \infty$, since by assumption $T^* - \overline{\lambda} \in SF_+^-(\mathcal{H})$, so that $\lambda \in E_0(T^*)$. Thus T^* satisfies property (ω). □

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