REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 52, No. 2, 2011, 1–22

HARDY SPACES ASSOCIATED WITH SEMIGROUPS OF OPERATORS

JORGE J. BETANCOR

ABSTRACT. This paper is a non exhaustive survey about Hardy spaces defined by semigroups of operators.

1. INTRODUCTION

This paper is a summary of the talk held by the author in the conference "X Encuentro de Analistas A. Calderón", that was celebrated in La Falda, Córdoba, in Argentina, in September 2010. Our purpose is to present a survey about Hardy spaces associated with semigroups of operators. Of course it is not possible to be exhaustive. There exist monographs about this topic (see [25], [43], and [50], amongst others) and almost every day a paper where Hardy spaces appear is written. This shows the great importance of the Hardy spaces.

In [42] Stein developed harmonic analysis associated to semigroups of operators. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. A family $\{T_t\}_{t>0}$ of operators is a C_0-semigroup of operators when the following conditions are satisfied:

(i) For every $t > 0$, the operator T_t is bounded from $L^2(\Omega, \mu)$ into itself.
(ii) $T_t \circ T_s = T_{t+s}$, $t, s > 0$.
(iii) For every $f \in L^2(\Omega, \mu)$, $\lim_{t\to 0^+} T_t f = f$, in $L^2(\Omega, \mu)$.

A C_0-semigroup $\{T_t\}_{t>0}$ is said to be a symmetric diffusion semigroup provided that, for every $t > 0$, we have that

(iv) T_t is a contraction in $L^p(\Omega, \mu)$, for every $1 \le p \le \infty$.
(v) T_t is selfadjoint in $L^2(\Omega, \mu)$.
(vi) $T_t f \ge 0$, for every $0 \le f \in L^p(\Omega, \mu)$ and $1 \le p \le \infty$.
(vii) For every $f \in L^2(\Omega, \mu)$, $\displaystyle\int_\Omega f d\mu = \int_\Omega T_t f d\mu$.

The classical heat and Poisson semigroups, Ornstein-Uhlenbeck (Gaussian) semigroup or Bessel semigroup considered in [36] are examples of symmetric diffusion semigroups.

Maximal operators, Littlewood-Paley g-functions and Laplace transform type multipliers associated with symmetric diffusion semigroups were studied in [42].

2000 *Mathematics Subject Classification.* 42C05 (primary), 42C15 (secondary).
Key words and phrases. semigroup of operators, Bessel, Laguerre, Hardy spaces.
This paper is partially supported by MTM2010/17974.

The maximal operator T_* associated with the semigroup of operators $\{T_t\}_{t>0}$ is defined by

$$T_* f = \sup_{t>0} |T_t f|.$$

The behavior of T_* in L^p-spaces is the following.

Theorem 1.1 ([42, p. 73])**.** *Suppose that $\{T_t\}_{t>0}$ is a C_0-semigroup of operators in $L^2(\Omega,\mu)$ that satisfies the properties (iv) and (v) above. Then, the maximal operator T_* is bounded from $L^p(\Omega,d\mu)$ into itself, for every $1<p\leq\infty$. Moreover, for every $f\in L^p(\Omega,\mu)$, $1<p\leq\infty$,*

$$\lim_{t\to 0^+} T_t f(x) = f(x), \quad (\mu)\text{-a.e. } x\in\Omega$$

In this point the question is: What is the behavior of T_* in $L^1(\Omega,\mu)$? This question has been answered in many cases: classical heat and Poisson semigroups, Ornstein-Uhlenbeck, Bessel and Laguerre semigroups, semigroups generated by quite general elliptic operators, ... In all these examples T_* is bounded from L^1 into $L^{1,\infty}$ but it is not bounded from L^1 into itself.

The Hardy space $H^1(\{T_t\}_{t>0})$ associated with the semigroup $\{T_t\}_{t>0}$ is defined by

$$H^1(\{T_t\}_{t>0}) = \{f\in L^1(\Omega,\mu) : T_*(f)\in L^1(\Omega,\mu)\}.$$

The norm $\|.\|_{H^1(\{T_t\}_{t>0})}$ in $H^1(\{T_t\}_{t>0})$ is given by

$$\|f\|_{H^1(\{T_t\}_{t>0})} = \|T_* f\|_1, \ \ f\in H^1(\{T_t\}_{t>0}).$$

For every semigroup $\{T_t\}_{t>0}$ it is usual to study different characterizations of the space $H^1(\{T_t\}_{t>0})$ (atomic representations, maximal functions, special singular integrals, area integrals, ...), $H^1(\{T_t\}_{t>0})$-boundedness of harmonic analysis operators (multipliers, singular integrals, ...), duality and interpolation, definitions and properties of the spaces $H^p(\{T_t\}_{t>0})$, $0<p<1$, amongst other problems.

We begin recalling definitions and results about classical Hardy spaces in Section 2. After this, we present in Section 3 the extension of H^p theory to homogeneous and non-homogeneous type spaces. In Section 3 we consider the Hardy spaces associated to second order linear differential operators. In the last years I have collaborated with J. Dziubanski (University of Wroclaw), G. Garrigós (University of Murcia), S. Molina (University of Mar del Plata), L. Rodríguez-Mesa (University of La Laguna) and J.L. Torrea (Autónoma University of Madrid) studying Hardy spaces in the Bessel and Laguerre settings. Our results about Bessel and Laguerre Hardy spaces are presented in Sections 4 and 5, respectively.

2. Classical Hardy spaces

The study of Hardy spaces in $\mathbb{R}^n$ was begun by Stein and Weiss [44]. Their theory was developed in connection with harmonic functions. In their celebrated paper [24], Fefferman and Stein introduced real variable methods in the analysis of H^p spaces (see [24, Section V]).

By $S(\mathbb{R}^n)$ we denote the Schwartz class of functions endowed with its usual Fréchet topology, and by $S(\mathbb{R}^n)'$ the dual space of $S(\mathbb{R}^n)$ that is called the space of tempered distributions. A tempered distribution f is said bounded when $f * \phi \in L^\infty(\mathbb{R}^n)$, for every $\phi \in S(\mathbb{R}^n)$.

If $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$, we define, for every $t > 0$, $\phi_t(x) = t^{-n}\phi(x/t)$, $x \in \mathbb{R}^n$. We consider the heat semigroup $\{W_t\}_{t>0}$ generated by the Laplace operator $\Delta = \sum_{j=1}^n \partial^2_{x_j}$ in $\mathbb{R}^n$, $n \geq 1$, i.e,

$$W_t(f) = f * h_{\sqrt{t}}, \;\; f \in L^p(\mathbb{R}^n), \;\; 1 \leq p \leq \infty,$$

where $h(x) = (4\pi)^{-n/2}\exp(-|x|^2/4)$, $x \in \mathbb{R}^n$, and the Poisson semigroup $\{P_t\}_{t>0}$, that is subordinated in the Bochner sense of $\{W_t\}_{t>0}$, defined by

$$P_t(f) = f * P_t, \;\; f \in L^p(\mathbb{R}^n), \;\; 1 \leq p \leq \infty,$$

where $P(x) = c_n \frac{1}{(1+|x|^2)^{(n+1)/2}}$, $x \in \mathbb{R}^n$, and $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$.

A crucial result is the following.

Theorem 2.1 ([24, Theorem 11])**.** *Let $0 < p \leq \infty$ and $f \in S(\mathbb{R}^n)'$. The following assertions are equivalent.*

(i) *There exists $\phi \in S(\mathbb{R}^n)$ such that $\int \phi(x)dx \neq 0$ and*

$$M_\phi(f) = \sup_{t>0} |f * \phi_t| \in L^p(\mathbb{R}^n).$$

(ii) *There exists $N \in \mathbb{N}$ such that*

$$M_N(f) = \sup_{\phi \in S_N} M_\phi(f) \in L^p(\mathbb{R}^n),$$

where

$$S_N = \Big\{\phi \in S(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha| \leq N} \Big|\frac{\partial^\alpha}{\partial x^\alpha}\phi(x)\Big|^2 dx \leq 1\Big\}.$$

(iii) *f is bounded and*

$$f^*(y) = \sup_{|x-y|<t} |(f * P_t)(x)| \in L^p(\mathbb{R}^n).$$

Definition 2.1. Let $0 < p \leq \infty$. A distribution $f \in S(\mathbb{R}^n)'$ is said to be in $H^p(\mathbb{R}^n)$ when f satisfies some of the (equivalently, all) conditions in Theorem 2.1.

Note that if $0 < p \leq \infty$ and $f \in S(\mathbb{R}^n)'$, then $f \in H^p(\mathbb{R}^n)$ if, and only if $W_*(f) = \sup_{t>0}|W_t(f)| \in L^p(\mathbb{R}^n)$. Moreover, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, for every $1 < p < \infty$.

An important characterization of the Hardy spaces in terms of a class of special functions called *atoms* was obtained by Coifman [9] in one dimension, and by Latter [33] in higher dimensions.

Definition 2.2. Let $0 < p \leq 1$. A measurable function a on $\mathbb{R}^n$ is an H^p-atom when a satisfies the following conditions:

(i) There exist $x_a \in \mathbb{R}^n$ and $r_a > 0$ such that supp $a \subset B(x_a, r_a)$ and $\|a\|_\infty \leq r_a^{-n/p}$,

(ii) For every $\beta \in \mathbb{N}^n$ such that $|\beta| \leq n(p^{-1}-1)$, $\displaystyle\int_{\mathbb{R}^n} x^\beta a(x)dx = 0$.

Theorem 2.2 ([9] and [33])**.** *Let $0 < p \leq 1$. A distribution $f \in S(\mathbb{R}^n)'$ is in $H^p(\mathbb{R}^n)$ if, and only if, $f = \sum_{k=0}^\infty \lambda_k a_k$, in $S(\mathbb{R}^n)'$, where a_k is a H^p-atom and $\lambda_k \in \mathbb{C}$, $k \in \mathbb{N}$, such that $\sum_{k=0}^\infty |\lambda_k|^p < \infty$. Moreover, if $f \in H^p(\mathbb{R}^n)$, then*

$$\|f\|_{H^p(\mathbb{R}^n)} \sim \inf_{f=\sum_{k=0}^\infty \lambda_k a_k} \Big(\sum_{k=0}^\infty |\lambda_k|^p\Big)^{1/p}.$$

Hardy spaces can also be characterized by Riesz transforms. For every $j = 1, \dots, n$ and $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the j-th Riesz transform $R_j(f)$ of f is defined by

$$R_j(f)(x) = \lim_{\varepsilon \to 0^+} c_n \int_{|x-y|>\varepsilon} \frac{|x_j - y_j|}{|x-y|^{n+1}} f(y)dy, \ \text{ a.e. } x \in \mathbb{R}^n.$$

As it is well known, for every $j = 1, \dots, n$, R_j is a bounded operator from $L^p(\mathbb{R}^n)$ into itself, $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, but it is not bounded from $L^1(\mathbb{R}^n)$ into itself.

We say that a distribution $f \in S(\mathbb{R}^n)'$ is restricted at infinity when there exists $r_0 > 1$ such that $f * \phi \in L^r(\mathbb{R}^n)$, for every $\phi \in S(\mathbb{R}^n)$ and $r_0 < r < \infty$.

Assume that $\varphi \in C^\infty(\mathbb{R}^n)$ is such that $\varphi(x) = 1$, $|x| \leq 1/2$, and $\varphi(x) = 0$, $|x| \geq 1$. Let $j = 1, \dots, n$. We denote by $K_j(x) = c_n x_j/|x|^{n+1}$, $x \in \mathbb{R}^n \setminus \{0\}$ and we define $K_{0,j} = K_j\varphi$, and $K_{\infty,j} = K_j(1-\varphi)$. Let $f \in S(\mathbb{R}^n)'$ be restricted at infinity. Since $K_{0,j}$ is a distribution with compact support, $f * K_{0,j} \in S(\mathbb{R}^n)'$. Also, we define the distribution $f * K_{\infty,j}$ by

$$\langle f * K_{\infty,j}, \phi \rangle = \langle f * \tilde{\phi}, \widetilde{K_{\infty,j}} \rangle, \ \ \phi \in S(\mathbb{R}^n).$$

Here, if $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$, we write $\tilde{\psi}(x) = \psi(-x)$, $x \in \mathbb{R}^n$. The j-th Riesz transform $R_j(f)$ is defined by

$$R_j(f) = f * K_{0,j} + f * K_{\infty,j}.$$

Definition 2.3. Let $\phi \in S(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$ and let $f \in S(\mathbb{R}^n)'$ be restricted at infinity. We say that the Riesz transform $R(f)$ of f is in $L^p(\mathbb{R}^n)$, $0 < p < \infty$, when

$$\sup_{t>0} \Big(\|f * \phi_t\|_p + \sum_{j=1}^n \|R_j(f) * \phi_t\|_p \Big) < \infty.$$

Note that this last property does not depend on ϕ.

Theorem 2.3 ([43, Proposition 3, p. 123])**.** *Let $1 - 1/n < p < \infty$. Suppose that $f \in S(\mathbb{R}^n)'$ is restricted at infinity. Then, $f \in H^p(\mathbb{R}^n)$ if, and only if, $R(f) \in L^p(\mathbb{R}^n)$. Moreover, if $\phi \in S(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x) \neq 0$, then*

$$\|f\|_{H^p(\mathbb{R}^n)} \sim \sup_{t>0} \Big(\|f * \phi_t\|_p + \sum_{j=1}^n \|R_j(f) * \phi_t\|_p \Big), \ \ f \in H^p(\mathbb{R}^n).$$

Hardy spaces $H^p(\mathbb{R}^n)$ can be described in other different ways: molecules, area integrals, Littlewood-Paley, ... (see [43]).

The dual space of $H^1(\mathbb{R}^n)$ was characterized in [24, Subsection 2.2] as the space $BMO(\mathbb{R}^n)$ of the bounded mean oscillation functions in $\mathbb{R}^n$ (also called John-Nirenberg space). A function $f \in L^1_{loc}(\mathbb{R}^n)$ is in $BMO(\mathbb{R}^n)$ when

$$\|f\|_* = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where the supremum is taken over all the balls B in $\mathbb{R}^n$. Here, for every ball B in $\mathbb{R}^n$, $|B|$ denotes the Lebesgue measure of B and $f_B = \int_B f(x)dx$.

The dual space $H^1(\mathbb{R}^n)'$ of $H^1(\mathbb{R}^n)$ can be identified in the natural way with the space $BMO(\mathbb{R}^n)$.

On the other hand, the dual space $H^p(\mathbb{R}^n)'$ of $H^p(\mathbb{R}^n)$ coincides with the Lipschitz space $L_{1/p-1}(\mathbb{R}^n)$, $0 < p < 1$.

The interested reader can find a complete study about classical Hardy spaces in Stein's monography [43].

3. Hardy spaces in spaces of homogeneous type

Coifman and Weiss [10] studied the theory of classical Hardy spaces and they isolated some of the measure theoretic and geometric properties in $\mathbb{R}^n$ that are fundamental in order to get the theory. They consider the so called spaces of homogeneous type.

Definition 3.1. A space of homogeneous type is a triple (X, μ, d) where

(i) X is a topological space.
(ii) μ is a Borel measure on X.
(iii) d is a quasimetric defined on $X \times X$.
(iv) μ satisfies the doubling property, i.e., there exists $C > 0$ such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)), \;\; x \in X \;\; and \;\; r > 0.$$

Examples of spaces of homogeneous type can be encountered in [10, p. 587-591].

Hardy spaces on spaces of homogeneous type are defined firstly by using atoms as follows.

Assume in the sequel that (X, μ, d) is a space of homogeneous type. Let $0 < p < q$ and $p \leq 1 \leq q \leq \infty$. A (μ)-measurable function a on X is a (p, q)-atom when

(i) There exist $x_0 \in X$ and $r_0 > 0$ such that $\mathrm{supp}(a) \subset B(x_0, r_0)$ and $\|a\|_q^q \leq \mu(B(x_0, r_0))^{1-q/p}$.
(ii) $\displaystyle\int_X a(x)d\mu(x) = 0$,

or, when $\mu(X) < \infty$, $a(x) = \mu(X)^{1/p}$, $x \in X$.

If $\alpha > 0$, by L_α we denote the usual Lipschitz space of exponent α defined by d. If, for every $j \in \mathbb{N}$, a_j is a (p, q)-atom and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=0}^\infty |\lambda_j|^p < \infty$, then the series $\sum_{j=0}^\infty \lambda_j a_j$ converges in $L'_{1/p-1}$, the dual space of $L_{1/p-1}$, when $0 < p < 1$, and in $L^1(X, \mu)$, when $p = 1$.

Definition 3.2. Let $0 < p < 1 \leq q \leq \infty$ (respectively, $1 < q \leq \infty$). The space $H^{p,q}(X,\mu)$ (respectively, $H^{1,q}(X,\mu)$) is the subspace of the dual $L'_{1/p-1}$ of $L_{1/p-1}$ (respectively, $L^1(X,\mu)$) consisting of all those linear functionals h on $L_{1/p-1}$ (functions in $L^1(X,\mu)$) admitting a representation as follows

$$h = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where, for every $j \in \mathbb{N}$, a_j is a (p,q)-atom and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$.

Moreover, we define

$$\|h\|_{H^{p,q}(X,\mu)} = \inf_{h=\sum_{j=0}^{\infty} \lambda_j a_j} (\sum_{j=0}^{\infty} |\lambda_j|^p)^{1/p}, \;\; h \in H^{p,q}(X,\mu).$$

On $H^{p,q}(X,\mu)$, $\|.\|_{H^{p,q}(X,\mu)}$ is a norm when $p=1$ and a quasinorm when $0 < p < 1$.

In [10, Theorem A] it was established that $H^{p,q}(X,\mu) = H^{p,\infty}(X,\mu)$, with equivalent quasinorms, for every $0 < p < q$ and $p \leq 1 \leq q \leq \infty$. The space $H^p(X,\mu)$ is defined as any one of the spaces $H^{p,q}(X,\mu)$, for $0 < p < q$ and $p \leq 1 \leq q \leq \infty$.

The dual of $H^1(X,\mu)$ is the space of bounded mean oscillation functions $BMO(X,\mu)$ on X, that is defined in a natural way, and, for every $0 < p < 1$ the dual of $H^p(X,\mu)$ is the Lipschitz space $L_{1/p-1}$ ([10, Theorem B]).

Macías and Segovia [34] and Uchiyama [49] gave characterizations for the $H^p(X,\mu)$ spaces by using maximal functions.

We now consider a space of homogeneous type (X,μ,d) satisfying, for a certain $C > 1$, that

$$\frac{r}{C} \leq \mu(B(x,r)) \leq r, \;\; x \in X \quad \text{and} \quad r \in (0,\mu(X)).$$

Also, we assume that there exists a nonnegative continuous function K defined on $(0,\infty) \times X \times X$ verifying the following properties:

(i) $K(r,x,y) = 0$, if $d(x,y) > r$.
(ii) For a certain $A > 0$, $K(r,x,y) > 1/A$.
(iii) $K(r,x,y) \leq 1$.
(iv) For a certain $\gamma > 0$, $|K(r,x,y) - K(r,x,z)| \leq (d(y,z)/r)^\gamma$.

For any $f \in L^1_{loc}(X,\mu)$, let

$$F(r,x,f) = \int_X K(r,x,y) f(y) \frac{d\mu(y)}{r}, \;\; (r,x) \in (0,\infty) \times X,$$

and, we define

$$f^+(x) = \sup_{r>0} |F(r,x,f)|, \;\; x \in X.$$

$$f^*(x) = \sup \Big| \int_X f(y) \varphi(y) \frac{d\mu(y)}{r} \Big|, \;\; x \in X,$$

where the supremum is taken over $r > 0$ and $\varphi \in C(X)$ such that $\|\varphi\|_\infty \leq 1$, $\operatorname{supp}(\varphi) \subset B(x,r)$, and

$$\sup_{x,y\in X, x\neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\gamma \leq r^{-\gamma}.$$

Macías and Segovia [34] and Uchiyama [49] proved the following results.

Theorem 3.1 ([34])**.** *If $f \in L^1(X,\mu)$ and $1/(1+\gamma) < p \leq 1$, then*

$$\|f^*\|_p/C \leq \|f\|_{H^p(X,\mu)} \leq C\|f^*\|_p,$$

where the constant $C > 1$ depends only on p and X.

Theorem 3.2 ([49])**.** *There exists $p_1 < 1$, only depending on X, such that, for any $f \in L^1(X)$ and $p > p_1$,*

$$\|f^*\|_p \leq C\|f^+\|_p,$$

where the constant $C > 0$ depends only on p and X.

An immediate consequence of these results is the next one.

Corollary 3.1. *For every $f \in L^1(X,\mu)$,*

$$\|f^+\|_1 \sim \|f^*\|_1 \sim \|f\|_{H^1(X,\mu)}.$$

Recently, L. Grafakos, D. Yang and their collaborators have studied Hardy spaces of a class of homogeneous spaces called RD-spaces (see [28] and [53], amongst others).

In the papers of Coifman and Weiss ([10]), Macías and Segovia ([34] and [35]), and Uchiyama ([49]), the main aspects of the harmonic analysis on spaces of homogeneous type are developed. There an interested reader can find more information about this topic.

In the last years the harmonic analysis on nonhomogeneous spaces have been studied by many authors (see, for instance, [37], [38], [39], [46], [47] and [48]). In this case the measure μ does not need to be doubling.

4. Hardy spaces associated to operators

In this section we present some results about Hardy spaces associated to operators. Most of this part can be encountered in the papers of Dziubanski and Zienkiewicz ([20], [21], [22] and [23]) and Hofmann, Lu, Mitrea, Mitrea and Yan ([31]).

4.1. **Hardy spaces and Schrödinger operators.** Let V be a nonnegative and locally integrable function in $\mathbb{R}^n$, $n \geq 3$, not identically zero. We define the sesquilinear form Q by

$$Q(u_1,u_2) = \int_{\mathbb{R}^n} \nabla u_1(x)\nabla u_2(x)dx + \int_{\mathbb{R}^n} V(x)u_1(x)u_2(x)dx,$$

with domain

$$D(Q) = \{(u_1,u_2) : u_j \in W^{1,2}(\mathbb{R}^n) \ \text{ and } \int_{\mathbb{R}^n} V(x)|u_j(x)|^2dx < \infty, \ j=1,2\}.$$

This symmetric form is closed. We denote by L_V the selfadjoint operator associated with Q. The domain of the operator L_V consists of all those functions $u \in W^{1,2}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} V(x)|u(x)|^2dx < \infty$ and there exists $v \in L^2(\mathbb{R}^n)$ such that $Q(v,\varphi) = \int_{\mathbb{R}^n} v(x)\overline{\varphi(x)}dx$, for every $\varphi \in W^{1,2}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} V(x)|\varphi(x)|^2dx < \infty$. Formally, we write $L_V = -\Delta + V$.

The operator $-L_V$ generates an analytic semigroup of operators $\{W_t^V\}_{t>0}$ defined, for every $t > 0$, by

$$W_t^V(f)(x) = \int_{\mathbb{R}^n} W_t^V(x,y)f(x)dx, \;\; f \in L^p(\mathbb{R}^n), \;\; 1 < p < \infty,$$

where, according to the Feynman-Kac formula

$$0 \le W_t^V(x,y) \le (4\pi t)^{-n/2} \exp\Big(-\frac{|x-y|^2}{4t}\Big), \;\; t > 0, \;\; x, y \in \mathbb{R}^n.$$

This semigroup $\{W_t^V\}_{t>0}$ is not conservative.

In Shen's paper [41] (see also [55]) some harmonic analysis operators associated with the operator L_V are analyzed. He assumed that the potential V satisfies a reverse Hölder inequality RH_q, where $q > n/2$. More precisely, we say that $0 \le V \in L^1_{loc}(\mathbb{R}^n)$ satisfies RH_q when, there exists $C > 0$ such that

$$\Big(\frac{1}{|B|}\int_B V(x)^q dx\Big)^{1/q} \le \frac{C}{|B|}\int_B V(x)dx,$$

for every ball B in $\mathbb{R}^n$. Here $|B|$ denotes the Lebesgue measure of B.

The auxiliary function $m(x,V)$ defined by

$$m(x,V) = \Big(\sup\Big\{r > 0 : r^{2-n}\int_{B(x,r)} V(y)dy \le 1\Big\}\Big)^{-1}.$$

plays an important role in this theory.

As usual the maximal operator W_*^V associated to the semigroup of operators $\{W_t^V\}_{t>0}$ is defined by $W_*^V(f) = \sup_{t>0}|W_t^V(f)|$.

Definition 4.1. A function $f \in L^1(\mathbb{R}^n)$ is in $H^1_V(\mathbb{R}^n)$ when $W_*^V(f) \in L^1(\mathbb{R}^n)$. The norm $\|.\|_{H^1_V(\mathbb{R}^n)}$ on $H^1_V(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^1_V(\mathbb{R}^n)} = \|W_*^V(f)\|_1, \;\; f \in H^1_V(\mathbb{R}^n).$$

In [21] Dziubanski and Zinkiewicz introduced the following class of atoms. For every $n \in \mathbb{Z}$, the set B_n is defined by

$$B_n = \{x \in \mathbb{R}^n : 2^{n/2} \le m(x,V) < 2^{(n+1)/2}\}.$$

Since $0 < m(x,V) < \infty$, $x \in \mathbb{R}^n$, we have $\mathbb{R}^n = \bigcup_{n\in\mathbb{Z}} B_n$.

Definition 4.2. A measurable function a on $\mathbb{R}^n$ is an atom for the Hardy space $H^1_V(\mathbb{R}^n)$ associated with the ball $B(x_0, r_0)$, where $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, when $\operatorname{supp}(a) \subset B(x_0,r_0)$, $\|a\|_\infty \le r_0^{-n}$ and, if $x_0 \in B_n$, then $r_0 \le 2^{1-n/2}$. Moreover, $\int_{\mathbb{R}^n} a(y)dy = 0$, provided that $x_0 \in B_n$ and $r_0 \le 2^{-1-n/2}$.

By using the theory of local Hardy spaces developed by Goldberg [29], Dziubanski and Zienkiewicz [13] obtained atomic representation for the elements of the space $H^1_V(\mathbb{R}^n)$.

Theorem 4.1 ([13, Theorem 1.5]). *Assume that V is not identically zero and $V \in RH_{n/2}$. Then, a function $f \in L^1(\mathbb{R}^n)$ is in $H^1_V(\mathbb{R}^n)$ if, and only if, $f = \sum_{k=0}^{\infty} \lambda_k a_k$, where, for every $k \in \mathbb{N}$, a_k is an atom for $H^1_V(\mathbb{R}^n)$ and $\lambda_k \in \mathbb{C}$ being $\sum_{k=0}^{\infty} |\lambda_k| < \infty$. Moreover, for every $f \in H^1_V(\mathbb{R}^n)$,*

$$\|f\|_{H^1_V(\mathbb{R}^n)} \sim \inf_{f=\sum_{k=0}^{\infty}\lambda_k a_k} \sum_{k=0}^{\infty} |\lambda_k|.$$

For every $j = 1, \dots, n$, the j-th Riesz transform $R_{V,j}$ associated to the Schrödinger operator L_V is defined by

$$R_{V,j} = \frac{\partial}{\partial x_j} L_V^{-1/2},$$

where the negative square root $L_V^{-1/2}$ is given by the functional calculus as follows:

$$L_V^{-1/2} f(x) = \int_{\mathbb{R}^n} K_V(x,y) f(y)\,dy, \tag{1}$$

being

$$K_V(x,y) = -\frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-1/2} \Gamma(x,y,\tau)\,d\tau.$$

Here, for every $\tau \in \mathbb{R}$, $\Gamma(x,y,\tau)$, $x,y \in \mathbb{R}^n$, represents the fundamental solution for the operator $L_V + i\tau$.

In [6] it was proved that, for every $j = 1, \dots, n$, the Riesz transform $R_{V,j}$ can be represented as a principal value integral operator on $C_c^\infty(\mathbb{R}^n)$, the space C^∞-functions in $\mathbb{R}^n$ that have compact support.

Proposition 4.1 ([6, Proposition 1.1]). *Let $j = 1, \dots, n$. Suppose that one of the following two conditions holds:*

(i) *$f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and $V \in RH_n$;*
(ii) *$f \in L^p(\mathbb{R}^n)$, $1 \le p < p_0$, where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, and $V \in RH_q$, $n/2 \le q < n$.*

Then, if $R_{V,j}(x,y) = \partial_{x_j} K_V(x,y)$, $x,y \in \mathbb{R}^n$, $x \neq y$, there exists the following limit

$$\lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_{V,j}(x,y) f(y)\,dy, \quad a.e. \quad x \in \mathbb{R}^n.$$

Moreover, if $f \in C_c^\infty(\mathbb{R}^n)$, then $L_V^{-1/2} f$ admits partial derivative with respect to x_j in almost everywhere $\mathbb{R}^n$ and

$$\frac{\partial}{\partial x_j} L_V^{-1/2} f(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_{V,j}(x,y) f(y)\,dy, \quad a.e. \quad x \in \mathbb{R}^n. \tag{2}$$

Moreover, Shen [41] established the L^p boundedness properties of the Riesz transforms $R_{V,j}$, $j = 1, \dots, n$.

Theorem 4.2 ([41, Theorems 0.5 and 0.8]). *Let $j = 1, \dots, n$. Suppose that one of the following two conditions holds:*

(i) $1 < p < \infty$, *and* $V \in RH_n$;
(ii) $1 < p < p_0$, *where* $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, *and* $V \in RH_q$, $n/2 \leq q < n$.

Then, $R_{V,j}$ *can be extended to* $L^p(\mathbb{R}^n)$ *as a bounded operator from* $L^p(\mathbb{R}^n)$ *into itself. Moreover,* $R_{V,j}$ *can be extended to* $L^1(\mathbb{R}^n)$ *as a bounded operator from* $L^1(\mathbb{R}^n)$ *into* $L^{1,\infty}(\mathbb{R}^n)$.

The Riesz transforms associated to the operator L_V characterize the Hardy spaces $H^1_V(\mathbb{R}^n)$.

Theorem 4.3 ([21, Theorem 1.7])**.** *If* $V \in RH_{n/2}$ *is a not identically zero and nonnegative potential, then a function* $f \in L^1(\mathbb{R}^n)$ *is in* $H^1_V(\mathbb{R}^n)$*, if and only if,* $R_{V,j}(f) \in L^1(\mathbb{R}^n)$, $j = 1, \dots, n$. *Moreover, we have that*

$$\|f\|_{H^1_V(\mathbb{R}^n)} \sim \|f\|_1 + \sum_{j=1}^{n} \|R_{V,j}(f)\|_1, \;\; f \in H^1_V(\mathbb{R}^n).$$

Czaja and Zienkiewicz ([11]) considered the one-dimensional Schrödinger operator $L_V = -\frac{d^2}{dx^2} + V$, where $0 \leq V \in L^1_{loc}(\mathbb{R})$, and V is not identically zero. They introduced a class of atomic Hardy spaces associated with a family of dyadic intervals. Let $\mathcal{I} = \{I_j\}_{j\in\mathbb{N}}$ be a cover of $\mathbb{R}$ by closed dyadic intervals with disjoint interiors. If I and J are two closed dyadic intervals in $\mathbb{R}$, we say that I and J are neighbors when $I \cap J$ has a single point. Assume that $\mathcal{I} = \{I_j\}_{j\in\mathbb{N}}$ satisfies the following property: There exists $C > 0$ such that, for every $j \in \mathbb{N}$, there exist its two neighbors I_{j_1} and I_{j_2} in $\mathcal{I}$ and $1/C \leq |I_j|/|I_{j_i}| \leq C$, $i = 1, 2$. We say a measurable function a on $\mathbb{R}$ is a $H^1_{\mathcal{I}}(\mathbb{R})$ atom if a is a classical atom supported in $(1+\alpha)I_j$, for some $j \in \mathbb{N}$, where $\alpha > 0$ is small enough (see [11]), or if $a = |I_j|\chi_{I_j}$, for some $j \in \mathbb{N}$. Here χ_J denotes the characteristic function of the set $J \subset \mathbb{R}$.

Definition 4.3. A function $f \in L^1(\mathbb{R})$ is in $H^1_{\mathcal{I}}(\mathbb{R})$ when $f = \sum_{k=1}^{\infty} \lambda_k a_k$, where, for every $k \in \mathbb{N}$, a_k is a $H^1_{\mathcal{I}}(\mathbb{R})$ atom and $\lambda_k \in \mathbb{C}$, being $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. The norm $\|.\|_{H^1_{\mathcal{I}}(\mathbb{R})}$ on $H^1_{\mathcal{I}}(\mathbb{R})$ is defined by

$$\|f\|_{H^1_{\mathcal{I}}(\mathbb{R})} = \inf_{f=\sum_{k=1}^{\infty} \lambda_k a_k} \sum_{k=0}^{\infty} |\lambda_k|, \;\; f \in H^1_{\mathcal{I}}(\mathbb{R}).$$

Theorem 4.4 ([11, Theorem 1.1])**.** *Assume that* $0 \leq V \in L^1_{loc}(\mathbb{R})$*, and* V *is not identically zero. Then, there exists a family* $\mathcal{I} = \{I_j\}_{j\in\mathbb{N}}$ *of dyadic intervals in* $\mathbb{R}$ *satisfying the above conditions such that* $H^1_{\mathcal{I}}(\mathbb{R}) = H^1_V(\mathbb{R})$.

In [23] Dziubanski and Zienkiewicz obtain atomic representation for the elements in $H^1_V(\mathbb{R}^n)$, $n \geq 3$, when the potential $V \geq 0$ has compact support and it belongs to $L^p(\mathbb{R}^n)$ for some $p > n/2$. In contrast with the case of V satisfying a reverse Hölder inequality or in the one dimensional case ([11]), the atoms for $H^1_V(\mathbb{R}^n)$ considered in [23] are not variants of local atoms.

In [12], [14], [17], [20], and [51] the interested reader can complete the information about Hardy spaces in the Schrödinger setting. This theory has been extended recently by Yang and Zhou [53] (see also [40]) defining the localized Hardy spaces.

When $L = -\Delta + V$ in $\mathbb{R}^n$, $n \geq 3$, and $0 \leq V \in L^1_{loc}(\mathbb{R}^n)$ satisfies RH_q, with $q > n/2$, the dual space of $H^1_V(\mathbb{R}^n)$ was characterized by Dziubanski, Garrigós, Martínez, Torrea and Zienkiewicz [16] as follows.

Definition 4.4. We say that a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ is in $BMO_V(\mathbb{R}^n)$ provided that there exists $C > 0$ such that the following two properties are satisfied:

(i) For every $x \in \mathbb{R}^n$ and $r > 0$,
$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - f_{B(x,r)}|dy \leq C,$$
where, as usual, $f_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$, and $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$; and

(ii) For every $x \in \mathbb{R}^n$ and $r \geq \gamma(x)$,
$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy \leq C.$$

Here, the critical radii $\gamma(x)$ is defined by
$$\gamma(x) = 1/m(x,V), \ \ x \in \mathbb{R}^n.$$

Theorem 4.5 ([16, Theorem 4])**.** *Let $n \geq 3$, and let $0 \leq V \in L^1_{loc}(\mathbb{R}^n)$ be satisfying RH_q, with $q > n/2$. Then, $(H^1_V(\mathbb{R}^n))' = BMO_V(\mathbb{R}^n)$.*

4.2. **Hardy spaces and operators satisfying Davies-Gaffney estimates.** Assume that (X, μ, d) is a space of homogeneous type where d is a metric defined on $X \times X$. Suppose also that

(i) L is a non-negative selfadjoint operator in $L^2(X,\mu)$.

(ii) The operator L generates an analytic semigroup $\{T_t\}_{t>0}$ which satisfies the Davies-Gaffney condition, that is, there exist $C, c > 0$ such that for every pair of open subsets $U1$ and $U2$ of X,
$$|\langle T_t f_1, f_2\rangle_{L^2(X,\mu)}| \leq C\exp\Big(-\frac{dist(U_1,U_2)^2}{ct}\Big)\|f_1\|_2\,\|f_2\|_2, \ \ t > 0,$$
for every $f_j \in L^2(X,\mu)$ with $\mathrm{supp}(f_j) \subset U_j$, $j = 1, 2$.

In order to define Hardy spaces associated with the operator L, the notion of atom in this context is introduced.

Definition 4.5. Let $M \in \mathbb{N}$. We say that a function $a \in L^2(X,\mu)$ is a $(1, 2, M)$-atom associated to the operator L if there exists a function b in the domain of the operator L^M and a ball $B = B(x_0, r_0)$ such that

(a) $a = L^M b$.

(b) $\mathrm{supp}(L^k b) \subset B$, $k = 0, 1, \dots, M$.

(c) $\|(r_0^2 L)^k b\|_2 \leq r_0^{2M}\mu(B)^{-1/2}$, $k = 0, 1, \dots, M$.

We take $M \in \mathbb{N}$ such that $M > n_0/4$, where
$$n_0 = \inf\Big\{n : \sup_{B \text{ ball in } X, \lambda\geq 1} \frac{\mu(\lambda B)}{\lambda^n \mu(B)} < \infty\Big\}.$$

The space $H^1_{L,at,M}(X)$ is defined as follows.

Definition 4.6. A measurable function f is in $\mathbb{H}^1_{L,at,M}(X)$ when $f=\sum_{k=0}^\infty \lambda_k a_k$, where the series converges in $L^2(X,\mu)$, and, for every $j\in N$, a_j is a $(1,2,M)$-atom and $\lambda_j\in\mathbb{C}$, being $\sum_{k=0}^\infty|\lambda_k|<\infty$. The norm $\|.\|_{\mathbb{H}^1_{L,at,M}(X)}$ on $\mathbb{H}^1_{L,at,M}(X)$ is given by

$$\|f\|_{\mathbb{H}^1_{L,at,M}(X)}=\inf_{f=\sum_{k=0}^\infty \lambda_k a_k}\sum|\lambda_j|,\;\; f\in\mathbb{H}^1_{L,at,M}(X).$$

The space $H^1_{L,at,M}(X)$ is the completion of $\mathbb{H}^1_{L,at,M}(X)$ with respect to $\|.\|_{\mathbb{H}^1_{L,at,M}(X)}$.

For every $f\in L^1(X,\mu)$ we consider the quadratic and nontangential area integral associated with the semigroup $\{T_t\}_{t>0}$ defined by

$$S_L(f)(x)=\Big(\int_{\Gamma(x)}|t^2LT_{t^2}(f)(y)|^2\frac{d\mu(y)}{\mu(B(x,t))}\frac{dt}{t}\Big)^{1/2},\;\; x\in X,$$

where $\Gamma(x)=\{(y,t)\in X\times(0,\infty):\, d(y,x)<t\}$.

Following [1] we consider the space $H^2_L(X)$ as the closure of the range $L(L^2(X,\mu))$ of the operator L on $L^2(X,\mu)$. The space $\mathbb{H}^1_{L,S}(X)$ is defined by

$$\mathbb{H}^1_{L,S}(X)=\{f\in H^2_L:\|S_L(f)\|_1<\infty\},$$

and the norm $\|.\|_{\mathbb{H}^1_{L,S}(X)}$ on $\mathbb{H}^1_{L,S}(X)$ by

$$\|f\|_{\mathbb{H}^1_{L,S}}=\|S_L(f)\|_1,\;\; f\in\mathbb{H}^1_{L,S}.$$

Definition 4.7. The Hardy space $H^1_{L,S}(X)$ is the completion of $\mathbb{H}^1_{L,S}(X)$ with respect to the norm $\|.\|_{\mathbb{H}^1_{L,S}(X)}$.

The Hardy spaces $H^1_{L,at,M}(X)$ and $H^1_{L,S}(X)$ coincide.

Theorem 4.6 ([31, Theorem 2.5])**.** . *Under the specified assumptions (i) and (ii) for the operator L, the topological space X and the constant M, we have that $H^1_{L,at,M}(X)=H^1_{L,S}(X)$ and*

$$\|f\|_{H^1_{L,at,M}(X)}\sim\|f\|_{H^1_{L,S}(X)},\;\; f\in H^1_{L,S}(X).$$

We now introduce a property about some Gaussian upper bounds for the kernel $T_t(x,y)$, $(t,x,y)\in(0,\infty)\times X\times X$ of the operator T_t, $t>0$. More precisely the property is the following:

(iii) For every $t>0$ the function $T_t(x,y)$ is a measurable function on $X\times X$ and there exists constants $C,c>0$ such that,

$$|T_t(x,y)|\le\frac{C}{\mu(B(x,\sqrt{t}))}exp\Big(-\frac{d(x,y)^2}{ct}\Big),\;\; a.e.\;\; x,y\in X.$$

Note that the property (iii) implies property (ii).

The space $\mathbb{H}^1_{L,max}(X)$ consists of all those $f\in L^2(X,\mu)$ such that $T_*f\in L^1(X,\mu)$, where $T_*f=\sup_{t>0}|T_t(f)|$. The norm $\|.\|_{\mathbb{H}^1_{L,max}(X)}$ in $\mathbb{H}^1_{L,max}(X)$ is defined by

$$\|f\|_{\mathbb{H}^1_{L,max}(X)}=\|T_*f\|_1,\;\; f\in\mathbb{H}^1_{L,max}(X).$$

Definition 4.8. The Hardy space $H^1_{L,max}(X)$ is the completion of $\mathbb{H}^1_{L,max}(X)$ with respect to the norm $\|.\|_{\mathbb{H}^1_{L,max}(X)}$.

In general, the spaces $H^1_{L,at,M}(X)$ and $H^1_{L,max}$ are not equal.

Theorem 4.7 ([31, Theorem 7.4])**.** *Assume that the operator L satisfies the conditions (i) and (iii). Then, for every $M \leq 1$, $H^1_{L,at,M}(X) \subset H^1_{L,max}(X)$.*

When $L = -\Delta + V$, the Schrödinger operator with potential $0 \leq V \in L^1_{loc}(\mathbb{R}^n)$, the equality $H^1_{L,at,M}(X) = H^1_{L,max}(X)$ holds, for every $M \geq 1$ ([31, Theorem 8.2]). Note that in [31] the class of atoms, in the case of the Schrödinger operator, is different to the one considered in [21].

We now define the space of BMO-type adapted to the operator L following the approach of [32]. Let $u = L^M v$ be a function in $L^2(X,\mu)$ where v belongs to the domain $D(L^M)$ of L^M. For every $\varepsilon > 0$ and $M \in \mathbb{N}$ the following norm is considered:

$$\|u\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} = \sup_{j\in\mathbb{N}} 2^{j\varepsilon}\mu(B(x_0,2^j))^{1/2}\sum_{k=0}^{M}\|L^k v\|_{L^2(U_j(B_0),\mu)},$$

where $B_0 = B(x_0,1)$, with $x_0 \in X$, and, for every $j \in \mathbb{N}$, $j \geq 1$, $U_j(B_0) = 2^jB_0 \setminus 2^{j-1}B_0$, being $U_0(B_0) = B_0$. Then, we set

$$\mathcal{M}_0^{1,2,M,\varepsilon}(L) = \{u = L^M v \in L^2(X,\mu) : \|u\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} < \infty\},$$

and

$$\mathcal{E}_M^L = \bigcap_{\varepsilon>0}(\mathcal{M}_0^{1,2,M,\varepsilon}(L))'.$$

Definition 4.9. Assume that $M \geq 1$ and the operator L satisfies the conditions (i) and (ii). An element $f \in \mathcal{E}_M^L$ is said to belong to the adapted space of bounded mean oscillation functions $BMO_{L,M}(X)$ if

$$\|f\|_{BMO_{L,M}(X)} = \sup_{B\subset X}\left(\frac{1}{\mu(B)}\int_B |(I - T_{r_B^2})^M f(x)|^2 d\mu(x)\right)^{1/2} < \infty,$$

where the supremun is taken over all the balls B in X. Here r_B denotes the radius of the ball B.

The analogue of Fefferman and Stein's duality result in this context is the following.

Theorem 4.8. *Suppose that $M \in \mathbb{N}$ and $M > n_0/4$. If L is an operator satisfying the conditions (i) and (ii), then $(H^1_{L,at,M}(X))^* = BMO_{L,M}(X)$.*

5. Hardy spaces associated with Bessel operators

Muckenhoupt and Stein ([36]) studied L^p-boundedness properties of maximal operators and Riesz transforms in the Bessel setting. In this case the measure space is $((0,\infty), x^{2\alpha}dx)$. By using pairs of appropriate conjugate functions defined by a Cauchy-Riemann type equations they consider Hardy spaces for the operator $\Delta_\alpha = -\frac{d^2}{dx^2} - \frac{2\alpha}{x}\frac{d}{dx}$, with $\alpha > 0$.

Dziubanski [13] defines Hardy spaces H^1 associated with the operator $L_{\alpha,V} = -\frac{1}{2}\frac{d^2}{dx^2} - \frac{\alpha}{2x}\frac{d}{dx} + V$, on $(0,\infty)$, where $\alpha > 1$. The potential V is assumed to be nonnegative, not identically zero, and that satisfies a reverse Hölder inequality with an exponent $q > (\alpha+1)/2$ with respect to the measure $d\mu(x) = \frac{2}{\alpha-1}x^\alpha dx$, that is, there exists $C > 0$ such that

$$\Big(\frac{1}{\mu(B)}\int_B V(y)^q d\mu(y)\Big)^{1/q} \leq \frac{C}{\mu(B)}\int_B V(y)d\mu(y),$$

for every interval $B \subset (0,\infty)$. Then, the operator $-L_{\alpha,V}$ generates a semigroup $\{W_t^{\alpha,V}\}_{t>0}$ of linear operators on $L^p((0,\infty), d\mu)$, $1 \leq p < \infty$. The maximal operator $W_*^{\alpha,V}$ defined by $\{W_t^{\alpha,V}\}_{t>0}$ is given by

$$W_*^{\alpha,V}(f) = \sup_{t>0}|W_t^{\alpha,V}(f)|.$$

Definition 5.1. A function $f \in L^1((0,\infty), d\mu)$ belongs to $H^1(L_{\alpha,V})$ when $W_*^{\alpha,V}f \in L^1((0,\infty), d\mu)$. The norm $\|.\|_{H^1(L_{\alpha,V})}$ on $H^1(L_{\alpha,V})$ is defined by

$$\|f\|_{H^1(L_{\alpha,V})} = \|W_t^{\alpha,V}(f)\|_{L^1((0,\infty),d\mu)}, \;\; f \in H^1(L_{\alpha,V}).$$

In order to introduce a class of atoms that allow to describe the elements of $H^1(L_{\alpha,V})$ the following function is introduced:

$$\rho(x) = \sup\Big\{r > 0 : \frac{r^2}{\mu(B(x,r))}\int_{B(x,r)} V(y)d\mu(y) \leq 1\Big\}, \;\; x \in (0,\infty),$$

where $B(x,r) = (x-r, x+r) \cap (0,\infty)$, $x, r \in (0,\infty)$.

Definition 5.2. A measurable function a is an $(1,\infty)$-atom when there exist $x_0, r_0 \in (0,\infty)$ such that $r_0 < \rho(x_0)$, $\mathrm{supp}(a) \subset B(x_0, r_0)$, and $\|a\|_\infty \leq \mu(B(x_0, r_0))^{-1}$. Moreover, if $r_0 < \rho(x_0)/4$, $\int_0^\infty a(y)d\mu(y) = 0$.

In the following an atomic characterization of the elements of $H^1(L_{\alpha,V})$ is established.

Theorem 5.1 ([13, Theorem 1.10])**.** *Assume that V is nonnegative, not identically zero, and that satisfies a reverse Hölder inequality with an exponent $q > (\alpha+1)/2$ with respect to the measure $d\mu(x) = \frac{2}{\alpha-1}x^\alpha dx$, with $\alpha > 1$. Then, a function $f \in L^1((0,\infty), d\mu)$ is in $H^1(L_{\alpha,V})$ if, and only if, $f = \sum_{k=1}^\infty \lambda_k a_k$, where, for every $k \in \mathbb{N}$, a_k is an $(1,\infty)$-atom and $\lambda_k \in \mathbb{C}$, being $\sum_{k=1}^\infty |\lambda_k| < \infty$. Moreover, we have that*

$$\|f\|_{H^1(L_{\alpha,V})} \sim \sup_{f=\sum_{k=1}^\infty \lambda_k a_k}\sum_{k=1}^\infty |\lambda_k|, \;\; f \in H^1(L_{\alpha,V}).$$

Betancor, Buraczewski, Fariña, Martínez and Torrea ([2]) investigated $L^p((0,\infty), dx)$ bounds for the Riesz transforms related to the Bessel operator S_α defined by $S_\alpha = -\frac{d^2}{dx^2} + \frac{\alpha^2-\alpha}{x^2}$, on $(0,\infty)$, where $\alpha > 0$. In [4] the Hardy spaces in the setting of Δ_α and S_α are analyzed.

The Poisson semigroup $\{P_t^\alpha\}_{t>0}$ generated by the operator $-\sqrt{\Delta_\alpha}$ is defined by

$$P_t^\alpha(f)(x) = \int_0^\infty P_t^\alpha(x,y) f(x) x^{2\alpha} dx, \quad x \in (0,\infty),$$

for every $f \in L^p((0,\infty), x^{2\alpha}dx)$, $1 \le p \le \infty$. Here,

$$P_t^\alpha(x,y) = \frac{2\alpha t}{\pi} \int_0^\pi \frac{(\sin\,\theta)^{2\alpha-1}}{(x^2+y^2+t^2-2xy\cos\,\theta)^{\alpha+1}} d\theta, \ \ t,x,y \in (0,\infty).$$

The maximal operator P_*^α is defined by $P_*^\alpha(f) = \sup_{t>0} |P_t^\alpha(f)|$.

Definition 5.3. Let $\alpha > 0$. The Hardy space $H^1_{max}(\Delta_\alpha)$ consists of all those functions $f \in L^1((0,\infty), x^{2\alpha}dx)$ such that $P_*^\alpha(f) \in L^1((0,\infty), x^{2\alpha}dx)$. The norm $\|.\|_{H^1_{max}(\Delta_\alpha)}$ is defined by

$$\|f\|_{H^1_{max}(\Delta_\alpha)} = \|f\|_{L^1((0,\infty),x^{2\alpha}dx)} + \|P_*^\alpha(f)\|_{L^1((0,\infty),x^{2\alpha}dx)}, \ \ f \in H^1_{max}(\Delta_\alpha).$$

The Riesz transform R_α in the Δ_α-setting was defined in [36] by using a notion of conjugation related to the operator Δ_α. In [2] it was established that, for every $f \in L^p((0,\infty), x^{2\alpha}dx)$, $1 \le p < \infty$,

$$R_\alpha(f)(x) = \lim_{\varepsilon\to 0} \int_{0,\,|x-y|>\varepsilon}^\infty R_\alpha(x,y) f(y) y^{2\alpha} dy, \ \ a.e. \ x \in (0,\infty),$$

where

$$R_\alpha(x,y) = -\frac{2\alpha}{\pi} \int_0^\pi \frac{(x-y\cos\,\theta)(\sin\,\theta)^{2\alpha-1}}{(x^2+y^2-2xy\cos\,\theta)^{\alpha+1}} d\theta, \ \ x,y \in (0,\infty).$$

R_α a Calderón-Zygmund operator with respect to the homogeneous space $((0,\infty), d, x^{2\alpha}dx)$, where d is the usual metric on $(0,\infty)$ (see [7]). We denote by $H^1_{CW}((0,\infty), x^{2\alpha}dx)$ the atomic Hardy space associated to the space of homogeneous type $((0,\infty), d, x^{2\alpha}dx)$ defined in [10] (see Section 3).

Definition 5.4. Let $\alpha > 0$. We say that a function $f \in L^1((0,\infty), x^{2\alpha}dx)$ is in $H^1_{Riesz}(\Delta_\alpha)$ when $R_\alpha \in L^1((0,\infty), x^{2\alpha}dx)$. The norm $\|.\|_{H^1_{Riesz}(\Delta_\alpha)}$ is defined by

$$\|f\|_{H^1_{Riesz}(\Delta_\alpha)} = \|f\|_{L^1((0,\infty),x^{2\alpha}dx)} + \|R_\alpha(f)\|_{L^1((0,\infty),x^{2\alpha}dx)}, \ \ f \in H^1_{Riesz}(\Delta_\alpha).$$

The three Hardy spaces that we have just defined for the operator Δ_α coincide.

Theorem 5.2 ([4, Theorem 1.7])**.** *Let $\alpha > 0$ and $f \in L^1((0,\infty), x^{2\alpha}dx)$. The following assertions are equivalent.*

(i) *$f \in H^1_{CW}((0,\infty), x^{2\alpha}dx)$.*
(ii) *$f \in H^1_{Riesz}(\Delta_\alpha)$.*
(iii) *$f \in H^1_{max}(\Delta_\alpha)$.*

Moreover, the corresponding norms are equivalent.

In the definition of the space $H^1_{max}(\Delta_\alpha)$ we can replace the Poisson integral by Hankel convolution operators ([4, Theorem 2.7]). The results in [4] were extended in [52] defining Hardy spaces $H^p(\Delta_\alpha)$, $0 < p < 1$. Also, the interested reader can take a look at [27].

The Poisson semigroup associated with the operator S_α is given by

$$\mathbb{P}_t^\alpha(f)(x) = \int_0^\infty \mathbb{P}_t^\alpha(x,y)f(y)dy, \;\; x \in (0,\infty),$$

for every $f \in L^p((0,\infty),dx)$, $1 \leq p \leq \infty$, where $\mathbb{P}_t^\alpha(x,y) = (xy)^\alpha P_t^\alpha(x,y)$, $t,x,y \in (0,\infty)$. We consider the maximal operator $\mathbb{P}_*^\alpha$ defined by $\mathbb{P}_*^\alpha(f) = \sup_{t>0}|\mathbb{P}_t^\alpha(f)|$.

The Riesz transform $\mathbb{R}_\alpha$ in the S_α-context is the following

$$\mathbb{R}_\alpha(f)(x) = x^\alpha R_\alpha(y^{-\alpha}f)(x), \;\; f \in L^2((0,\infty),dx),$$

$\mathbb{R}_\alpha$ is a Calderón-Zygmund operator in the space $((0,\infty),dx,d)$, where as above d denotes the usual metric in $(0,\infty)$.

The Hardy spaces $H^1_{max}(S_\alpha)$ and $H^1_{Riesz}(S_\alpha)$ are defined in the usual way.

In [26], Fridli introduced the following class of atoms.

Definition 5.5. A measurable function a on $(0,\infty)$ is said to be an Fr-atom if

(i) $a = \frac{1}{\delta}\chi_{(0,\delta)}$, for some $\delta > 0$, where $\chi_{(0,\delta)}$ denotes the characteristic function of the interval $(0,\delta)$, or

(ii) a is a classic atom associated to a certain bounded interval $I \subset (0,\infty)$.

By using Fr-atoms Fridli defined, in the usual way, the atomic Hardy space $H^1_{Fr}(0,\infty)$.

In the context of the Bessel operator S_α we have also the coincidence of the three Hardy spaces. Note that the atomic Hardy spaces associated with Δ_α and S_α are defined by different class of atoms.

Theorem 5.3 ([4, Theorem 1.10])**.** *Let $\alpha > 0$. Then, $H^1_{Fr}(0,\infty) = H^1_{max}(S_\alpha) = H^1_{Riesz}(S_\alpha)$ and the corresponding norms are equivalent.*

In [4] a Hardy type inequality on $H^1_{Fr}(S_\alpha)$ for the Hankel transformation defined by

$$h_\alpha(f)(y) = \int_0^\infty \sqrt{xy}J_{\alpha-1/2}(xy)f(x)dx, \;\; y \in (0,\infty),$$

is established. Also, multipliers and transplantation operators involving Hankel transforms are investigated in $H^1_{Fr}(0,\infty)$ ([4, Theorem 4.11]).

6. Hardy spaces associated with Laguerre operators

Hardy spaces in the Laguerre settings have been studied by Dziubanski ([22] and [23]), Betancor, Dziubanski and Garrigós ([3]), and Dziubanski and Preisner ([19]).

For every $n \in \mathbb{N}$, the n-th Laguerre polynomial of type α is defined by ([45])

$$L_n^\alpha(x) = \frac{1}{n!}e^x x^{-\alpha}\frac{d^n}{dx^n}(e^{-x}x^{\alpha+n}), \;\; x \in (0,\infty).$$

Hardy spaces associated with three classes of Laguerre functions have been studied.

6.1. **Laguerre functions** $\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$**.** Let $\alpha > 0$. We consider, for every $n \in \mathbb{N}$, the Laguerre function $\mathcal{L}_n^\alpha$ defined by

$$\mathcal{L}_n^\alpha(x) = \Big(\frac{n!}{\Gamma(n+\alpha+1)}\Big)^{1/2} L_n^\alpha(x) e^{-x/2} x^{\alpha/2},\ \ x \in (0,\infty).$$

The family $\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2((0,\infty), dx)$. For every $n \in \mathbb{N}$, $\widetilde{\mathcal{L}}^\alpha \mathcal{L}_n^\alpha = \mu_n \mathcal{L}_n^\alpha$, where $\mu_n = n + (\alpha+1)/2$ and

$$\widetilde{\mathcal{L}}^\alpha = -\Big(x\frac{d^2}{dx^2} + \frac{d}{dx} - \Big(\frac{x}{4} + \frac{\alpha^2}{4x}\Big)\Big),\ \ x \in (0,\infty).$$

We define the operator $\mathcal{L}^\alpha$ by

$$\mathcal{L}^\alpha(f) = \sum_{n=0}^{\infty} \mu_n c_n(f) \mathcal{L}_n^\alpha,\ \ f \in D(\mathcal{L}_n^\alpha),$$

where, for every $n \in \mathbb{N}$, $c_n(f) = \int_0^\infty f(y)\mathcal{L}_n^\alpha(y)dy$, and

$$D(\mathcal{L}^\alpha) = \{f \in L^2((0,\infty), dx) : \sum_{n=0}^{\infty} |\mu_n c_n(f)|^2 < \infty\}.$$

The semigroup of operators $\{W_t^\alpha\}_{t>0}$ generated by $\mathcal{L}^\alpha$ is given, for every $f \in L^p((0,\infty), dx)$, $1 \leq p \leq \infty$, by

$$W_t^\alpha(f)(x) = \int_0^\infty W_t^\alpha(x,y) f(y) dy,\ \ x \in (0,\infty),$$

where

$$W_t^\alpha(x,y) = \sum_{n=0}^{\infty} e^{-\mu_n t} \mathcal{L}_n^\alpha(x) \mathcal{L}_n^\alpha(y),\ \ t, x, y \in (0,\infty).$$

The maximal operator W_*^α is defined by $W_*^\alpha(f) = \sup_{t>0} |W_t^\alpha(f)|$.

Definition 6.1. Let $\alpha > 0$. The space $H^1_{max}(\mathcal{L}^\alpha)$ consists of all those functions $f \in L^1((0,\infty), dx)$ such that $W_*^\alpha(f) \in L^1((0,\infty), dx)$. The norm $\|.\|_{H^1_{max}(\mathcal{L}^\alpha)}$ on $H^1_{max}(\mathcal{L}^\alpha)$ is defined by

$$\|f\|_{H^1_{max}(\mathcal{L}^\alpha)} = \|f\|_1 + \|W_*^\alpha(f)\|_1,\ \ f \in H^1_{max}(\mathcal{L}^\alpha).$$

In [14] it was proved that $H^1_{max}(\mathcal{L}^\alpha)$ can be characterized by atomic representations. The notion of atom introduced in [14] depends on the following auxiliary function: $\rho(x) = \frac{1}{8}\min(x,1)$, $x \in (0,\infty)$.

Definition 6.2. A measurable function a is said to be an $H^1(\mathcal{L}^\alpha)$-atom if there exist $x_0, R_0 \in (0,\infty)$, such that $R_0 \leq \rho(x_0)$, $\mathrm{supp}(a) \subset (x_0 - R_0, x_0 + R_0) \cap (0,\infty)$, and $\|a\|_\infty \leq R_0^{-1}$. Moreover, if $R_0 \leq \rho(x_0)/2$, $\int_0^\infty a(y)dy = 0$.

The space $H^1_{at}(\mathcal{L}^\alpha)$ is defined in the usual way by using $H^1(\mathcal{L}^\alpha)$-atoms.

Theorem 6.1 ([14, Theorem 1.2])**.** *Let $\alpha > 0$. Then, $H^1_{max}(\mathcal{L}^\alpha) = H^1_{at}(\mathcal{L}^\alpha)$ and the corresponding norms are equivalent.*

Note that the last theorem says that the space $H^1_{max}(\mathcal{L}^\alpha)$ is actually independent of α.

The Riesz transform $\mathcal{R}_\alpha$ in the $\mathcal{L}_\alpha$-setting is defined by

$$\mathcal{R}_\alpha = \delta_\alpha(\mathcal{L}^\alpha)^{-1/2}, \quad \text{where } \delta_\alpha = \sqrt{x}\frac{d}{dx} + \frac{1}{2}(\sqrt{x} - \frac{\alpha}{\sqrt{x}}),$$

on $span\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$. Here, the negative square root $(\mathcal{L}^\alpha)^{-1/2}$ of $\mathcal{L}^\alpha$ is given by

$$(\mathcal{L}^\alpha)^{-1/2} f = \sum_{n=0}^{\infty} \mu_n^{-1/2} c_n(f)\mathcal{L}_n^\alpha, \;\; f \in L^2((0,\infty), dx).$$

$\mathcal{R}_\alpha$ can be extended to $L^p((0,\infty), dx)$, for every $1 \leq p < \infty$, as a bounded operator from $L^p((0,\infty), dx)$ into itself, when $1 < p < \infty$, and from $L^1((0,\infty), dx)$ into $L^{1,\infty}(0,\infty)$ (see [30]).

The Hardy space for $\mathcal{L}^\alpha$ can be characterized by the Riesz transform $\mathcal{R}_\alpha$.

Definition 6.3. A function $f \in L^1((0,\infty), dx)$ is in the space $H^1_{Riesz}(\mathcal{L}^\alpha)$ when $\mathcal{R}_\alpha(f) \in L^1((0,\infty), dx)$. The norm $\|.\|_{H^1_{Riesz}(\mathcal{L}^\alpha)}$ on $H^1_{Riesz}(\mathcal{L}^\alpha)$ is defined by

$$\|f\|_{H^1_{Riesz}(\mathcal{L}^\alpha)} = \|f\|_1 + \|\mathcal{R}_\alpha(f)\|_1, \;\; f \in H^1_{Riesz}(\mathcal{L}^\alpha).$$

Theorem 6.2 ([3, Theorem 1.2])**.** *Let $\alpha > 0$. Then, $H^1_{max}(\mathcal{L}^\alpha) = H^1_{Riesz}(\mathcal{L}^\alpha)$ and the corresponding norms are equivalent.*

6.2. **Laguerre functions $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$.** Let $\alpha > 0$. For every $n \in \mathbb{N}$, the Laguerre function φ_n^α is defined by

$$\varphi_n^\alpha(x) = \sqrt{2x}\mathcal{L}_n^\alpha(x^2), \;\; x \in (0,\infty).$$

The system $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2((0,\infty), dx)$. Moreover, for every $n \in \mathbb{N}$, φ_n^α is an eigenfunction of the operator

$$\widetilde{L}^\alpha = -\frac{1}{2}\Big(x\frac{d^2}{dx^2} - x^2 - \frac{1}{x^2}\Big(\alpha^2 - \frac{1}{4}\Big)\Big), \;\; x \in (0,\infty).$$

More precisely, $\widetilde{L}^\alpha \varphi_n^\alpha = \lambda_n \varphi_n^\alpha$, where $\lambda_n = 2n + \alpha + 1$, for every $n \in \mathbb{N}$.

The Hardy space $H^1(L^\alpha)$ was investigated in [3] and [14].

6.3. **Laguerre functions $\{\ell_n^\alpha\}_{n\in\mathbb{N}}$.** Let $\alpha > -1$. The Laguerre function ℓ_n^α is defined by

$$\ell_n^\alpha(x) = x^{-\alpha/2}\mathcal{L}_n^\alpha(x), \;\; x \in (0,\infty).$$

The sequence $\{\ell_n^\alpha\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2((0,\infty), dx)$. We have that, for every $n \in \mathbb{N}$, $\widetilde{\mathbb{L}}^\alpha \ell_n^\alpha = \gamma_n \ell_n^\alpha$, where $\gamma_n = n + (\alpha+1)/2$ and

$$\widetilde{\mathbb{L}}^\alpha = -\Big(x\frac{d^2}{dx^2} + (\alpha+1)\frac{d}{dx} - \frac{x}{4}\Big), \;\; x \in (0,\infty).$$

The Hardy space $H^1(\mathbb{L}_\alpha)$ has been studied in [15] and [18].

References

[1] P. Auscher, A. McIntosh and E. Russ, *Hardy spaces of differential forms on Riemaniann manifolds*, J. Geom. Anal., **18** (2008), 192-248. 12

[2] J.J. Betancor, D. Buracewski, J.C. Fariña, M.T. Martínez and J.L. Torrea, *Riesz transforms related to Bessel operators*, Proc. Roy. Soc. Edinburgh Sect. A, **137** (2007), 701-725. 14, 15

[3] J.J. Betancor, J. Dziubański, and G. Garrigós, *Riesz transform characterization of Hardy spaces associated with certain Laguerre expansions*, Tohoku Math. J., **62** (2010), 215-231. 16, 18

[4] J.J. Betancor, J. Dziubański, and J.L. Torrea, *On Hardy spaces associated with Bessel operators*, J. Anal. Math., **107** (2009), 195-219. 14, 15, 16

[5] J.J. Betancor, J.C. Fariña, L. Rodríguez-Mesa, A. Sanabria, and J.L. Torrea, *Transference between Laguerre and Hermite settings*, J. Func. Anal., **254** (2008), 826-850.

[6] J.J. Betancor, J.C. Fariña, E. Harboure, and L. Rodríguez-Mesa, *L^p-boundedness properties of variation and oscillation operators in the Schrödinger setting*, preprint. 9

[7] J.J. Betancor, J.C. Fariña, and A. Sanabria, *On Littlewood-Paley functions associated with Bessel operators*, Glasgow Math. J., **51** (1) (2009), 55-70. 15

[8] J.J. Betancor, S. Molina, and L. Rodríguez-Mesa, *Area Littlewood-Paley functions associated with Hermite and Laguerre operators*, Potential Analysis, **34**, (4) (2011), 345-369.

[9] R.R. Coifman, *A real variable characterization of H^p*, Studia Math., **51** (1974), 269-274. 3, 4

[10] R.R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83** (1977), 569-645. 5, 6, 7, 15

[11] W. Czaja and J. Zienkiewicz, *Atomic characterization on the Hardy space $H^1_L(\mathbb{R})$ of one dimensional Schrödinger operators with nonegative potentials*, Proc. Amer. Math. Soc., **136** (1) (2008), 89-94. 10

[12] J. Dziubański, *Atomic decomposition of H^p spaces associated with some Schrödinger operators*, Indiana Univ. Math. J., **47** (1998), 75-98. 10

[13] J. Dziubański, *Hardy spaces associated with semigroups generated by Bessel operators*, Houston J. Math., **34** (2008), no. 1, 205-234. 9, 14

[14] J. Dziubański, *Hardy spaces for Laguerre expansions*, Constructive Approx., **27** (2008), 269-287. 10, 17, 18

[15] J. Dziubański, *Atomic decomposition of Hardy spaces associated with certain Laguerre expansions*, J. Fourier Anal. Appl., **15** (2009), no. 2, 129-152. 18

[16] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. **249** (2005), no. 2, 329–356. 11

[17] J. Dziubański, and M. Preisner, *On Riesz transform characterization of H^1 spaces associated with some Schrödinger operators*, Potential Analysis, DOI: 10.1007/s11118-010-9202-0. 10

[18] J. Dziubański, and M. Preisner, *Riesz transform characterization of Hardy spaces associated with Schrödinger operators with compactly supported potentials*, Arkiv för Matematik, **48**, (2) (2010), 301-310. 18

[19] J. Dziubański, and M. Preisner, *Riesz transform characterization of H^1 spaces associated with certain Laguerre expansions*, preprint. 16

[20] J. Dziubański, and J. Zienkiewicz, *Hardy spaces associated to with some Schrödinger operators*, Studia Math., **126** (2) (1997), 149-160. 7, 10

[21] J. Dziubański, and J. Zienkiewicz, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Revista Mat. Iberoamericana **15** (2) (1999), 279-296. 7, 8, 10, 13

[22] J. Dziubański, and J. Zienkiewicz, *H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes*, Colloq. Math. **98** (1) (2003), 5–38. 7, 16

[23] J. Dziubański, and J. Zienkiewicz, *Hardy space H^1 for Schrödinger operators with compactly supported potentials*, Annali di Matematica, **184** (2005), 315-326. 7, 10, 16

[24] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137-193. 2, 3, 5

[25] G. Folland and E. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton, 1982. 1

[26] S. Fridli, *Hardy spaces generated by an integrability condition*, J. Approx. Theory, **113** (2001), 91-109. 16

[27] J. García-Cuerva, *Weighted H^p-spaces*, Dissertations Math. (Rozprawy Mat.), **162** (1979). 15

[28] L. Grafakos, L. Liu, and D. Yang, *Maximal function caracterization of Hardy spaces on RD-spaces and their applications*, Sci. China Ser. A, **51** (2008), 2253-2284. 7

[29] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J., 46 (1978), 27-42. 9

[30] E. Harboure, J.L. Torrea, and B. Viviani, *Riesz transforms for Laguerre expansions*, Indiana Univ. Math. J., **55** (2006), 999-1014. 18

[31] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, Memoirs of the A.M.S. 7, 12, 13

[32] S. Hofmann and S. Mayboroda, *Hardy and BMO spaces associated to divergence form elliptic operators*, Math. Ann., **344** (2009), 37-116. 13

[33] R.H. Latter, *A decomposition of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math., **62** (1977), 92-102. 3, 4

[34] R. Macías and C. Segovia, *A decomposition into atoms of distributions on spaces in homogeneous type*, Adv. Math., **33** (1979), 271-309. 6, 7

[35] R. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Advances in Math., **33** (1979), 257-270. 7

[36] B. Muckenhoupt and E. Stein, *Classical expansions and their relations to conjugate harmonic functions*, Trans. Amer. Math. Soc., **118** (1965), 17-92. 1, 13, 15

[37] F. Nazarov, S. Treil and A. Volberg, *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces*, Internat. Math. Res. Notices, 1998, 463-487. 7

[38] F. Nazarov, S. Treil and A. Volberg, *Accretive system -theorems on nonhomogeneous spaces*, Duke Math. J., **113** (2002), 259-312. 7

[39] F. Nazarov, S. Treil and A. Volberg, *The Tb -theorem on non-homogeneous spaces*, Acta Math. **190** (2003), 151-239. 7

[40] R. Jiang, D. Yang, and Yuan Zhou, *Localized Hardy spaces associated with operators*, Applicable Analysis, **88** (9) (2009), 1409 - 1427. 10

[41] Z. W. Shen, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, 513–546. 8, 9

[42] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970. 1, 2

[43] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, **43** (1993), xiv+695, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. 1, 4, 5

[44] E. Stein and G. Weiss, *On the theory of harmonic functions of several variables, I. The theory of H^p-spaces*, Acta Math., **103** (1960), 25-62. 2

[45] G. Szegö, *Orthogonal polynomials*, Colloquium Publ., Vol. XXIII, Amer. Math. Soc., Providence, R.I., 1975. 16

[46] X. Tolsa, *BMO, H^1, and Calderón-Zygmund operators for non doubling measures*, Math. Ann., **319** (2001), 89-149. 7

[47] X. Tolsa, *A proof of the weak (1,1) inequality for singular integrals with non doubling measures based on a Calderón-Zygmund decomposition*, Publ. Mat., **45** (2001), 163-174. 7

[48] X. Tolsa, *The atomic space H^1 for non doubling measures in terms of a maximal operator*, Trans. Amer. Math. Soc., **355** (2003), 315-348. 7

[49] A. Uchiyama, *A maximal function characterization of H^p on the spaces of homogeneous type*, Trans. Amer. Math. Soc., **262** (2) (1980), 579-592. 6, 7

[50] A. Uchiyama, *Hardy Spaces on the Euclidean Space*, Springer Monographs in Mathematics, Springer, 2001. 1

[51] H. Wang, *The boundedness of some integral operators on weighted Hardy spaces associated with Schrödinger operators*, arXiv:1103.1715v1. 10

[52] Dachun Yang and Dongyong Yang, *Real variable characterizations of Hardy spaces associated with Bessel operators*, Anal. Appl. (Singap.) (to appear). 15

[53] D. Yang and Y. Zhou, *Radial maximal function characterizations of Hardy spaces on RD-spaces and their applications*, Math. Anal., **346** (2010), 307-333. 7, 10

[54] D. Yang and Y. Zhou, *Localized Hardy spaces related to admissible functions on RD-spaces and applications to Schrödinger operators*, Trans. Amer. Math. Soc., **363** (2011), 1197-1239.

[55] J. Zhong, *Harmonic analysis for some Schrödinger type operators*, Ph. D. Thesis, Princeton University, 1993. 8

Jorge J. Betancor
Departamento de Análisis Matemático
Universidad de la Laguna
Campus de Anchieta, Avda. Astrofísico Francisco Sánchez, s/n
38271 La Laguna (Sta. Cruz de Tenerife), España
jbetanco@ull.es

Recibido: 29 de abril de 2011
Aceptado: 1 de noviembre de 2011