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SOLUTION OF TROESCH'S PROBLEM USING HE'S POLYNOMIALS

SYED TAUSEEF MOHYUD-DIN

ABSTRACT. In this paper, we apply He's polynomials for finding the approximate solution of the Troesch's problem which arises in the confinement of a plasma column by radiation pressure and applied physics. The proposed technique proved to be very effective and is easier to implement as compare to decomposition method.

1. Introduction

This paper is devoted to the study of Troesch's problem which arises in the confinement of a plasma column by radiation pressure and applied physics, see [1]. The Troesch's problem has been investigated by several researchers using different points of view because of its applications in applied and physical sciences [1]. The basic motivation of this paper is to use He's polynomials for the approximate solution of this problem. These polynomials are calculated from He's homotopy perturbation method (HPM) which was developed and formulated by He by merging the standard homotopy and perturbation, see [2, 23] and the references therein. The He's polynomials were first introduced by Ghorbani et. al. [2, 3] by splitting the nonlinear term into a series of polynomials. Moreover, it was proved [2, 3] that He's polynomials are compatible with the Adomian's polynomials but are easier to calculate and are more user friendly. The proposed polynomials are calculated from homotopy perturbation method (HPM), [2, 23]. Numerical results show the complete reliability of the proposed technique.

2. Homotopy Perturbation Method

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{1}$$

where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2}$$

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where F (u) is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{3}$$

we have

$$H(u, 0) = F(u), \quad H(u, 1) = L(u).$$

This shows that $H(u, p)$continuously traces an implicitly defined curve from a starting point H (v_0, 0) to a solution function H (f, 1). The embedding parameter monotonically increases from zero to unit as the trivial problem F (u) = 0 is continuously deforms the original problem L (u) = 0. The embedding parameter p ∈(0, 1] can be considered as an expanding parameter [2, 23]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [4, 11] to obtain

$$u = \sum_{i=0}^{\infty} p^i\, u_i = u_0 + p\, u_1 + p^2 u_2 + p^3 u_3 + \ \cdots\,, \tag{4}$$

if $p \to 1$, then (4) corresponds to (2) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{5}$$

It is well known that series (5) is convergent for most of the cases and also the rate of convergence is dependent on L (u); see [4, 11]. We assume that (5) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [2, 3], He's HPM considers the solution, $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x) = \sum_{i=0}^{\infty} p^i\, u_i = u_0 + p\, u_1 + p^2 u_2 + \cdots\,,$$

and the method considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i\, H_i = H_0 + p\, H_1 + p^2 H_2 + \cdots\,,$$

where H_n's are the so-called He's polynomials [2, 3], which can be calculated by using the formula

$$H_n(u_0, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N(\sum_{i=0}^{n} p^i u_i) \right)_{p=0}, \qquad n = 0, 1, 2, \ldots.$$

3. Numerical Applications

In this section, we use He's polynomials for the approximate solution of Troesch's problem, which is given by:

$$u''(x) = \beta \sinh(\beta\, u(x)), \qquad 0 \leqslant x \leqslant 1,$$

with boundary conditions

$$u(0) = 0, u(1) = 1.$$

Applying the convex homotopy

$$\begin{aligned} u_0 + pu_1 + p^2 u_2 + \cdots &= x \\ &+ p \int_0^x \int_0^x \beta^2 \Bigg((u_0 + pu_1 + \cdots) \\ &+ \frac{1}{3!} (u_0 + pu_1 + \cdots)^3 + \frac{1}{5!} (u_0 + pu_1 + \cdots)^5 + \cdots \Bigg)\, dx\, dx. \end{aligned}$$

Comparing the coefficients of like powers of p

$$p^{(0)} : u_0(x) = x,$$

$$p^{(1)} : u_1(x) = -\beta \left(\frac{x}{\beta} - \frac{\sinh(x\beta)}{\beta^2} \right),$$

$$\begin{aligned} & p^{(2)} : u_2(x) = \tfrac{1}{72\beta}(-147x\beta + 6x^3\beta^3 - 216x\beta\cosh(x\beta) - 9x\beta\cosh(2x\beta) \\ & +405\sinh(x\beta) + 36x^2\beta^2\sinh(x\beta) + 18\sinh(2x\beta) + \sinh(3x\beta)), \\ & \vdots, \end{aligned}$$

where p^is are the He's polynomials. The series solution is given by

$$\begin{aligned} u(x) = x - \beta & \left(\frac{x}{\beta} - \frac{\sinh(x\beta)}{\beta^2} \right) \\ & + \frac{1}{72\beta} \Bigg(-147x\beta + 6x^3\beta^3 - 216x\beta\cosh(x\beta) - 9x\beta\cosh(2x\beta) \\ & + 405\sinh(x\beta) + 36x^2\beta^2\sinh(x\beta) + 18\sinh(2x\beta) + \sinh(3x\beta) \Bigg) + \cdots . \end{aligned}$$

Table 3.1 and 3.2 exhibit the exact solution along with the errors obtained by using He's polynomials. Higher accuracy can be obtained by adding some more polynomials.

TABLE 3.1. Numerical solution for Troesch's problem $\beta = 0.5$.

x	Exact Solution	VIM	*Error
0.1	0.095176902	0.100042	-0.004865
0.2	0.1906338691	0.200334	-0.0097
0.3	0.286653403	0.301128	-0.014475
0.4	0.3835229288	0.402677	-0.019154
0.5	0.4815373854	0.505241	-0.023704
0.6	0.5810019749	0.609082	-0.02808
0.7	0.6822351326	0.71447	-0.032235
0.8	0.7855717867	0.821682	-0.03611
0.9	0.8913669875	0.931008	-0.039641
1.0	0.9999999999	1.04274	-0.04274

**Error = Exact solution – Series solution*

TABLE 3.2. Numerical solution for Troesch's problem $\beta = 1$.

x	Exact Solution	VIM	*Error
0.1	0.0817969966	0.100167	-0.100167
0.2	0.1645308709	0.201339	-0.036808
0.3	0.2491673608	0.304541	-0.055374
0.4	0.3367322092	0.410841	-0.074109
0.5	0.428347161	0.521373	-0.093026
0.6	0.5252740296	0.637362	-0.112088
0.7	0.6289711434	0.760162	-0.131191
0.8	0.7411683782	0.891287	-0.150119
0.9	0.8639700206	1.03246	-0.16849
1.0	1.00000000020	1.18565	-0.18565

**Error = Exact solution – series solution*

4. Conclusion

In this paper, we used He's polynomials (which are obtained by applying He's homotopy perturbation method) for finding the approximate solution of Troesch's problem. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials is a clear advantage of this algorithm over the decomposition method.

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Syed Tauseef Mohyud-Din
HITEC University Taxila Cantt, Pakistan
syedtauseefs@hotmail.com

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