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IMPORTANCE OF ZAK TRANSFORMS FOR HARMONIC ANALYSIS

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Abstract. In engineering and applied mathematics, Zak transforms have been effectively used for over 50 years in various applied settings. As Gelfand observed in a 1950 paper, the variable coefficient Fourier series ideas articulated in Andre Weil's famous book on integration lead to an exceedingly elementary proof of the Plancherel Theorem for LCA groups. The transform for functions on $\mathbb{R}$ appearing in Zak's seminal 1967 paper is actually a special case of the LCA group transforms earlier introduced by Weil; Zak states this explicitly in his 1967 paper but the mathematical community nonetheless chose to name the transform for him.

In brief, the properties of Zak transforms are simply reflections of elementary Fourier series properties and the Plancherel Theorem for non-compact LCA groups is an immediate consequence of the fact that Fourier transforms are averages of Zak transforms. It is remarkable that only a small handful of mathematicians know this proof and that all textbooks continue to give much harder and less transparent proofs for even the case of the group $\mathbb{R}$. Generalized Zak transforms arise naturally as intertwining operators for various representations of Abelian groups and allow formulation of many appealing theorems.

Remark: The results discussed below represent joint work by E. Hernandez, H. Sikic, G. Weiss, and the author.

1. The Abelian Group Plancherel Theorem

1.1. **Overview.** In textbooks on real analysis, one can find a variety of proofs of the Plancherel Theorem for $\mathbb{R}^n$, $n \in \mathbb{N}$. All are lengthy, non-elementary, and technical, e.g.:

⋆ use of complex analysis to compute Fourier transforms of Gaussian functions followed by use of approximate identities defined by Gaussians to extend the Fourier transform from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to a unitary operator on $L^2(\mathbb{R}^n)$;

⋆ alternative use of the Hermite function orthonormal basis for $L^2(\mathbb{R}^n)$ and the computation that Hermite functions are eigenfunctions of the Fourier transform with eigenvalues which are fourth roots of 1;

⋆ reversion to the 19th century interpretation of Fourier transforms as Riemann sum limits of Fourier series and justification of this approach by somewhat delicate dominated convergence arguments.

In fact, as we will show, none of this is necessary. The proof of the Plancherel Theorem for $\mathbb{R}^n$ via Zak transforms uses only basic Fourier series ideas and applies with only syperficial changes in notation to every locally compact, Abelian group (LCA group).

1.2. **Notations and Definitions.** Let $G = (G, +)$ be an additive LCA group.

(i) The dual group $\hat{G} = (\hat{G}, +)$ of G is the essentially unique LCA group for which there is a continuous, bi-additive homomorphism $(\xi, x) \mapsto \xi \cdot x$ from $\hat{G} \times G$ into $\mathbb{R}/\mathbb{Z}$ such that, with $e_\xi(x) = e_x(\xi) \equiv e^{2\pi i \xi \cdot x}$, every continuous homomorphism from G (respectively, $\hat{G}$) into the multiplicative group $\{z \in \mathbb{C} : |z| = 1\}$ is of the form e_ξ for some $\xi \in \hat{G}$ (respectively, of the form e_x for some $x \in G$). Existence of $\hat{G}$ is shown in many standard texts, e.g. [R]. [For $G = \mathbb{R}^n =$ additive group of $n \times 1$ real column matrices, it is convenient to take $\hat{G}$ to be the group of $1 \times n$ real row matrices with $\xi \cdot x$ a matrix product and similarly with all other Abelian Lie groups.]

(ii) A lattice in G is a topologically discrete subgroup $\mathcal{L} \subset G$ for which $T_{\mathcal{L}} = G/\mathcal{L}$ is compact in the quotient topology (e.g., the integer lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$).

Existence of lattices in G follows from Weil's structural theorem. The connected component G_0 of 0 in G is the direct sum of a maximal, connected subgroup K and a subgroup isomorphic and homomorphic to $\mathbb{R}^n$, $n \geq 0$, with G/G_0 discrete.

(iii) Given a lattice $\mathcal{L}$, there is a unique Haar measure $\mu = \mu_{\mathcal{L}}$ on G assigning mass 1 to every $\mathcal{L}$-tiling domain $C \subset G$ (thus C is Borel measurable and G is the disjoint union of the translates of C by members of $\mathcal{L}$ — if we wish, we can take C to have compact closure). Then $\mathcal{L}^\perp = \{j \in \hat{G} : \forall k \in \mathcal{L},\ j \cdot k$ is the zero element in $\mathbb{R}/\mathbb{Z}\}$ is a lattice in $\hat{G}$ called the lattice dual of $\mathcal{L}$ and there is a unique Haar measure $\hat{\mu} = (\hat{\mu})_{\mathcal{L}^\perp}$ on $\hat{G}$ assigning mass 1 to every $\mathcal{L}^\perp$ — tiling domain. Also, μ (respectively, $\hat{\mu}$) induces normalized Haar measure on the compact group $T_{\mathcal{L}} = G/\mathcal{L}$ (respectively, on $T_{\mathcal{L}^\perp} = \hat{G}/\mathcal{L}^\perp$) and $\{e_j : j \in \mathcal{L}^\perp\}$ is an orthonormal basis for $L^2(T_{\mathcal{L}})$ (respectively, $\{e_k : k \in \mathcal{L}\}$ is an orthonormal basis for $L^2(T_{\mathcal{L}^\perp})$.

Examples. In the $\mathbb{R}^n$ case with $\mathcal{L} = \mathbb{Z}^n$, $[0,1)^n$ is a $\mathcal{L}$-tiling domain and the matrix transpose map takes $\mathbb{Z}^n$ to $(\mathbb{Z}^n)^\perp$, so μ is Lebesgue measure on $\mathbb{R}^n$ and $\hat{\mu}$ is Lebesgue measure on $(\mathbb{R}^n)^\wedge$. When G is an n=dimensional Abelian Lie group with finitely many connected components, there is a finite group F for which each of the compact groups $T_{\mathcal{L}} = G/\mathcal{L}$ is isomorphic and homeomorphic to the product of F and the standard n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $n \geq 0$.

(iv) Using the notations in (iii), let $\mathcal{M}_{T_{\mathcal{L}} \times T_{\mathcal{L}^\perp}}$ and $(\mathcal{M}_{T_{\mathcal{L}} \times T_{\mathcal{L}^\perp}})^\sim$ be the spaces of $\mathbb{C}$-valued measurable functions Φ and $\Phi^\sim$ on $G \times \hat{G}$ such that, for all $(k, j) \in \mathcal{L} \times \mathcal{L}^\perp$

and all $(x,\xi) \in G \times \hat{G}$,

$$\Phi(x+k, \xi+j) = e_k(\xi)\Phi(x,\xi) \quad \text{a.e.,} \tag{1.1}$$

$$\Phi^{\sim}(x+k, \xi+j) = e_{-j}(x)\Phi(x,\xi) \quad \text{a.e.,} \tag{1.2}$$

and the $\mathcal{L} \times \mathcal{L}^{\perp}$ periodic functions $|\Phi|$, $|\Phi^{\sim}|$ are in $L^2(T_{\mathcal{L}} \times T_{\mathcal{L}^{\perp}})$.

Note that the magnitudes of Φ and $\Phi^{\sim}$ are periodic in both variables by (1.1) and (1.2). The norms of these functions are understood to be the L^2 norms of their magnitudes as functions on the compact group $T_{\mathcal{L}} \times T_{\mathcal{L}^{\perp}}$ relative to the normalized Haar measure induced by $\mu \times \hat{\mu}$.

(v) Using the notations in (iii), for $f \in L^2(G,\mu)$ and $g \in L^2(\hat{G},\hat{\mu})$, the Zak transforms $Z_{\mathcal{L}}f$ and $Z^{\sim}_{\mathcal{L}^{\perp}}g$ of f and g are the a.e. well defined Fourier series expressions

$$(Z_{\mathcal{L}}f)(x,\cdot) = \sum_{k\in\mathcal{L}} f(x+k)e_{-k}(\cdot) \tag{1.3}$$

$$(Z^{\sim}_{\mathcal{L}^{\perp}}g)(\cdot,\xi) = \sum_{j\in\mathcal{L}^{\perp}} g(\xi+j)e_j(\cdot) \tag{1.4}$$

Note that not only are the roles of x and ξ reversed in (1.3) and (1.4) but, as in (1.1) and (1.2), we also have a change of sign in the exponents.

1.3. **Theorem.** Using the above notations, for each choice of $\mathcal{L}$,

(i) $f \mapsto Z_{\mathcal{L}}f$ is a unitary map from $L^2(G,\mu)$ onto $\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^{\perp}}}$ whose inverse is a.e. well defined by

$$f(x) = ((Z_{\mathcal{L}})^{-1}\Phi)(x) = \int_{T_{\mathcal{L}^{\perp}}} \Phi(x,\xi)d\hat{\mu}(\xi); \tag{1.5}$$

(ii) $g \mapsto Z^{\sim}_{\mathcal{L}^{\perp}}g$ is a unitary map from $L^2(\hat{G},\hat{\mu})$ onto $(\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^{\perp}}})^{\sim}$ whose inverse is a.e. well defined by

$$g(\xi) = ((Z^{\sim}_{\mathcal{L}^{\perp}})^{-1}\Phi^{\sim})(x) = \int_{T_{\mathcal{L}}} \Phi^{\sim}(x,\xi)d\mu(x); \tag{1.6}$$

(iii)

$$\Phi^{\sim}(x,\xi) = (\mathcal{U}\Phi)(x,\xi) = e^{-2\pi i\xi x}\Phi(x,\xi) \tag{1.7}$$

defines a unitary map $\mathcal{U}$ from $\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^{\perp}}}$ onto $(\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^{\perp}}})^{\sim}$.

Proof. (i) For each $\mathcal{L}$-tiling domain $C \subset G$, translation invariance of μ gives

$$\|f\|^2_{L^2(G,\mu)} = \int_C \sum_{k\in\mathcal{L}} |f(x+k)|^2 \tag{1.8}$$

so $(f(x+k))_{k\in\mathcal{L}} \in \ell^2(\mathcal{L})$ for a.e. $x \in G$. Since $\{e_{-k} : k \in \mathcal{L}\}$ is an orthonormal basis for $L^2(T_{\mathcal{L}^{\perp}})$, $(Z_{\mathcal{L}}f)(x,\cdot) \in L^2(T_{\mathcal{L}^{\perp}})$ for a.e. x and a simple change of summation index argument shows that $Z_{\mathcal{L}}f$ satisfies the transformation condition (1.1) with (1.8) then yielding $Z_{\mathcal{L}}f \in \mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^{\perp}}}$. Moreover, (1.5) holds since $f(x)$ is the 0^{th} Fourier coefficient of the $\mathcal{L}^{\perp}$-periodic function $(Z_{\mathcal{L}}f)(x,\cdot)$. Conversely, when we start with $\Phi \in \mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^{\perp}}}$ and f is defined by (1.5), the transformation law (1.1) implies that, for a.e. x, $f(x+k)$ is the $(-k)^{th}$ Fourier coefficient of the $L^2(T_{\mathcal{L}^\perp})$ function $\Phi(x,\cdot)$ from which it follows that $f \in L^2(G,\mu)$ and $\Phi = Z_{\mathcal{L}} f$.

(ii) We merely repeat the arguments in (i) with the roles of x and ξ reversed and using the transformation law (1.2) in place of (1.1).

(iii) is merely an elementary computation showing that $\mathcal{U}$ converts the transformation law (1.1) for Φ to the transformation law (1.2) for $\Phi^\sim$ along with the trivial observation that $\mathcal{U}$ doesn't change magnitudes. □

1.4. **Corollary 1 (The Plancherel Theorem for LCA Groups).** Using the above notations, for each choice of a lattice $\mathcal{L} \subset G$ and corresponding dual lattice $\mathcal{L}^\perp \subset \hat{G}$,

(i) the unitary map $\mathcal{F}_G = (Z^\sim_{\mathcal{L}^\perp})^{-1} \circ \mathcal{U} \circ Z_{\mathcal{L}}$ from $L^2(G,\mu)$ onto $L^2(\hat{G},\hat{\mu})$ is described on the dense subspace $L^1(G,\mu) \cap L^2(G,\mu) \subset L^2(G,\mu)$ by

$$(\mathcal{F}_G f)(\xi) = \hat{f}(\xi) = \int_G f(x) e_{-\xi}(x) d\mu(x); \tag{1.9}$$

(ii) when $g \in L^1(\hat{G},\hat{\mu}) \cap L^2(\hat{G},\hat{\mu})$,

$$\begin{aligned} ((\mathcal{F}_G)^{-1} g)(x) &= ((Z_{\mathcal{L}})^{-1} \circ \mathcal{U}^{-1} \circ Z_{\mathcal{L}^\perp} g)(x) \\ &= (\mathcal{F}_{\hat{G}} g)(-x) = \int_{\hat{G}} g(\xi) e_x(\xi) d\hat{\mu}(\xi) \end{aligned} \tag{1.10}$$

[In particular, of course, (i) not only proves the existence of a unique unitary extension to $L^2(G)$ of the Fourier transform $f \mapsto \hat{f}$ on $(L^1 \cap L^2)(G,\mu)$ but gives an explicit expression for this extension, (ii) gives the standard formula relating $\mathcal{F}_{\hat{G}}$ to the inverse of $\mathcal{F}_G$, and (i) and (ii) show that the only pairs of Haar measures μ, $\hat{\mu}$ on G, $\hat{G}$ for which the Plancherel Theorem holds are $\mu = \mu_{\mathcal{L}}$, $\hat{\mu} = (\hat{\mu})_{\mathcal{L}^\perp}$ for some dual lattice pair $\mathcal{L}$, $\mathcal{L}^\perp$.]

Proof. (i) For $f \in L^1(G,\mu) \cap L^2(G,\mu)$, we use the definitions of $Z_{\mathcal{L}} f$ and $\mathcal{U}$ in (1.3) and (1.7) along with the inversion formula (1.6) for $Z_{\mathcal{L}^\perp}$ to obtain, for each choice of a $\mathcal{L}$-tiling domain $C \subset G$,

$$\begin{aligned} ((Z^\sim_{\mathcal{L}^\perp})^{-1} \circ \mathcal{U} \circ Z_{\mathcal{L}} f(\xi) &= {}_C e^{-2\pi i \xi \cdot x} (Z_{\mathcal{L}} f)(x,\xi) d\mu(x) \\ &= \int_C \sum_{k \in \mathcal{L}} f(x+k) e^{-2\pi i \xi \cdot (x+k)} \\ &= \text{(by translation invariance of } \mu\text{)} \\ &\quad\ \int_G f(y) e_{-\xi}(y) d\mu(y) \\ &= \hat{f}(\xi). \end{aligned}$$

(ii) follows from a similar computation using (1.4) and (1.5) in place and (1.6), the only changes being reversal of the roles of G and $\hat{G}$ changes in the exponents for $Z^\perp_{\mathcal{L}^\perp}$ and $\mathcal{U}^{-1}$. □

1.5. **Corollary 2.** (Poisson Summation Formula) When f satisfies the smoothness and decay properties needed to have both $Z_{\mathcal{L}}f$ and $Z^{\sim}_{\mathcal{L}^{\perp}}\hat{f}$ pointwise well-defined and jointly continuous in an open neighborhood of (0,0),

$$\sum_{k\in\mathcal{L}} f(k) = \sum_{j\in\mathcal{L}^{\perp}} \hat{f}(j). \tag{1.11}$$

Proof. From Theorem 1.3 and Corollary 1, $Z^{\sim}_{\mathcal{L}^{\perp}}\hat{f} = \mathcal{U}Z_{\mathcal{L}}f$. Since $(\mathcal{U}Z_{\mathcal{L}}f)(0,0) = (Z_{\mathcal{L}}f(0,0) = \sum_{k\in\mathcal{L}} f(k)$ while $(Z^{\sim}_{\mathcal{L}^{\perp}}\hat{f})(0,0) = \sum_{j\in\mathcal{L}^{\perp}} \hat{f}(j)$, we obtain (1.11). □

1.6. **Remarks.** Corollary 2 is not surprising since all standard proofs of the Poisson Summation Formula rest on lattice periodization of Fourier integrals and that is precisely what is going on with Zak transforms. Zak transforms can be viewed as discretizations of Fourier integrals and, for the case $G = \mathbb{R}$, can be compared with other discretizations such as the short-time Fourier transform and the Discrete Cosine transform. However, Corollary 1 yields the intriguing converse statement that Fourier integrals are just averages of Zak transforms for any choice of a lattice. Since periodization techniques have been used for over 100 years in harmonic analysis, and since A. Weil's 1940 book [W] on integration on locally compact topological groups alludes to a proof of the Plancherel Theorem for classical Abelian groups via Fourier series ideas with I. Gelfand being sufficiently impressed by this approach to sketch Weil's argument for $\mathbb{R}$ in a 1950 paper [G] on eigenfunction expansions, it is surprising that only a small handful of mathematicians have paid any attention. Perhaps part of the reason is that the Zak transform for $\mathbb{R}$ is often presented as a somewhat arcane way to turn $L^2(\mathbb{R})$ into $L^2(\mathbb{T}^2)$ and its applications are customarily described as part of the discretization machinery germane to certain problems in mathematical physics and applied harmonic analysis. Indeed, Zak's motivation for the transform he introduced in 1967 was to provide a tool for some problems in quantum mechanics. The above discussion is intended to suggest that the Zak transform ought to be seen as a fundamental tool for every aspect of Abelian harmonic analysis with the Fourier transform being just a by-product of Zak transforms and with passage to Zak transform image spaces for calculations equivalent to but often considerably less technical than passage to Fourier domains. We will illustrate this point of view in a forthcoming expository article on square integrable sampling functions, showing how use of the intermediate Zak space $\mathcal{M}_{T_{\mathcal{L}}\times T_{\mathcal{L}^{\perp}}}$ makes unnecessary the customary separate distinction between sampling functions with compact support in the time domain and those with compact support in the Fourier domain. To say the least, such illustrations substantially change the perspective on Fourier transforms and suggest that introductory courses in real analysis should follow-up standard coverage of elementary measure theory and Fourier series for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with definition of the Zak transform $Z_{\mathbb{Z}}$ and at least a sketch of the above argument showing how $Z_{\mathbb{Z}}$ leads quickly and "painlessly" to the Plancherel Theorem for $\mathbb{R}$.

2. Generalized Zak Transforms for Abelian Group Representations.

2.1. **Overview.** The isometry and transformation condition properties discussed above for $Z_{\mathcal{L}}$ are succinctly expressed in the language of group representations by saying that the unitary map $Z_{\mathcal{L}}$ intertwines the restriction to $\mathcal{L}$ of the regular representation $f(\cdot) \mapsto f(x+\cdot)$ of G on $L^2(G,\mu)$ with the modulation representatioin $\Phi(x,\cdot) \mapsto e_\ell(\cdot)\Phi(x,\cdot)$ of $\mathcal{L}$ on the Zak space $\mathcal{M}_{T_{\mathcal{L}} \times T_{\mathcal{L}^\perp}}$. This suggests going on to define and apply generalized Zak transforms intertwining certain unitary representations of LCA groups with modulation representations. One can also look at operator-valued analogs for non Abelian groups, the limitation being that a non-Abelian discrete group $\mathcal{L}$ has a Plancherel Formula if and only if $\mathcal{L}$ is a finite extension of an Abelian group. We won't take time below to discuss non-Abelian generalizations.

2.2. **General Setting for (Abelian) Zak transforms.** (i) $(\ell, x) \mapsto \ell \cdot x$ is a stability-free action of a countable additive group $\mathcal{L}$ on a set X. Thus $k \cdot (\ell \cdot x) = (k+\ell) \cdot x$ for all $x \in X$ and all $k, \ell \in \mathcal{L}$ with $\ell \cdot x = x \Leftrightarrow \ell = 0$.

(ii) There is a σ-finite measure ν on X for which $L^2(X,\nu)$ is a separable Hilbert space and for which ν is quasi $\mathcal{L}$-invariant in the sense that, for each $\ell \in \mathcal{L}$, $x \mapsto \ell \cdot x$ is measurable and we have a Radon-Nikodym derivative $J_\ell(x) = \dfrac{d\nu(\ell \cdot x)}{d\nu(x)}$ defined and > 0 for a.e. x. Then, by the chain rule for Radon-Nikodym derivatives,

$$J_{\ell+k}(x) = J_\ell(k \cdot x) J_k(x) \quad \text{a.e.}$$

and

$$(D_\ell f)(x) = J_\ell(x)^{\frac{1}{2}} f(\ell \cdot x)$$

defines a unitary representation D of $\mathcal{L}$ on $L^2(X,\nu)$.

(iii) The action is regular in the sense that there exists a measurable set C such that X is the disjoint union of the set $\ell \cdot C$, $\ell \in \mathcal{L}$. Hence, X is also the disjoint union of the orbits $\mathcal{L} \cdot x$ as x ranges over C. (Obviously, C plays the role of a $\mathcal{L}$-tiling domain for the special case when $\mathcal{L}$ is a lattice in $X = G$ and $\ell \cdot x = \ell + x$).

Remarks: (i) In practice, we start with a continuous action of a non-discrete LCA group G on a locally compact, Hausdorff space X' (perhaps a topological manifold), take $\mathcal{L}$ to be a lattice in G, take C to be a Borel subset of X' for which $\mathcal{L} \cdot C$ is a ν -null set and $(\ell \cdot C) \cap C = \emptyset$ for each $\ell \in \mathcal{L} \setminus \{0\}$, then take $X = \mathcal{L} \cdot C$. But C could then be replaced by any measurable subset of X containing exactly one point from each $\mathcal{L}$-orbit.

(ii) In general, when a quasi $\mathcal{L}$-invariant measure ν exists, one can construct a finite $\mathcal{L}$-invariant measure μ which is equivalent to ν in the usual measure sense. But, for examples of actions of discrete Abelian matrix groups on $\mathbb{R}^n$ and, more generally, actions by commuting manifold diffeomorphisms along the integral curves of commuting vector fields, there will be a natural choice for ν, e.g. Lebesgue measure on $\mathbb{R}^n$ and the measure defined by a Riemannian volume form in the manifold case. In such cases, replacement of ν by μ is artificial and doesn't add anything new.

2.3. **Notations and Definitions.** In the above general setting:

(i) $(\hat{\mathcal{L}},+)$ is the compact, additive LCA group dual to $\mathcal{L}$ and $\hat{\mu}$ is normalized Haar measure on $\hat{\mathcal{L}}$ with $\{e_\ell : \ell \in \mathcal{L}\}$ the orthonormal basis of $L^2(\hat{\mathcal{L}},\hat{\mu})$ defined as in §1 by $e_\ell(\xi) = e^{2\pi i\ell\cdot x}$.

(ii) For $\psi \in L^2(X,\nu)$, the generalized Zak transform $Z\psi$ of ψ relative to the action of $\mathcal{L}$ and the measure ν is the $L^2(\hat{\mathcal{L}},\hat{\mu})$ -valued function on X well defined ν-a.e. by

$$Z\psi(x,\cdot) = \sum_{\ell\in\mathcal{L}}(D_\ell\psi)(x)e_{-\ell}(\cdot). \tag{2.1}$$

2.4. **Remarks:** The computations we made earlier in the case of the translation action of $\mathcal{L} \subset G$ on G generalize easily to yield the following:

$$\text{(i)} \qquad (ZD_\ell\psi)(x,\xi) = e_\ell(\xi)(Z\psi)(x,\xi) \qquad \text{a.e.} \tag{2.2}$$

so $\psi \mapsto Z\psi$ intertwines the unitary representation D of $\mathcal{L}$ on $L^2(X,\mu)$ and the modulation representation of $\mathcal{L}$ on the image under Z of $L^2(X,\mu)$;

(ii) For any choice of an orbit-cross section set $C \subset X$ as above,

$$\begin{aligned}\int_C\int_{\hat{\mathcal{L}}}|Z\psi(x,\xi)|^2d\hat{\mu}(\xi)d\nu(x) &= \int_C\sum_{\ell\in\mathcal{L}}J_\ell(x)|\psi(\ell\cdot x)|^2d\nu(x)\\ &= \sum_{\ell\in\mathcal{L}}\int_{\ell\cdot C}|\psi(y)|^2d\nu(y) = \|\psi\|^2_{L^2(X,\nu)}\end{aligned} \tag{2.3}$$

Defining the initial expression in (2.3) to be $\|Z\psi\|^2_{\mathcal{M}}$, it follows that Z is an isometry from $L^2(X,\nu)$ onto the Hilbert space $\mathcal{M}$ of measurable functions Φ from $X \times \hat{\mathcal{L}}$ into $\mathbb{C}$ satisfying the transformation condition $D_\ell(\Phi(\cdot,\xi)) = e_\ell(\xi)\Phi(\cdot,\xi)$ a.e. and $\|\Phi\|^2_{\mathcal{M}} = \int_C\int_{\hat{\mathcal{L}}}|\Phi(x,\xi)|^2d\hat{\mu}(\xi)d\nu(x) < \infty$. Indeed, for $\Phi \in \mathcal{M}$, $f = Z^{-1}\Phi$ is a.e. well defined by

$$f(x) = \int_{\hat{\mathcal{L}}}\Phi(x,\xi)d\hat{\mu}(\xi).$$

2.5. **More Notations and Definitions.** In the context of 2.2 – 2.4, for $\phi,\psi \in L^2(X,\nu)$,

(i) the bracket function $[\phi,\psi] = [\phi,\psi]_D$ is the member of $L^1(\hat{\mathcal{L}},\hat{\mu})$ well defined a.e. by

$$[\phi,\psi](\xi) = \int_C Z\phi(x,\xi)\overline{Z\psi(x,\xi)}d\nu(x); \tag{2.4}$$

[Note that by the computations in 2.4 and use of the Cauchy-Schwartz inequality, $(\phi,\psi) \mapsto [\phi,\psi]$ is a bounded, sesquilinear, Hermitian symmetric map from $L^2(X,\nu)\times L^2(X,\nu)$ into $L^1(\hat{\mathcal{L}},\hat{\mu})$ and has the positive semi-definite property $[\psi,\psi] \geq 0$.]

(ii) p_ψ is the $L^1(\hat{\mathcal{L}},\hat{\mu})$ weight function $[\psi,\psi]$;

(iii) supp $p_\psi = \{\xi : p_\psi(\xi) \neq 0\}$ (well defined modulo a $\hat{\mu}$-null set);

(iv) when $\psi \in L^2(X,\nu)\setminus\{0\}$, the D-cyclic subspace $<\psi>_D$ is the closure in $L^2(X,\nu)$ of the span of $\mathcal{B}_\psi = \{D_\ell\psi : \ell \in \mathcal{L}\}$.

Theorem 2.1. *Using the above notations, for ϕ, ψ non-zero members of $L^2(X, \nu)$ and $\ell \in \mathcal{L}$,*

$$(i) \qquad [D_\ell \phi, \psi] = e_\ell[\phi, \psi] = [\phi, D_{-\ell}\psi];$$

$$(ii) \qquad < D_\ell \phi, \psi >_{L^2(X,\nu)} = \int_{\hat{\mathcal{L}}} e_\ell[\phi, \psi] d\hat{\mu} =< \phi, D_\ell \psi >_{L^2(X,\nu)};$$

$$(iii) \qquad < \phi >_D \perp < \psi >_D \Leftrightarrow [\phi, \psi] = 0 \qquad a.e.;$$

$$(iv) \qquad \phi \mapsto \mathcal{J}_\psi(\phi) = \frac{[\phi, \psi]}{p_\psi^{\frac{1}{2}}} \chi_{\text{supp } p_\psi}$$

is a unitary map from $< \psi >_D$ onto the closed subspace $\mathcal{ZH}_\psi$ of $L^2(\hat{\mathcal{L}}, \hat{\mu})$ consisting of members of $L^2(\hat{\mathcal{L}}, \hat{\mu})$ which vanish a.e. off supp p_ψ. *In particular, using (i), $\mathcal{J}_\psi$ intertwines D on $< \psi >_D$ with the modulation representation of $\mathcal{L}$ on $\mathcal{H}_\psi$.*

Sketch of the Proof. (i) and (ii) are easy calculations using the transformation and isometry properties of Z, (iii) follows easily from (ii), and (iv) is another routine calculation using (i), (iii), and the "inner product" properties of $[\cdot, \cdot]$.

Corollary 2.7. For each non-zero $\psi \in L^2(X, \mu)$, the spanning set $\mathcal{B}_\psi$ for $< \psi >_D$

(i) is an orthonormal basis $\Leftrightarrow p_\psi = 1$ a.e.;

(ii) is a Riesz basis $\Leftrightarrow$ both $\|p_\psi\|_\infty$ and $\|\frac{1}{p_\psi}\|_\infty$ are finite;

(iii) is a frame $\Leftrightarrow$ there are positive constants A, B with

$$A\chi_{\text{supp } p_\psi} \leq p_\psi \leq B\chi_{\text{supp } p_\psi} \qquad \text{a.e.}$$

(where we take $A = B = 1$, $\mathcal{B}_\psi$ is said to be a Parseval frame).

Proof. Immediate from the properties of modulation representations and the fact that $\mathcal{J}_\psi(D_\ell \psi) = e_\ell p_\psi^{\frac{1}{2}}$. □

Remark: The above list of connections between properties of the generating set $\mathcal{B}_\psi$ for $< \psi >_D$ and properties of the weight function p_ψ can be expanded considerably to discuss many other connections between $\mathcal{B}_\psi$ and p_ψ (e.g., see [HSWW]). Also, as discussed by Heil and Powell in [HP], the non-averaged weight function $q_\psi = |Z\psi|$ controls the properties of the Gabor system generated by ψ in the $\mathbb{R}^n$ case.

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