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SOBOLEV SPACES DIVERSIFICATION

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ABSTRACT. This work attempts to be an overview of a variety of results concerning Sobolev spaces associated to some orthonormal systems, particularly the Hermite and Laguerre operators settings.

1. INTRODUCTION

This work is based on a talk given in "X Encuentro de Analistas A. P. Calderón" at La Falda, Córdoba, Argentina, August 25-28, 2010. The talk was about some results obtained in [5] and [6] about Sobolev spaces associated to the Hermite and Laguerre operators.

Hermite-Sobolev spaces already appear in [23] for the case $p = 2$ where an expansion type definition was used. For $p > 1$ an approach considering *derivatives* was presented in [5]. These results where extended in [6] and the new ideas where used to describe Laguerre-Sobolev spaces.

Sobolev spaces associated to other operators where also studied in the last years creating an interesting diversity. Examples of these are [2], [3], [11] and [15] among others.

In the classical theory, given a multi-index $\alpha = (\alpha_j)_{j=1}^d$ of non-negative integers, we denote the operator

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}},$$

where $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, and the derivatives $\frac{\partial}{\partial x_i}$, $i = 1, \dots, d$, are taken in the weak sense. Then, for $1 \le p \le \infty$, the classical *Sobolev space* of order $k \in \mathbb{N}_0$ is defined by

$$W^{k,p} = \{f \in L^p(\mathbb{R}^d) : \ \frac{\partial^\alpha}{\partial x^\alpha} f \in L^p(\mathbb{R}^d), \ |\alpha| \le k\}.$$

The space $W^{k,p}$ is a separable Banach space with the norm

$$\|f\|_{k,p} = \sum_{|\alpha| \le k} \left\| \frac{\partial^\alpha}{\partial x^\alpha} f \right\|_p,$$

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where $\|\cdot\|_p$ denotes the usual $L^p(\mathbb{R}^d)$ norm. It is also well known that the set C_c^∞ (the set of functions with infinitely many derivatives and compact support) is a dense subspace (see [18]).

If we start with $f \in L^2(\mathbb{R}^d)$, by the formula

$$\left(\frac{\partial}{\partial \zeta_j} f\right)\widehat{}(\zeta) \;=\; -2\pi i \zeta_j \widehat{f}(\zeta),$$

we have $f \in W^{k,2}$ if and only if $(1+|\zeta|^2)^{k/2}\widehat{f} \in L^2(\mathbb{R}^d)$.

Therefore, it is reasonable to define for $s > 0$

$$W^{s,p} = \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\widehat{f}(\zeta)|^2 (1+|\zeta|^2)^s d\zeta < \infty \right\}.$$

Now, using

$$[(I-\Delta)f]\widehat{}(\zeta) = (1+4\pi^2|\zeta|^2)\widehat{f},$$

we have for $s > 0$,

$$[(I-\Delta)^{-s/2} f]\widehat{}(\zeta) = (1+4\pi^2|\zeta|^2)^{-s/2}\widehat{f}(\zeta).$$

The operator $(I-\Delta)^{-s/2}$, with $s > 0$, is called the Bessel potential of order s. Therefore, $f \in W^{s,2}$ if and only if there exists $g \in L^2(\mathbb{R}^d)$ such that $(I-\Delta)^{-s/2} g = f$, that is to say

$$W^{2,p} = (I-\Delta)^{-s/2}(L^2(\mathbb{R}^d)).$$

Bessel potentials are bounded operators on $L^p(\mathbb{R}^d)$, $p \geq 1$ (see [18], Chapter V).

For $p \geq 1$ and $s > 0$, the *potential space* of order s and integrability p is defined as

$$L^p_s = (I-\Delta)^{-s/2}(L^p(\mathbb{R}^d))$$

with the norm

$$\|f\|_{L^p_s} = \|g\|_p,$$

where g is such that $(I-\Delta)^{-s/2} g = f$.

In [18] it was proven that for a positive integer k and $1 < p < \infty$, the space $W^{k,p} = L^p_k$.

A more general setting might be constructed considering a second order differential operator $\mathbf{L}$ self-adjoint with respect to a measure μ.

Sometimes, it is possible to obtain a factorization of $\mathbf{L}$ as

$$\mathbf{L} = \sum_i \partial_i \tilde{\partial}_i,$$

where ∂_i and $\tilde{\partial}_i$ are first order differential operators, and then to define the *Sobolev space* of order k and integrability p associated to $\mathbf{L}$ as

$$W^{k,p} = \{ f \in L^p(\mu) : \; (\partial_i)^j f \in L^p(\mu) \text{ and } (\tilde{\partial}_i)^j f \in L^p(\mu), \; j \leq k \}.$$

When $s > 0$ the definition of the *potential space* of order s and integrability p could be given by

$$L^p_s = (I+\mathbf{L})^{-s/2}(L^p(\mu)).$$

In the case that $\mathbf{L}^{-s/2}$ is bounded on $L^p(\mu)$, we have

$$L^p_s = \mathbf{L}^{-s/2}(L^p(\mu)). \tag{1}$$

It is reasonable to expect that for a positive integer k it follows $W^{k,p} = L^p_k$, and this tell us that ∂_i and $\tilde{\partial}_i$ have the right to be called *derivatives* associated to $\mathbf{L}$.

2. Hermite setting

The Hermite operator is defined as

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^d. \tag{2}$$

Its eigenvectors are the Hermite functions, which form an orthonormal basis for $L^2(\mathbb{R}^d)$. In $\mathbb{R}$ they are defined for $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ as

$$h_n(t) = \frac{H_n(t)\, e^{-t^2/2}}{(2^n n! \pi^{1/2})^{1/2}}, \;\; t \in \mathbb{R},$$

where H_n is the Hermite polynomial of order n (see [21]).

In $\mathbb{R}^d$ given a multi-index $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the Hermite function of order α is defined using the unidimensional ones by

$$h_\alpha(x) = \Pi_{j=1}^d h_{\alpha_j}(x_j), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

As we said, the Hermite functions are eigenvectors of H (see [22]) satisfying

$$H h_\alpha = (2|\alpha| + d)\, h_\alpha,$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$.

Some other interesting properties of Hermite functions are the following. Their proofs can be found in [19] and [22].

Proposition 1. *If $M \in \mathbb{N}$ and $f \in C_c^\infty$, then there exists a constant $C_{M,f} > 0$ such that*

$$\left| \int_{\mathbb{R}^d} f\, h_\alpha \right| \le C_{M,f}\, (|\alpha| + 1)^{-M}, \quad \alpha \in \mathbb{N}^d.$$

Proposition 2. *If $1 \le p < \infty$ and $w \in A_p$, then there exist constants $\epsilon_p > 0$ and C_w such that*

$$\|h_\alpha\|_{L^p(w)} \le C_w\, (|\alpha| + 1)^{\epsilon_p}.$$

Proposition 3. *As $n \to \infty$ the Hermite functions satisfy the estimates:*

i) $\|h_n\|_p \sim n^{\frac{1}{2p} - \frac{1}{4}}, \quad 1 \le p < 4,$

ii) $\|h_n\|_p \sim n^{-\frac{1}{8}} \log(n), \quad p = 4,$

iii) $\|h_n\|_p \sim n^{\frac{1}{6p} - \frac{1}{12}}, \quad 4 < p \le \infty.$

The Hermite operator can be factorized as

$$H = \frac{1}{2} \sum_{j=1}^d A_j A_{-j} + A_{-j} A_j, \tag{3}$$

where

$$A_j = \frac{\partial}{\partial x_j} + x_j \quad \text{and} \quad A_{-j} = A_j^* = -\frac{\partial}{\partial x_j} + x_j, \quad 1 \le j \le d.$$

The operators A_j and A_{-j}, are called *annihilation* and *creation* operators respectively because of their behavior over Hermite functions, in fact, for $1 \le j \le d$,

$$A_j h_\alpha = \sqrt{2\alpha_j}\, h_{\alpha - e_j}, \qquad A_{-j} h_\alpha = \sqrt{2(\alpha_j + 1)}\, h_{\alpha + e_j}, \tag{4}$$

where e_j is the jth-coordinate vector in $\mathbb{N}_0^d$. In the context of the Hermite operator, the notion of *derivatives* is given by these operators.

Given $p \in (1, \infty)$ and $k \in \mathbb{N}$, the Hermite-Sobolev space of order k, denoted by $W^{k,p}$, is the set of functions $f \in L^p(\mathbb{R}^d)$ such that

$$A_{j_1} \cdots A_{j_m} f \in L^p(\mathbb{R}^d),\ 1 \le |j_1|, \ldots, |j_m| \le d,\ 1 \le m \le k,$$

with the norm

$$\|f\|_{W^{k,p}} = \sum_{1 \le |j_1|, \ldots, |j_m| \le d,\ 1 \le m \le k} \|A_{j_1} \cdots A_{j_m} f\|_p + \|f\|_p.$$

This definition was rewritten in [6] considering only annihilation operators proving they are enough to define the Hermite-Sobolev space. In the same work, the authors deal with weighted Sobolev spaces for weights in the Muckenhoupt class A_p, defined for $1 < p < \infty$, as the set of weights (non-negative and locally integrable functions) w such that

$$\left(\int_B w \right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}} \right)^{1/p'} \ \le\ C|B|, \tag{5}$$

for every ball $B \subset \mathbb{R}^d$; and for $p = 1$, A_1 is defined as those weights w satisfying the condition

$$w(B) \ \sup_B w^{-1} \ \le\ C|B|, \tag{6}$$

for every ball $B \subset \mathbb{R}^d$.

Given a weight w, $k \in \mathbb{N}$ and $p \ge 1$, the Hermite-Sobolev space of order k, denoted by $W^{k,p}(w)$, is defined as the set of functions $f \in L^p(w)$ such that

$$\overbrace{A_j \cdots A_j}^{m \text{ times}} f = A_j^m f \ \in\ L^p(w),\ 1 \le m \le k, 1 \le j \le d,$$

with the norm

$$\|f\|_{W^{k,p}(w)} = \sum_{j=1}^{d} \sum_{1 \le m \le k} \|A_j^m f\|_{L^p(w)} + \|f\|_{L^p(w)}.$$

On the other hand, in order to define a potential space like (1), for $a > 0$ we define the operator

$$H^{-a} f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tH} f(x)\, t^a \, \frac{dt}{t}, \quad x \in \mathbb{R}^d, \ \ f \in \mathfrak{F}, \tag{7}$$

where $\{e^{-tH}\}_{t\geq 0}$ is the heat semi-group associated to H, and $\mathfrak{F}$ denotes the set of linear combinations of Hermite functions.

Remark 1. If $a > 0$ and $\alpha \in \mathbb{N}_0^d$, by using the Γ function and the fact

$$e^{-tH} h_\alpha = e^{-t(2|\alpha|+d)} h_\alpha,$$

we have

$$H^{-a} h_\alpha(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tH} h_\alpha(x)\, t^a \,\frac{dt}{t} = (2|\alpha| + d)^{-a} h_\alpha(x), \quad x \in \mathbb{R}^d.$$

The operator H^{-a} has a kernel K_a with exponential behavior far from the diagonal as the following proposition shows. The proof can be found in [5].

Proposition 4. *The operator H^{-s}, $s > 0$, has an integral representation*

$$H^{-s} f(x) = \int_{\mathbb{R}^d} K_s(x, y) f(y) dy,\ x \in \mathbb{R}^d,$$

where $K_s(x, y)$ is positive and symmetric. Moreover,

$$K_a(x, y) \leq C\, \phi_a(|x - y|), \quad x, y \in \mathbb{R}^d, \tag{8}$$

where $\phi_a(r)$, for $r \geq 0$, is defined by

$$\phi_a(r) = \begin{cases} \frac{\chi_{\{r<1\}}(r)}{r^{d-2a}} + e^{\frac{-r^2}{4}}\, \chi_{\{r\geq 1\}}(r), & \text{if } a < \frac{d}{2}, \\ \log\left(\frac{e}{r}\right)\, \chi_{\{r<1\}}(r) + e^{\frac{-r^2}{4}}\, \chi_{\{r\geq 1\}}(r), & \text{if } a = \frac{d}{2}, \\ \chi_{\{r<1\}}(r) + e^{\frac{-r^2}{4}}\, \chi_{\{r\geq 1\}}(r), & \text{if } a > \frac{d}{2}. \end{cases}$$

In [6] it was proven that H^{-a} a bounded operator on $L^p(w)$ for a weight $w \in A_p$ as the following theorem states.

Theorem 1. *Let $1 \leq p < \infty$ and $a > 0$. If $w \in A_p$, then the operator H^{-a} is bounded on $L^p(w)$.*

Remark 2. The results in [4] (Theorem 4 therein) suggest that there should be more weights for the boundedness of H^{-a} in $L^p(w)$. Those classes of weights allow power weights $w(x) = |x|^\gamma$ without restriction on γ.

For the unweighted case L^p-L^q inequalities where obtained in [5].

Theorem 2. *Let a, d such that $0 < a < d$, then:*

i) There exists a constant C such that

$$\|H^{-a/2} f\|_q \leq C\|f\|_1,$$

for all $f \in L^1(\mathbb{R}^d)$ if and only if $1 \leq q < \frac{d}{d-a}$.

ii) There exists a constant C such that

$$\|H^{-a/2} f\|_\infty \leq C\|f\|_p,$$

for all f in $L^p(\mathbb{R}^d)$ if and only if $p > \frac{d}{a}$.

iii) *There exists a constant C such that*

$$\|H^{-a/2}f\|_q \leq C\|f\|_\infty,$$

for all $f \in L^\infty(\mathbb{R}^d)$ if and only if $q > \frac{d}{a}$.

iv) *There exists a constant C such that*

$$\|H^{-a/2}f\|_1 \leq C\|f\|_p,$$

for all $f \in L^p(\mathbb{R}^d)$ if and only if $1 \leq p < \frac{d}{d-a}$.

v) *If $1 < p < \infty$, $1 < q < \infty$ and $\frac{1}{p} - \frac{a}{d} \leq \frac{1}{q} < \frac{1}{p} + \frac{a}{d}$, then there exists a constant C such that*

$$\|H^{-a/2}f\|_q \leq C\|f\|_p,$$

for all $f \in L^p(\mathbb{R}^d)$.

In the Hermite setting the potential space of order a and integrability p is defined as $\mathcal{L}_a^p(w) = H^{-a/2}(L^p(w))$, with respect to an absolute continuous measure $w(x)dx$, being w a weight in a class where $H^{-a/2}$ results bounded. From Theorem 1 it is enough to ask $w \in A_p$.

A norm on $\mathcal{L}_a^p(w)$ is given by

$$\|f\|_{\mathfrak{L}_a^p(w)} = \|g\|_{L^p(w)},$$

where g is such that $H^{-a/2}g = f$.

Remark 3. It is easy to see that $H^{-a/2}$ is one to one (see [5]), and this assures that the space $\mathfrak{L}_a^p$ is well defined for $p \in [1, \infty)$ and $a > 0$. As $\mathfrak{F} = H^{-a/2}(\mathfrak{F})$ then $\mathfrak{F}$ is a dense space of $\mathfrak{L}_a^p$.

A fundamental devise of the theory is the family of Hermite-Riesz transforms of order $m \in \mathbb{N}$, associated to H, defined by

$$R_J^m = A_{j_1} \dots A_{j_m} H^{-m/2}, \text{ where } J = (j_1, \dots, j_m), 1 \leq |j_i| \leq d,\ 1 \leq i \leq m.$$

In the case $j_1 = \dots = j_m = j$, these operators will be denoted by R_j^m. The case $m = 1$ was introduced by S. Thangavelu (see [22]). He proved that they are bounded operators in $L^p(\mathbb{R}^d)$. Also in [19] and [20], it was shown that the operators R_J^m are Calderón-Zygmund operators and as a consequence they are bounded in $L^p(w)$ for $w \in A_p$, $1 < p < \infty$.

We shall now present some expected properties of the spaces $\mathfrak{L}_a^p(w)$ appearing in [6].

Theorem 3. *Let $w \in A_p$, $1 < p < \infty$, and $a > 0$.*

i) *If $t > a$, then $\mathfrak{L}_t^p(w) \subset \mathfrak{L}_a^p(w) \subset L^p(w)$ with continuous inclusions. Moreover, $\mathfrak{L}_a^p(w)$ and $\mathfrak{L}_t^p(w)$ are isometrically isomorphic.*

ii) *If $t > 0$, then $H^{-t/2}$ maps $\mathfrak{L}_a^p(w)$ into $\mathfrak{L}_{a+t}^p(w)$.*

iii) *If $a > 1$ and $1 \leq |j| \leq d$, then A_j is bounded from $\mathfrak{L}_a^p(w)$ into $\mathfrak{L}_{a-1}^p(w)$.*

iv) *The operators R_J^m are bounded on $\mathfrak{L}_a^p(w)$.*

For the unweighted case some comparison with the classical Sobolev spaces is presented in the following result.

Theorem 4. *Let $a > 0$ and $p \in (1, \infty)$. Then*

i) $\mathfrak{L}^p_a \subset L^p_a$.
ii) $\mathfrak{L}^p_a \neq L^p_a$.
iii) If $f \in L^p_a$ *and has compact support, then* f *belongs to* $\mathfrak{L}^p_a$.

Remark 4. The results in [10] about Sobolev spaces associated to Schrödinger operators with a polynomial potential, implies in particular that f belongs to $\mathfrak{L}^p_a$ if and only if $f \in L^p_a$ and $|x|^{2a} f \in L^p$.

The following structural theorem was proved in [6]. A fundamental part of the proof is due to the boundedness of higher order Riesz transforms given in Theorem 3.

Theorem 5. *Let $k \in \mathbb{N}$, $1 < p < \infty$, and $w \in A_p$. Then,*

$$W^{k,p}(w) = \mathfrak{L}^p_k(w)$$

and the norms $\|\cdot\|_{W^{k,p}(w)}$ *and* $\|\cdot\|_{\mathfrak{L}^p_k(w)}$ *are equivalent.*

3. Laguerre setting

For $\alpha > -1$ and $n \in \mathbb{N}_0$, the Laguerre polynomial of order n and type α, is defined by

$$L^\alpha_n = \frac{x^\alpha e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}),$$

and the Laguerre function of order n and type α is

$$\mathcal{L}^\alpha_n(y) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right)^{1/2} e^{-y/2} y^{\alpha/2} L^\alpha_n(y), \quad y \in \mathbb{R}^+, \; n \in \mathbb{N}_0. \tag{9}$$

For each $\alpha > -1$, $\{\mathcal{L}^\alpha_n\}_{n=0}^\infty$ is an orthonormal system in $L^2((0,\infty))$ and satisfies

$$L_\alpha \mathcal{L}^\alpha_n = \left(n + \frac{\alpha+1}{2} \right) \mathcal{L}^\alpha_n, \; n \in \mathbb{N}_0,$$

where L_α is the Laguerre operator, self-adjoint in the set $\mathcal{C}_c(0,\infty)$, defined by

$$L_\alpha = -y \frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y \in (0,\infty). \tag{10}$$

The operator L_α can be written as

$$L_\alpha = (\delta^\alpha)^* \delta^\alpha + \frac{(\alpha+1)}{2},$$

where

$$\delta^\alpha = \sqrt{x} \frac{d}{dx} + \frac{1}{2} \left(\sqrt{x} - \frac{\alpha}{\sqrt{x}} \right) \text{ and } (\delta^\alpha)^* = -\sqrt{x} \frac{d}{dx} + \frac{1}{2} \left(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}} \right). \tag{11}$$

As was shown in [6] the operator δ^α plays the role of *derivative* in the Laguerre setting. The action of these operators on Laguerre functions is given by

$$\delta^\alpha(\mathcal{L}^\alpha_0) = 0, \qquad \delta^\alpha(\mathcal{L}^\alpha_n) = -\sqrt{n}\, \mathcal{L}^{\alpha+1}_{n-1}, \qquad \text{for } n \geq 1, \text{ and} \tag{12}$$

$$(\delta^\alpha)^*(\mathcal{L}^{\alpha+1}_n) = -\sqrt{n+1}\, \mathcal{L}^\alpha_{n+1} \qquad \text{for } n \geq 0. \tag{13}$$

In this section we will deal with power weights y^γ in order to proof boundedness on $L^p(\mathbb{R}^+, y^\gamma dy)$ of some operators associated to L_α, (see [1]). The range for the exponent γ will be

$$-\frac{\alpha}{2}p - 1 < \gamma < p - 1 + \frac{\alpha}{2}p, \tag{14}$$

where $\alpha > -1$, $1 < p < \infty$.

Under this hypothesis it is known (see [21] Theorem 5.7.1) that the set S_α of finite linear combinations of Laguerre functions is dense in $L^p((0,\infty), y^\gamma dy)$.

At first sight, the natural way of defining a Sobolev space should be, iterating the *derivative* δ^α, as the set of functions f in $L^p(\mathbb{R}^+, y^\gamma dy)$ such that $(\delta^\alpha)^m f \in L^p(\mathbb{R}^+, y^\gamma dy),\ 0 \le m \le k$. We shall denote this space by $\mathcal{W}_\alpha^{k,p}(y^\gamma)$, and the norm is

$$\|f\|_{\mathcal{W}_\alpha^{k,p}(y^\delta)} = \sum_{m=0}^{k} \|(\delta^\alpha)^m f\|_{L^p(\mathbb{R}^+, y^\delta dy)}.$$

We will see later that this space is not appropriate as a Sobolev space for L_α when $-1 < \alpha < 0$.

It was found in [6] that the right space that plays the role of a Sobolev space defined via derivatives in the Laguerre setting is called Laguerre-Sobolev spaces, denoted by $W_\alpha^{k,p}(y^\gamma)$, defined by the sets of functions f in $L^p(\mathbb{R}^+, y^\gamma dy)$ such that

$$\delta^{\alpha+m} \circ \ldots \circ \delta^{\alpha+1} \circ \delta^\alpha f \in L^p(\mathbb{R}^+, y^\gamma dy),\ 0 \le m \le k-1.$$

The norm on $W_\alpha^{k,p}(y^\gamma)$ is given by

$$\|f\|_{W_\alpha^{k,p}(y^\gamma)} = \|f\|_{p,\gamma} + \sum_{m=0}^{k-1} \left\|\delta^{\alpha+m} \circ \ldots \circ \delta^{\alpha+1} \circ \delta^\alpha f\right\|_{p,\gamma}.$$

These spaces are the right spaces for the theory as the following theorem shows (see [6]).

Theorem 6. *Let $\alpha > -1$, $1 < p < \infty$, $k \in \mathbb{N}$ and γ satisfying* (14). *Then,*

$$W_\alpha^{k,p}(y^\gamma) = \mathfrak{W}_\alpha^{k,p}(y^\gamma),$$

and the norms are equivalent.

The Riesz transforms were defined in [12], for $\alpha > -1$, by

$$R_\alpha = \delta^\alpha (L_\alpha)^{-1/2} \qquad \text{and} \qquad \tilde{R}_\alpha = (\delta^\alpha)^* (L_{\alpha+1})^{-1/2}.$$

In [13] it was proved that those operators are bounded on $L^p(\mathbb{R}^+, y^\gamma dy)$ for γ satisfying (14). Given a positive integer k and $\alpha > -1$ we define the higher order Riesz transform of order k as

$$R_\alpha^k = \left(\delta^{\alpha+k-1} \circ \cdots \circ \delta^{\alpha+1} \circ \delta^\alpha\right)(L_\alpha)^{-k/2}$$

and

$$\tilde{R}_\alpha^k = \left((\delta^\alpha)^* \circ (\delta^{\alpha+1})^* \circ \cdots \circ (\delta^{\alpha+k-1})^*\right)(L_{\alpha+k})^{-k/2}.$$

Observe that $R_\alpha^1 = R_\alpha$ and $\tilde{R}_\alpha^1 = \tilde{R}_\alpha$.

It was proved in [6] the following boundedness result for the operators R_α^k and $\tilde{R}_\alpha^k$.

Theorem 7. *Let $k \in \mathbb{N}$, $1 < p < \infty$, $\alpha > -1$ and γ satisfying (14). The operators R^k_α and $\tilde{R}^k_\alpha$ are bounded on $L^p(\mathbb{R}^+, y^\gamma dy)$.*

As in the Hermite setting we have the following structural theorem for the spaces $\mathfrak{W}^p_{\alpha,a}(y^\gamma)$.

Theorem 8. *Let $\alpha > -1$, $1 < p < \infty$, $a > 0$, and γ satisfying (14).*

i) If $t > a$, then $\mathfrak{W}^p_{\alpha,a}(y^\gamma) \subset \mathfrak{W}^p_{\alpha,t}(y^\gamma) \subset L^p(y^\gamma)$ with continuous inclusions. Moreover, $\mathfrak{W}^p_{\alpha,a}(y^\gamma)$ and $\mathfrak{W}^p_{\alpha,t}(y^\gamma)$ are isometrically isomorphic .
ii) If $t > 0$, then $(L_\alpha)^{-t/2}$ maps $\mathfrak{W}^p_{\alpha,a}(y^\gamma)$ into $\mathfrak{W}^p_{\alpha,a+t}(y^\gamma)$.
iii) If $a > 1$, then δ^α is bounded from $\mathfrak{W}^p_{\alpha,a}(y^\gamma)$ into $\mathfrak{W}^p_{\alpha+1,a-1}(y^\gamma)$.
iv) The operators R^k_α, are bounded from $\mathfrak{W}^p_{\alpha,s}(y^\gamma)$ into $\mathfrak{W}^p_{\alpha+k,s}(y^\gamma)$.

Proposition 5. *Let $\alpha > -1$, $1 < p < \infty$, $k \in \mathbb{N}$ and γ satisfying (14). Then S_α is a dense subspace of $W^{k,p}_\alpha(y^\gamma)$.*

A *Riesz transform* of higher order k associated to L_α could be defined iterating the *derivative* δ^α as $(\delta^\alpha)^k (L_\alpha)^{-k/2}$. This operator has the same boundedness properties of R^k_α.

Theorem 9. *Let $\alpha > -1$ and k a positive integer. Then the* Riesz transforms *$(\delta^\alpha)^k (L_\alpha)^{-k/2}$ are bounded in $L^p(y^\gamma dy)$ for γ and p satisfying (14).*

When $-1 < \alpha \le 0$, despite the boundedness of $(\delta^\alpha)^k (L_\alpha)^{-k/2}$ in $L^p(y^\gamma dy)$ for γ satisfying (14), the spaces $\mathcal{W}^{k,p}_\alpha(y^\delta)$ are different from the potential spaces $\mathfrak{W}^p_{\alpha,k}(y^\delta)$ as the following theorem shows.

Theorem 10. *Let p be in the range $1 < p < \infty$.*

i) If $\alpha > -1$, and γ satisfies (14), then $\mathfrak{W}^{k,p}_\alpha(y^\gamma) \subset \mathcal{W}^{k,p}_\alpha(y^\gamma)$.
ii) If $-1 < \alpha \le 0$, then $\mathfrak{W}^{2,2}_\alpha \neq \mathcal{W}^{2,2}_\alpha$.
iii) If $\alpha > 0$, and γ satisfies

$$-\frac{\alpha-1}{2}p - 1 < \gamma < p - 1 + \frac{\alpha-1}{2}p. \tag{15}$$

then $\mathfrak{W}^{2,p}_\alpha(y^\gamma) = \mathcal{W}^{2,p}_\alpha(y^\gamma)$,

4. Other Laguerre function systems

It is possible to translate the concepts and results of the previous section to other Laguerre systems. For instance, we shall consider the orthonormal system in $L^2((0,\infty), dy)$ given by $\varphi^\alpha_k(y) = \mathfrak{L}^\alpha_k(y^2)(2y)^{1/2}$, where $\mathfrak{L}^\alpha_k$ are the functions defined in (9). The functions φ^α_k are eigenfunctions of the operator

$$\mathbf{L}_\alpha = \frac{1}{4}\left\{-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\left(\alpha^2 - \frac{1}{4}\right)\right\},$$

since

$$\mathbf{L}_\alpha(\varphi^\alpha_k) = \left(k + \frac{\alpha+1}{2}\right)\ \varphi^\alpha_k.$$

The operator $\mathbf{L}_\alpha$ can be written as

$$\mathbf{L}_\alpha = (\mathbf{D}_\alpha)^*\mathbf{D}_\alpha - \Big(\frac{\alpha+1}{2}\Big),$$

with $\mathbf{D}^\alpha = \frac{1}{2}\left\{\frac{d}{dy} + y - \frac{1}{y}(\alpha+\frac{1}{2})\right\}$ and $(\mathbf{D}^\alpha)^* = \frac{1}{2}\left\{-\frac{d}{dy} + y - \frac{1}{y}(\alpha+\frac{1}{2})\right\}$. The operator $(\mathbf{D}^\alpha)^*$ is the formal adjoint of $\mathbf{D}^\alpha$ with respect to the Lebesgue measure. The behavior of those operators over φ_k^α is

$$\mathbf{D}^\alpha(\varphi_k^\alpha) = -\sqrt{k}\varphi_{k-1}^{\alpha+1} \quad \text{and} \quad (\mathbf{D}^{\beta-\mathbf{1}})^*(\varphi_k^\beta) = -\sqrt{k+1}\varphi_{k+1}^{\beta-1}.$$

As in Section 3 the Riesz transforms can be defined as

$$\mathbf{R}_\alpha^k = \mathbf{D}^{\alpha+\mathbf{k}-\mathbf{1}} \circ \cdots \circ \mathbf{D}^\alpha(\mathbf{L}_\alpha)^{-k/2}, \quad \text{alternatively} \quad (\mathbf{D}^\alpha)^k(\mathbf{L}_\alpha)^{-k/2},\ \alpha > -1.$$

If V is the operator defined by $Vf(y) = (2y)^{1/2}f(y^2)$ and $2\gamma = \eta + \frac{p}{2} - 1$, then $\|Vf\|_{L^p(y^\eta\,dy)} = 2^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L^p(y^\gamma\,dy)}$ and the following proposition holds.

Proposition 6. *Let $1 < p < \infty$, $2\gamma = \eta + \frac{p}{2} - 1$ and T an operator defined over the set of finite linear combination of Laguerre functions $\{\mathfrak{L}_k^\alpha\}_k$. The operator T has a bounded extension from $L^p((0,\infty), y^\gamma dy)$ into $L^p((0,\infty), y^\gamma dy)$ if and only if the operator $\mathbf{T} = V\,T\,V^{-1}$ has a bounded extension from $L^p((0,\infty), y^\eta dy)$ into $L^p((0,\infty), y^\eta dy)$.*

An easy consequence of the above proposition, and Theorems 7 and 9 is the following.

Theorem 11. *Let $\alpha > -1$ and let f be a finite linear combination of Laguerre functions $\{\mathfrak{L}_k^\alpha\}_k$.*

i) $e^{-tL_\alpha}f = V^{-1}e^{-t\mathbf{L}_\alpha}Vf$,
ii) $(L_\alpha)^{-s}f = V^{-1}(\mathbf{L}_\alpha)^{-s}Vf$, *for all* $s > 0$,
iii) $\delta^\alpha f = V^{-1}\mathbf{D}^\alpha Vf$,
iv) $R_\alpha^k f = V^{-1}\mathbf{R}_\alpha^k Vf$.

Proposition 7. *Let $\alpha > -1$, $1 < p < \infty$, and η be real number. Let S be any one of the operators $\mathbf{L}^{-s}$, $s > 0$, $\mathbf{R}_\alpha^k$, $(\mathbf{D}^\alpha)^k\mathbf{L}^{-k/2}$, $s > 0$. Then the operator $\mathbf{S}$ has a bounded extension from $L^p((0,\infty), y^\eta dy)$ into itself, for η satisfying*

$$-1 - \alpha p - \frac{p}{2} < \eta < \alpha p + \frac{3p}{2} - 1. \tag{16}$$

Now in the same way as in Section 3, we can define potential spaces and Sobolev spaces for the class of Laguerre functions $\{\varphi_k^\alpha\}_k$, $\alpha > -1$. Thus, given $\alpha > -1$, $1 < p < \infty$, $s > 0$ and η satisfying (16), we define

$$\mathfrak{U}_{\alpha,s}^p(y^\eta) = (\mathbf{L}_\alpha)^{-s/2}[L^p(\mathbb{R}^+, y^\eta dy)]$$

with the norm $\|f\|_{\mathfrak{U}_{\alpha,a}^p(y^\eta)} = \|g\|_{p,\eta}$, where $(\mathbf{L}_\alpha)^{-a/2}g = f$.

We shall denote by $U_\alpha^{k,p}(y^\gamma)$, the set of functions f in $L^p(\mathbb{R}^+, y^\eta dy)$ such that

$$\mathbf{D}^{\alpha+m} \circ \ldots \circ \mathbf{D}^{\alpha+1} \circ \mathbf{D}^\alpha f \in L^p(\mathbb{R}^+, y^\eta dy),\ 0 \le m \le k-1,$$

with the norm

$$\|f\|_{U_\alpha^{k,p}(y^\eta)} = \|f\|_{p,\eta} + \sum_{m=0}^{k-1} \left\|\mathbf{D}^{\alpha+m} \circ \ldots \circ \mathbf{D}^{\alpha+1} \circ \mathbf{D}^{\alpha} f\right\|_{p,\eta}.$$

Finally, $\mathcal{U}_\alpha^{k,p}(y^\gamma)$, will denote the set of functions f in $L^p(\mathbb{R}^+, y^\eta dy)$ such that $(\mathbf{D}^\alpha)^m f \in L^p(\mathbb{R}^+, y^\eta dy)$, $0 \le m \le k$, with the norm

$$\|f\|_{\mathcal{U}_\alpha^{k,p}(y^\eta)} = \sum_{m=0}^{k} \|(\mathbf{D}^\alpha)^m f\|_{p,\eta}.$$

The following theorems are direct consequences of Theorems 6, 10 and Propositions 7 and 11.

Theorem 12. *Let $\alpha > -1$, $1 < p < \infty$, $k \in \mathbb{N}$ and η satisfies (16).*

i) $U_{\alpha,\eta}^{k,p} = \mathfrak{U}_{\alpha,\eta}^{k,p}$, and the norms are equivalent.

ii) Let η satisfying $-\frac{\alpha}{2}p - 1 < \eta < p - 1 + \frac{\alpha}{2}p$. Then $\mathfrak{U}_\alpha^{k,p}(y^\eta) \subset \mathcal{U}_\alpha^{k,p}(y^\eta)$.

iii) Let $-1 < \alpha \le 0$. Then $\mathfrak{U}_\alpha^{2,2} \ne \mathcal{U}_\alpha^{2,2}$.

iv) If η satisfies

$$-1 - (\alpha - 1)p - \frac{p}{2} < \eta < (\alpha - 1)p + \frac{3p}{2} - 1, \tag{17}$$

then $\mathfrak{U}_\alpha^{2,p}(y^\eta) = \mathcal{U}_\alpha^{2,p}(y^\eta)$.

Analogous results could be obtained for the systems of Laguerre functions $\ell_k^\alpha(y) = \mathfrak{L}_k^\alpha(y) y^{-\alpha/2}$ and $\psi_k^\alpha(y) = \sqrt{2} y^{-\alpha} \mathcal{L}_k^\alpha(y^2)$, $\alpha > -1$. These systems are eigenfunctions of the differential operators

$$\mathbb{L}_\alpha = -y \frac{d^2}{dy^2} - (\alpha + 1) \frac{d}{dy} + \frac{y}{4}.$$

and

$$\mathfrak{L}_\alpha = -\frac{1}{4}\left\{ \frac{d^2}{dy^2} + \left(\frac{2\alpha + 1}{y}\right) \frac{d}{dy} - y^2 \right\}.$$

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References

[1] I. Abu-Falahah, R. A. Macías, C. Segovia, and J. L. Torrea. Transferring strong boundedness among Laguerre orthogonal systems. *Proc. Indian Acad. Sci. Math. Sci.*, 119(2):203–220, 2009. 30

[2] Jorge J. Betancor. Sobolev spaces associated to Bessel operators and Hankel translations commuting operators. *Acta Sci. Math. (Szeged)*, 66(1-2):227–243, 2000. 23

[3] Jorge J. Betancor, Juan C. Fariña, Lourdes Rodríguez-Mesa, Ricardo Testoni, and José L. Torrea. A choice of Sobolev spaces associated with ultraspherical expansions. *Publ. Mat.*, 54(1):221–242, 2010. 23

[4] B. Bongioanni, E. Harboure, and O. Salinas. Classes of weights related to Schrödinger operators. *J. Math. Anal. Appl.*, 373(2):563–579, 2011. 27

[5] B. Bongioanni and J. L. Torrea. Sobolev spaces associated to the harmonic oscillator. *Proc. Indian Acad. Sci. Math. Sci.*, 116(3):337–360, 2006. 23, 27, 28

[6] B. Bongioanni and J. L. Torrea. What is a Sobolev space for the Laguerre function systems? *Studia Math.*, 192(2):147–172, 2009. 23, 26, 27, 28, 29, 30

[7] L. Carleson. *Some analytic problems related to statistical mechanics*. Number 779 in Lecture Notes in Math. Springer, 1980.

[8] M. G. Cowling. *Pointwise behaviour of solutions to Schrödinger equations*. Number 992 in Lecture Notes in Math. Springer, 1983.

[9] B. Dahlberg and C. Kenig. *A note on the almost everywhere behavior of solutions to the Schrödinger equation*. Number 908 in Lecture Notes in Math. Springer, 1982.

[10] Jacek Dziubański and Paweł Głowacki. Sobolev spaces related to Schrödinger operators with polynomial potentials. *Math. Z.*, 262(4):881–894, 2009. 29

[11] Piotr Graczyk, Jean-J. Loeb, Iris A. López P., Adam Nowak, and Wilfredo O. Urbina R. Higher order Riesz transforms, fractional derivatives, and Sobolev spaces for Laguerre expansions. *J. Math. Pures Appl. (9)*, 84(3):375–405, 2005. 23

[12] E. Harboure, J. L. Torrea, and B. Viviani. Riesz transforms for Laguerre expansions. *Indiana Univ. Math. J.*, 55(3):999–1014, 2006. 30

[13] Eleonor Harboure, Carlos Segovia, José L. Torrea, and Beatriz Viviani. Power weighted L^p-inequalities for Laguerre-Riesz transforms. *Ark. Mat.*, 46(2):285–313, 2008. 30

[14] Sanghyuk Lee. On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$. *Int. Math. Res. Not.*, 21pp, Art. ID 32597, 21, 2006.

[15] I.A. López and W.O. Urbina. On some functions of the Littlewood-Paley theory for γ_d and a characterization of Gaussian Sobolev spaces of integerer order. *Revista de la Unión Matemática Argentina*, 45(2):41–53, 2004. 23

[16] Keith M. Rogers and Sanghyuk Lee. The free and Hermite Schrödinger equations: an equivalence. To appear.

[17] Per Sjölin. Regularity of solutions to the Schrödinger equation. *Duke Math. J.*, 55(3):699–715, 1987.

[18] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1970. 24

[19] K. Stempak and J. L. Torrea. Poisson integrals and Riesz transforms for Hermite function expansions with weights. *Journal of Functional Analysis*, 202:443–472, 2003. 25, 28

[20] K. Stempak and J. L. Torrea. Higher Riesz transforms and imaginary powers associated to the harmonic oscillator. *Acta Math. Hungar.*, 111(1-2):43–64, 2006. 28

[21] G. Szegö. *Orthogonal Polinomials*, volume XXIII of *American Mathematical Society Colloquium Publications*. American Mathematical Society, 1939. 25, 30

[22] S. Thangavelu. *Lectures on Hermite and Laguerre expansions*, volume 42 of *Mathematical Notes*. Princeton University Press, NJ, 1993. 25, 28

[23] S. Thangavelu. On regularity of twisted spherical means and special Hermite expansions. *Proc. Indian Acad. Sci.*, 103:303–320, 1993. 23

[24] Luis Vega. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc.*, 102(4):874–878, 1988.

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